On hypersemigroups

Niovi KEHAYOPULU*

TBAE, The Scientific and Technological Research Council of Turkey, Turkey

Abstract: This is from the paper "Hypergroupes canoniques valués et hypervalués" by J. Mittas in Mathematica Balkanica 1971: "The concept of hypergroup introduced by Fr. MARTY in 1934 [Actes du Congrès des Math. Scand. Stocholm 1935, p. 45] is as follows: "A hypergroup is a nonempty set H endowed with a multiplication xy such that, for every $x, y, z \in H$, the following hold: (1) $xy \subseteq H$; (2) $x(yz) = (xy)z$ and (3) $xH = Hx = H$. The first condition expresses that the multiplication is an *hyperoperation* on H, in other words, the composition of two elements x, y of H is a subset of H. It is very easy to prove that for any $x, y \in H$, we have $xy \neq \emptyset$." Although according to Mittas "it is very easy to prove that $xy \neq \emptyset$", this is not possible. The notation $x(yz)$ has a meaning of course if we identify the x by $\{x\}$ and define an operation between sets. The authors working on hypersemigroups added in the definition by Mittas, the following: $x(yz) = (xy)z$ means that $\bigcup_{u \in yz} xu = \bigcup_{v \in xy} vz$. But we never use this last equality in the papers on hypersemigroups in which we always use the $x(yz) = (xy)z$. As a result, most of the results of ordered hypersemigroups are copies from corresponding results on ordered semigroups in which the multiplication "$\cdot$" has been replaced by "$\circ$".

Key words: (ordered) hypersemigroup, ordered semigroup, right ideal, right ideal element, regular, intra-regular

1. Introduction

According to the recent bibliography (2013-today), the theory of hypersemigroups is one of the most active fields of research in the area of hyperalgebraic structures having many applications in automata, probability, geometry, topology, cryptography and coding theory, lattices, binary relations, graphs, hypergraphs, theory of fuzzy and rough sets and other branches of science such as biology, chemistry, and physics (see, [Bijan Davvaz. Semihypergroup theory. Elsevier/Academic Press, London, 2016. viii+156 pp; MR3588314 Reviewer: Jian Tang; Zbl 13f6.20002 Reviewer: Dariush Heidari]; also [Piergiulio Corsini. Hyperstructures and some of the most recent applications. J. Hyperstruct. 2017 (6), Spec. 13th AHA, 1–13; MR3733523 Reviewer: Jian Tang]). The aim of the present paper is to give the right information about the hypersemigroups. Examples illustrate the results.

This is from Zentralblatt:

Marty, F.

On a generalization of the notion of group. (Sur une généralization de la notion de groupe.) (French) [Zbl 0012.05303]

8. Skand. Mat.-Kongr., Stockholm 1934, 45-49 (1935).

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*Correspondence: niovi.kehayopulu-tbae@tubitak.gov.tr

A set with four operations, right and left products, right and left quotients, is a hypergroup if the multiplications are associative. The operations need not be unique. The author shows that uniqueness of one division implies uniqueness of the other and of multiplication. A completely regular hypergroup contains a unit and the inverse of every element. Multiplication of the sets of conjugates in a group is defined so that they form a completely regular hypergroup. The hypergroup of automorphism of a rational fraction belongs, in this sense, to the Galois group of an algebraic extension of the rational numbers.

Reviewer: Engström (New Haven)

In all published papers on ordered hypersemigroups almost all published papers on this structure (related or not, correct or not) are cited in references, information on ordered semigroups from which the results on ordered hypersemigroups are obtained are omitted and the introduction is the same in all of them. Begin with Marty, then "hundreds of papers appeared ..." etc. Can we ever imagine something like that in any paper on algebra (groups, semigroups etc.) or in another structure? It seems like the authors search for an excuse for studying this structure. We have to do that of course with special care for survey articles or books where we have to give complete information about the bibliography but not for every paper on ordered hypersemigroups where the papers used in the articles should be cited. The cited articles on ordered semigroups are usually different than the articles used in the papers. Even if are cited, they are cited among so many (not related) articles on ordered hypersemigroups that is almost impossible for the editors-referees to evaluate them correctly.

An le-semigroup is a semigroup S at the same time a lattice with a greatest element, usually denoted by e ($e \geq a \ \forall a \in S$) such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for every $a, b, c \in S$. An ordered semigroup is a semigroup S at the same time an ordered set such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for every $c \in S$. Every le-semigroup is an ordered semigroup (Birkhoff, Fuchs). We have seen in [16] that from many results on le-semigroups corresponding results on ordered semigroups can be obtained. Then, the results on hypersemigroup can be obtained in a very easy way from ordered semigroups (see, for example [27]). It is enough to prove few necessary lemmas, like the analogous of the property $(A](B] \subseteq (AB]$ of the ordered semigroup, in case of an ordered hypersemigroup.

Most of the papers on ordered semigroups can be transferred to ordered hypersemigroups. It is just a transfer, useful for students, and nice structure, but one can never say that this is something important as there is no any independent, original idea in it. All the results come from ordered semigroups, every example can be obtained from a corresponding result of an ordered semigroup but, many results (not to say all) in the bibliography, are based on definitions without any sense.

This is from [Mittas J. Hypergroupes canoniques valués et hypervalués. Mathematica Balkanica 1971; 1: 181–185] [30]: "the notion of hypergroups due to Fr. Marty in 1933 (according to the literature, it should be 1934) is the following: A hypergroup is a nonempty set H endowed with a multiplication xy such that, for every $x, y, z \in H$, we have

(1) $xy \subseteq H$;

(2) $x(yz) = (xy)z$;

(3) $xH = Hx = H$.

It is very easy to prove that $xy \neq \emptyset$ for every $x, y \in H$."

As one can immediately see, this definition has no sense and certainly it is not possible to prove that $xy \neq \emptyset$. There is no any explanation in his paper what the second and third conditions mean. In the same paper, Mittas defines the hypergroup using addition "$+$" instead of "$\cdot$" as well and gives the first and the

second definition as follows:

(1) $x + y \subseteq H$;

(2) $x + (y + z) = (x + y) + z$

again, without any explanation what the $x + (y + z)$ and $(x + y) + z$ mean.

Regarding the "associativity" of the hyperoperation $\circ$, P. Corsini defines at the same way as Mittas as $x \circ (y \circ z) = (x \circ y) \circ z$ for all x, y, z [Hypergroupes et groupes ordonnés. Rendiconti del Seminario Matematico della Università di Padova 1972; 48: 189–204 (1973)]. One can also find the associativity in [P. Corsini. Sur les homomorphismes d'hypergroupes. Rendiconti del Seminario Matematico della Università di Padova 1974; 52: 117–140 (1975)] (again without any explanation what the notations $x \circ (y \circ z)$ and $(x \circ y) \circ z$ mean). According to MK Sen, R. Ameri and G. Chowdhury [Fuzzy hypersemigroups. Soft Computing 2008; 12 (9): 891–900] a semihypergroup is an hypergroupoid $(H, \circ)$ that is associative i.e. $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$ and refer to [Corsini. Prolegomena of hypergroup theory. 1979; Aviani Editore]. According to T. Vougiouklis [Representations of hypergroups by generalized permutations. Algebra Universalis 1992; 29 (2): 172–183], a hypergroup is a set H equipped with a hyperoperation $\cdot : H \times H \to \mathcal{P}(H)$ (If $A, B \subseteq H$, we set $A \cdot B = \bigcup_{a \in A, b \in B} ab$. If $A = \{a\}$, we write $A \cdot B = aB$) which is associative: $x(yz) = (xy)z$ for all $x, y, z \in H$ and satisfies the reproductive axiom: $hH = Hh = H$ for all $h \in H$. In 1996 Corsini explains what the $(x \circ y) \circ z = x \circ (y \circ z)$ means by adding "i.e. $\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$" [Hypergraphs and hypergroups. Algebra Universalis 1996; 35 (4): 548–555]. In 2000, the concept of hypersemigroup has been given by Davvaz and N.S. Poursalavati [Semihypergroups and $\mathcal{S}$-hypersystems. Pure Math. Appl. 2000; 1: 43–49] as follows: "A hyperstructure is a set H together with a map $\cdot : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of all nonempty subsets of H. A hyperstructure $(H, \cdot)$ is called *semihypergroup* if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y, z in H, where $A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b$ for all $A, B \subseteq H$" and refer, among others to [P. Corsini. Prolegomena of hypergroup theory. Supplement to Riv. Mat. Pura Appl. Aviani Editore, Tricesimo, 1993. 215 pp.] and [B. Davvaz. Topics in algebraic hyperstructures. Ph.D. Thesis. Tarbiat Modarres University 1998 (Persian). The associativity without any explanation has been also given by Davvaz in [Characterizations of sub-semihypergroups by various triangular norms. Czechoslovak Mathematical Journal 2005; 55 (4): 923–932]. According to [B. Davvaz. Strong regularity and fuzzy strong regularity in semihypergroups. The Korean Journal of Computational and Applied Mathematics 2000; 7 (1): 205–213] Marty defined a semihypergroup as a hyperstructure $(H, \cdot)$ such that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y, z in H, where $A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b$ for all $A, B \subseteq H$ in [Marty. Sur une generalization de la notion de groupe, 8iem Congress Math. Scandinaves, Stocholm 1934: 45–49]. As one can immediately see Marty could never give (and never gave) such a definition. Without any doubt, Mittas understood it this way. In 2013, B. Davvaz, A. Dehghan Nezhad and M.M. Heidari [Inheritance examples of algebraic hyperstructures. Information Sciences 2013; 224: 180–187] explain what the associativity relation mean (the same given in 1996 by Corsini).

After some earlier (not related to recent 1969-today) work on multigroups by F. Marty (1934), H.S. Wall (1937), J. Kuntzmann (1937), Oystein Ore (1937), M. Dresher (1938), L.W. Griffiths (1938), W. Prenowitz, (1943) (mainly on projective geometries and descriptive geometries as multigroups), hypergroups have been studied by Mittas beginning (1969) and Corsini (1972). Since then, hundreds of papers on hyperstructures

appeared (as the authors working on this structure say), and in recent years many groups in the world investigate hypersemigroup in research programs based on this definition.

To pass from a semigroup to an hypersemigroup we have to put "$\circ$" instead of the multiplication "$\cdot$" of the semigroup. To see the connection of this definition with the definition of semigroup, let us write down the well known definition of semigroup: A semigroup is a nonempty set S for which there exists a mapping $\cdot : S \times S \to S$ (that is, an operation on S) such that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in S$. If we get this definition and put "$\circ$" instead of "$\cdot$", then we have the following: An hypersemigroup is a nonempty set S for which there exists a mapping $\circ : S \times S \to \mathcal{P}^*(S)$ such that

$$a \circ (b \circ c) = (a \circ b) \circ c \tag{1.1}$$

for every $a, b, c \in S$.

Since the mapping $\circ$ assigns to each element (a, b) of $S \times S$ a nonempty subset (and not an element) of S, it is called "hyperoperation" (it might be called "operation" as well).
$\circ : S \times S \to \mathcal{P}^*(S)$ clearly means that

(1) if $a, b \in S$, then $\emptyset \neq a \circ b \subseteq S$ and

(2) if $(a, b), (c, d) \in S \times S$ such that $a = c$ and $b = d$, then $a \circ b = c \circ d$.

As we have already said, the equality $a \circ (b \circ c) = (a \circ b) \circ c$ has been explained later beginning by Corsini and even later by other authors as follows:

$$\bigcup_{u \in b \circ c} a \circ u = \bigcup_{v \in a \circ b} v \circ c \tag{1.2}$$

(not given by Mittas) while the property (1.2) is written just to explain what the (1.1) means; otherwise we had to stop at this point and no further investigation on hypersemigroups was possible. But we never use the property (1.2) in the text in which we always use the property (1.1). If we used the (1.2), then no problem about (1.1) was ever possible -but we NEVER use it.

In a report of one of my recent papers, the referee of Turk J Math wrote: "In an algebraic hyperstructure the result between two elements of the support set is a set, but in the same time, there is a very clear meaning of the hyperoperation between an element and a set, and in particular between two subsets of the support set. For this reason, *the meaning of writing $a \circ (b \circ c)$ is very very clear and correct.*" However, the first part of the above two sentences does not show how one can conclude (for this reason.....) that the meaning of writing $a \circ (b \circ c)$ is very very clear and correct.

If the concept of property (1.1) was true then, for any elements $a, b, c, d, e, ...$ of S, we could write of course $a \circ b \circ c \circ d \circ e \circ ...$. But property (1.1) is just a notation for which an explanation is needed. We mean, it is not enough to replace the multiplication "$\cdot$" of a semigroup by "$\circ$" of the hypersemigroup to pass from a semigroup to hypersemigroup. So condition (1.1) should be written in a suitable way to pass from semigroups to hypersemigroups. This being so, the following should be noted as the first step to correct property (1.1): As a is an element, $b \circ c$ is a set and "$\circ$" an operation between elements, the $a \circ (b \circ c)$ in property (1.1) has no meaning, the expression $(a \circ b) \circ c$ has no sense as well.

Operations between elements and sets are considered as the same operation in the bibliography. We can do that, if this is just a definition or if it is convenient for the article. But we can never do that for a research

article on ordered hypersemigroups (hypersemigroups, hypergroups, hyperrings) where the two operations are used continuously at the same time in the same result.

These are from the referees of submitted papers of mine in 2015: Referee: "I do not understand what the author is trying to say. But really these are the same operations! I strongly recommend rejection." Another referee: "I do not understand the meaning of this paper. These two hyperoperations are the same so why this strange notation?" Another Referee: "Unfortunately this new hyperoperation coincides with the old. No any difference between these two operations!" Another referee: "As in groups, the star operation should be identified with basic hyperoperation." On a given hypergroupoid (H, o) is constructed a new hyperoperation "." Referee in 2017: "as in groups, the star operation should be identified with basic hyperoperation." The editor: "Your star operation is an additional operation defined on hyperstructures. But on the other hand, this operation is a natural extension of basic hyperoperation so it should be identified with this operation. Similarly as in group theory, operation on elements is identified with operation on cosets. Also the complex operation $\{x\} \circ \{y\} = \{x \circ y\}$ is denoted as $x \circ y$." (As one can see, the last phrase "$\{x\} \circ \{y\} = \{x \circ y\}$ is denoted as $x \circ y$" has no sense). For a paper of mine entitled "Fuzzy sets in $\vee e$-hypergroupoids (2016)", the referee wrote: "The References are not adequate, must include papers on hyperstructures (in particular hypergroups) connected with relations and fuzzy sets in particular those recently published, including the book on fuzzy hyperstructures" while paper was not related to hypergroups or with the recently published books.

In spite of that, phrases, like "The operation "$\circ$" is extended to subsets of S in a *natural way* as $A \circ B := \bigcup_{a \in A, b \in B} a \circ b$" (see, for example, [1, 2, 33, 36])

or

"if A, B are nonempty subsets of S, *then* $A \circ B = \bigcup_{a \in A, b \in B} ab$" [7–9, 29] make this very simple definition incomprehensible. In what sense the operation $\circ$ between elements has been extended in a "*natural way*" to subsets? It is natural extension for a semigroup where $A \cdot B$ is defined by $\{a \cdot b \mid a \in A, b \in B\}$ but there is no any natural extension for an hypersemigroup dealing with sets and elements at the same time. What that "then" means? Is it a conclusion? Is it a definition?

For a groupoid $(S, \cdot)$ we have one operation corresponding to each $(a, b) \in S \times S$ the unique element ab of S. For an hypergroupoid S we have two "operations". One of them is the "operation" between the elements of S called *hyperoperation* as it maps the set $S \times S$ into the set of nonempty subsets of S and the other is the operation between the nonempty subsets of S. As the "operation" between the elements of S is usually denoted by $\circ$, we will show the operation between the nonempty subsets of S by $*$.

According to the bibliography, A^n is the set $\underbrace{A \circ A \circ A \circ ... \circ A}_{n}$

What is the A^n? We are not in a semigroup where the meaning of A^n as $\underbrace{A \circ A \circ A \circ ... \circ A}_{n}$ is clear and simple. In an ordered hypergroupoid,

$A \circ A$ is the set $\bigcup_{a,b \in A} a \circ b$.

$(A \circ A) \circ A$ is the set $\bigcup_{x \in A \circ A, y \in A} x \circ y = \bigcup_{\substack{x \in \bigcup_{a,b \in A} a \circ b, y \in A}} x \circ y$.

$\left((A \circ A) \circ A\right) \circ A$ is the set $\bigcup_{t \in (A \circ A) \circ A, z \in A} t \circ z = \bigcup_{\substack{t \in \bigcup x \circ y, z \in A \\ x \in \bigcup_{a,b \in A} a \circ b, y \in A}} t \circ z$.

$$\left(\left(\left(A \circ A\right) \circ A\right) \circ A\right) \circ A \text{ is the set } \bigcup_{u \in ((A \circ A) \circ A) \circ A, v \in A} u \circ v = \bigcup_{\substack{u \in \bigcup t \circ z, v \in A \\ t \in \bigcup x \circ y, z \in A \\ x \in \bigcup a \circ b, y \in A \\ a, b \in A}} u \circ v$$

and this, after proving the associativity of $\circ$ (that it should corrected to $*$, first appeared in [21]).

Even if we identify the singletons $\{x\}$ by the element x they contain, expressions like $a \circ b \circ C \circ d \circ F \circ G$ have no sense (unless a satisfactory explanation for this notation is given). The quasi-ideal of an ordered groupoid can be extended to ordered hypergroupoid, in a natural way, as follows: This is a nonempty set Q such that (1) if $x \in (Q \circ S] \cap (S \circ Q]$, then $x \in Q$ and (2) if $a \in Q$ and $S \ni b \le a$, then $b \in Q$. In other words, if $x \le t$ for some $t \in a \circ u$, $a \in Q, u \in S$ and $x \le h$ for some $h \in v \circ b$, $v \in S, b \in Q$, then $x \in Q$ and 2) if $a \in Q$ and $S \ni b \le a$, then $b \in Q$. In the bibliography, the definition of the (m, n)-quasi ideal of an ordered hypersemigroup S is given (as in ordered semigroups), as a nonempty subset Q of S such that (1) $(Q^m \circ S] \cap (S \circ Q^n] \subseteq Q$ and (2) if $a \in Q$ and $S \ni b \le a$, then $b \in Q$. But the definition of $Q^m \circ S$ $((S \circ Q^n))$ that leads to a suitable equivalent definition (as we did for $m = n = 1$) is missing here. That means, that we have to stop at this point and investigation on this topic cannot go further (we are not in an ordered semigroup where the meaning of Q^n is obvious).

We use the terms left (right) ideal, bi-ideal, quasi-ideal instead of left (right) hyperideal, bi-hyperideal, quasi-hyperideal and so on, and this is because in this structure there are no two kind of left ideals, for example, to distinguish them as left ideal and left hyperideal. The left ideal in this structure is that one that corresponds to the left ideal of semigroups, to left ideal element of le-semigroups. Similarly, we can use the term "subsemigroup of an hypersemigroup". If we use the word "hyper" for every term in this structure, then we might say hyperregular, hyperintra-regular (something we never say). The Abstract of papers presented in the Summer Meeting East Lansing, Michigan, September 2–5, 1952 have been published in Bull Amer Math Soc. In the Abstract No. 575 [R.A. Goods, D.R. Hughes. Associated groups for a semigroup, p. 624-625] one can find the concept of a bi-ideal of a semigroup S as a subsemigroup B of S satisfying $BSB \subseteq B$. Later S. Lajos used the term "generalized bi-ideal" for the results in which the subsemigroup was not necessary. It is better to use the term bi-ideal instead of generalized by ideal given by Lajos. This is because almost all results hold without the assumption that bi-ideals are subsemigroups (that is, for generalized bi-ideal as most of the authors, based on papers of Lajos say), so we avoid to say "generalized bi-ideal", by calling it "bi-ideal" and only in some very special cases we use the term "subidempotent bi-ideal element" (if we work on poe-semigroups), and "bi-ideal that is a subsemigroup of S at the same time" (if we work on ordered semigroups). S. Lajos, the author who intensively studied the bi-ideals, changed the concept "generalized bi-ideal" to "bi-ideals" in our last joint papers. Besides, taking into account that in a poe-semigroup, every left (right) ideal element is a quasi-ideal element and every quasi-ideal element is a bi-ideal element; and that in an ordered semigroup every left (right) ideal is a quasi-ideal and every quasi-ideal is a bi-ideal (the same if we replace the word bi-ideal with "generalized bi-ideal"), the concept "bi-ideal" instead of "generalized bi-ideal" is certainly much better. For hypersemigroups we use the similar definitions.

We always use the term "hypersemigroup" instead of "semihypergroup". In one of my papers in 2017 that I changed it to "semihypergroup" to satisfy the referees and the editor and publish a paper written in the usual way (with the intention to come back to this paper in a forthcoming one), one of the referees wrote: "Davvaz applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of ordered semigroups. In connection with the terminology "(ordered)

semihypergroup", there is a somewhat chaos in logical flow. One can see that the author used the terminology "(ordered) hypersemigroup" in her several papers. Of course, in this paper, she uses the terminology "(ordered) semihypergroup". In considering the hyper version of (ordered) semigroup, the terminology "(ordered) hypersemigroup" will be more logical approach as we use "hypergroup", "hyperring", hyper BCK-algebra etc. as a generalization (hyper version) of group, ring, BCK-algebra etc. If two terminologies "(ordered) semihypergroup" and "(ordered) hypersemigroup" are same, then it is better to use "(ordered) hypersemigroup" than "(ordered) semihypergroup". The hyper theory of the process "groupoid $\rightarrow$ semigroup $\rightarrow$ group" is the process "hypergroupoid $\rightarrow$ hypersemigroup $\rightarrow$ hypergroup."

Another referee in 2017: "The term "semihypergroup" is wrong. Should be "hypersemigroup.""

2. Definitions, examples

First of all, the definition of the hypersemigroup in the bibliography should be corrected and the goal is to show that it is not enough to get the proof of a result on ordered semigroup and replace the multiplication "·" of the semigroup with ∘ to pass to an ordered hypersemigroup. From every example on an ordered semigroup given by a table of multiplication and an order, in a very easy way, a corresponding example of an ordered hypersemigroup can be obtained. Therefore, we do not have to search for such examples directly for an ordered hypersemigroup which is a much more complicated structure. For any such example, reference should be made to the original example of the ordered semigroup. If we take an example of an ordered semigroup given by a multiplication table and an order and delete one of its elements (so that it is again an ordered semigroup), then again the corresponding example of ordered hypersemigroup does not belong to the authors of the hypersemigroup and a correct reference should be given.

Definition 2.1 *[21] An hypergroupoid is a nonempty set H with an hyperoperation*

$$\circ : H \times H \to \mathcal{P}^*(H) \mid (a, b) \to a \circ b$$

on H and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A, B) \to A * B$$

on $\mathcal{P}^*(H)$ (induced by the hyperoperation $\circ$) defined by

$$A * B = \bigcup_{a \in A, b \in B} a \circ b \tag{2.1}$$

($\mathcal{P}^*(H)$ is the set of all nonempty subsets of H).

It is easy to see that the operation $*$ in (2.1) is well defined.

As a consequence, we have the following which we often use:

    (1) if $x \in A * B$, then $x \in a \circ b$ for some $a \in A$, $b \in B$.

    (2) if $x \in a \circ b$ for some $a \in A$, $b \in B$, then $x \in A * B$; that is $a \in A$, $b \in B$ imply $a \circ b \subseteq A * B$.

We have $\{x\} * \{y\} = x \circ y$. We also have $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$ for any nonempty subsets A and C of H.

Definition 2.2 *[21] An hypergroupoid H is called hypersemigroup if*

$$\{x\} * (y \circ z) = (x \circ y) * \{z\} \tag{2.2}$$

for every $x, y, z \in H$.

If it is convenient and no confusion is possible, we can identify the $\{x\}$ with its element and write, for short, $x * (y \circ z) = (x \circ y) * z$.

For two nonempty subsets A, B of H we write $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \le b$.

Definition 2.3 *[6, 23] An hypersemigroup H is called an ordered hypersemigroup if there is an order relation $\le$ on H such that $a \le b$ implies $a \circ c \preceq b \circ c$ and $c \circ a \preceq c \circ b$ for every $c \in H$.*

Before we start working on hypersemigroups, we certainly have to show that the operation $*$ is associative. This has been first appeared in [21]; see also [23, 24]. It is very essential since, unless this associativity relation, in an expression of the form $A_1 * A_2 * * * A_n$ of elements of $\mathcal{P}^*(S)$ we do not know where to put parentheses and investigation on (ordered) hypersemigroups is not possible. A shorter and better proof than that one given in [24] is as follows: Let A, B, C nonempty subsets of S. Then $A*(B*C) = (A*B)*C$. Indeed: If $x \in A*(B*C)$, then $x \in a \circ u$ for some $u \in B * C$. Since $u \in B * C$, $u \in b \circ c$ for some $b \in B$, $c \in C$. Then we have

$$
\begin{aligned}
x \in a \circ u &= \{a\} * \{u\} \subseteq \{a\} * (b \circ c) \\
&= (a \circ b) * \{c\} \text{ (by (2.2))} \\
&\subseteq (A * B) * C
\end{aligned}
$$

and so $A * (B * C) \subseteq (A * B) * C$. Similarly $(A * B) * C \subseteq A * (B * C)$ and equality holds. So we can write symbols of the form $A * B * C * D * E.....$ without using parentheses.

Proposition 2.4 *Every semigroup $(S, \cdot)$ is an hypersemigroup. Every ordered semigroup $(S, \cdot, \le)$ is an ordered hypersemigroup.*

Proof Let $\circ$ be the mapping of $S \times S$ into $\mathcal{P}^*(S)$ defined by:

$$\circ : S \times S \to \mathcal{P}^*(S) \mid (x, y) \to x \circ y = \{xy\}.$$

Then $(S, \circ)$ is an hypersemigroup. Indeed, if $x, y, z \in S$, then we have

$$
\begin{aligned}
\{x\} * (y \circ z) &= \{x\} * \{yz\} \text{ ($*$ is an operation)} \\
&= x \circ yz = \{x(yz)\} = \{(xy)z\} = (xy) \circ z \\
&= \{xy\} * \{z\} = (x \circ y) * \{z\}.
\end{aligned}
$$

(see also [24]).

Let now $(S, \cdot, \le)$ be an ordered semigroup and $\circ$ the hyperoperation on S defined by $x \circ y = \{xy\}$. Then $(S, \circ, \le)$ is an ordered hypersemigroup. Indeed: Let $a \le b$, $c \in S$ and $x \in a \circ c$. Then $x = ac \le bc \in b \circ c$ and so $a \circ c \preceq b \circ c$. If $x \in c \circ a$, then $x = ca \le cb \in c \circ b$ and so $c \circ a \preceq c \circ b$.

Another proof of this proposition is as follows: Let $\circ$ be the mapping of $S \times S$ into $\mathcal{P}^*(S)$ defined by:

$$\circ : S \times S \to \mathcal{P}^*(S) \mid (a, b) \to a \circ b = \{a, b, ab\}.$$

Then $(S, \circ)$ is an hypersemigroup [4].

Let now $(S, \cdot, \leq)$ be an ordered semigroup and $\circ$ the hyperoperation on S defined by $a \circ b = \{a, b, ab\}$. Then $(S, \circ, \leq)$ is an ordered hypersemigroup. Indeed: Let $a \leq b$, $c \in S$ and $x \in a \circ c$. Then $x = a$ or $x = b$ or $x = ac$. If $x = a$ then, for the element $b \in b \circ c$, we have $a \leq b$. If $x = b$ then, for the element $b \in b \circ c$, we have $b \leq b$. If $x = ac$ then, for the element $bc \in b \circ c$, we have $ac \leq bc$. $\qquad\square$

Example 2.5 *We consider the semigroup given by Table 1. Applying Proposition 2.4, from this semigroup, we get the hypersemigroups given by Tables 2 and 3. Table 1 together with the order $\leq= \{(a, b)\}$ defines an ordered semigroup. Again by Proposition 2.4, Table 2 with the order $\leq= \{(a, b)\}$ and Table 3 with the same order define ordered hypersemigroups as well.*

Table 1. The multiplication table of the semigroup of the Example 2.5.

$\cdot$	a	b	c
a	a	b	a
b	a	b	a
c	a	b	c

Table 2. The hypersemigroup of Example 2.5 applying the first part of Proposition 2.4.

$\circ$	a	b	c
a	$\{a\}$	$\{b\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{a\}$
c	$\{a\}$	$\{b\}$	$\{c\}$

Table 3. The hypersemigroup of Example 2.5 applying the second part of Proposition 2.4.

$\circ$	a	b	c
a	$\{a\}$	$\{a, b\}$	$\{a, c\}$
b	$\{a, b\}$	$\{b\}$	$\{a, b, c\}$
c	$\{a, c\}$	$\{b, c\}$	$\{c\}$

It has been proved by Corsini, Shabir, and Mahmood [4] that every ordered semigroup with the hyperoperation $a \circ b = \{t \in S \mid t \leq ab\}$ is an hypersemigroup. According to Kehayopulu [25], this is not only an hypersemigroup, but an ordered hypersemigroup. So we have the following proposition that is very useful for applications. In fact, using Proposition 2.6 from each example of an ordered semigroup given by a table of multiplication and an order a corresponding example of an ordered hypersemigroup can be obtained.

Proposition 2.6 *[4, 25] Let $(S, \cdot, \leq)$ be an ordered semigroup and $\circ$ the hyperoperation on S defined by*

$$\circ : S \times S \to \mathcal{P}^*(S) \mid (a, b) \to a \circ b := \{t \in S \mid t \leq ab\}.$$

Then $(S, \circ, \leq)$ is an ordered hypersemigroup.

Many examples of ordered hypersemigroups in the literature have been obtained from ordered semigroups, while the papers from which these examples have been obtained are usually not cited in References. We will give few of them.

Example 2.7 *According to Example 17 by Farooq, Khan, and Davvaz [5] the set $S = \{a, b, c, d\}$ endowed with the hyperoperation given by Table 4 and Figure 1 is an ordered hypersemigroup.*

Table 4. The hyperoperation of Example 2.7.

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a\}$
d	$\{a\}$	$\{a, d\}$	$\{a\}$	$\{a\}$
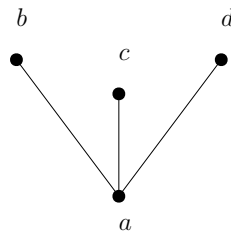
Figure 1. The order of Example 2.7.

We observe that this is not an hypersemigroup. Indeed,

$(c \circ c) * \{b\} = \{a, b\} * \{b\} = (a \circ b) \cup (b \circ b) = \{a\} \cup \{a, b\} = \{a, b\}$ *but*

$\{c\} * (c \circ b) = \{c\} * \{a\} = c \circ a = \{a\}$

and so $(c \circ c) * \{b\} \neq \{c\} * (c \circ b)$.

If this example was correct, then it should be constructed by the ordered semigroup given by Table 5 and the same Figure 1 in the way indicated in Proposition 2.6. But Table 5 does not define a semigroup. Using the Light's associativity test, we can see it immediately by Table 6.

Apparently, the authors of the above article constructed the example from a published example of an ordered semigroup deleting one element, presented it as their own, and have not observed that that was not an ordered semigroup any more.

Example 2.8 *According to Example 3.2 by Tang, Davvaz, and Xie in [34], Table 7 and Figure 2 define an ordered hypersemigroup and "with a small amount of effort one can verify that the sets $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$ and S are quasi-ideal of S but the sets $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$ are not ideals of S." It is not necessary to verify*

Table 5. Table 5 of Example 2.7.

·	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	a	b	a
d	a	d	a	a

Table 6. Table 5 of Example 2.7 is not a semigroup.

(b)	a	b	c	a
a	a	a	a	a
b	a	b	c	a
a	a	a	a	a
d	a	d	a	a

anything, since this example has been obtained from the ordered semigroup given by Table 8 and the same Figure 2 by Kehayopulu in [12] and, according to the methodology described in [25], it is an ordered hypersemigroup and it has the same right, left, bi-ideals and quasi-ideals with the ordered semigroup in [12]. For this example, the authors refer to Pibaljommee and Davvaz [31].

Table 7. The hyperoperation of Example 2.8.

∘	a	b	c	d	e
a	{a}	{a}	{a, b, c}	{a}	{a, b, c}
b	{a}	{a}	{a, b, c}	{a}	{a, b, c}
c	{a}	{a}	{a, b, c}	{a}	{a, b, c}
d	{a, b, d}	{a, b, d}	S	{a, b, d}	S
e	{a, b, d}	{a, b, d}	S	{a, b, d}	S
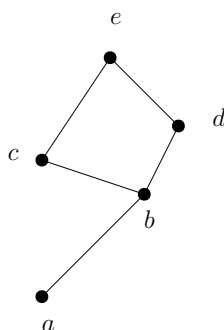
Figure 2. The order of Example 2.8.

Example 2.9 *According to Corsini, Shabir, and Mahmood [4], Table 9 with Figure 3 define an ordered hypersemigroup. This is also the Example 2.5 by Pibaljommee, Wannatong, and Davvaz [32] who refer to [4].*
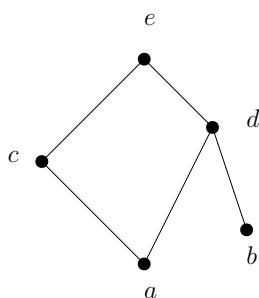
Table 8. The ordered semigroup of Example 2.8.

·	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

The same example is the Example 2 by Tipachot and Pibaljommee [35] who refer to Pibaljommee, Wannatong, and Davvaz [32] while this example has been obtained, in the way indicated in [25], from the example given by Table 10 and the same Figure 3 that appeared in [15]. The authors present it as their own.

Table 9. The hyperoperation of Example 2.9.

∘	a	b	c	d	e
a	$\{a\}$	$\{a,b,d\}$	$\{a\}$	$\{a,b,d\}$	$\{a,b,d\}$
b	$\{a\}$	$\{b\}$	$\{a\}$	$\{a,b,d\}$	$\{a,b,d\}$
c	$\{a\}$	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	S
d	$\{a\}$	$\{a,b,d\}$	$\{a\}$	$\{a,b,d\}$	$\{a,b,d\}$
e	$\{a\}$	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	S

Figure 3. The order of Example 2.9.

Table 10. The ordered semigroup of Example 2.9.

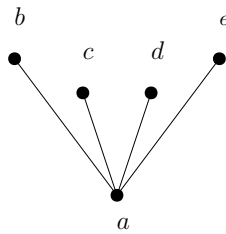| ·	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Example 2.10 *According to Example 18 by Farooq, Khan, and Davvaz [5], the set $S = \{a, b, c, d, e\}$ endowed with the hyperoperation given by Table 11 and Figure 4 is an ordered hypersemigroup. This is also the Example 4 by Changphas and Davvaz [3]. It is the Example 2.6 by Pibaljommee, Wannatong, and Davvaz in [32]. It is the Example 5.2 by Kamali Ardekani and Davvaz [10] as well. According to Changphas, Davvaz, Kamali Ardekani, Pibaljommee, and Wannatong it is easy to see that the quasi-ideals of $(S, \circ, \leq)$ are the sets*

$\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{a, b, e\}$, $\{a, c, e\}$ *and* S.

This ordered hypersemigroup has been constructed by the ordered semigroup given Table 12 and the same Figure 4 in the way indicated in Proposition 2.6. In fact, it is not easy to check that this is an ordered hypersemigroup or to find the set of all quasi-ideals of S, while one can see it immediately, without any effort directly by the example of ordered semigroup given by Kehayopulu in [18]; not cited in the above articles by Davvaz et al.

Table 11. The hyperoperation of Example 2.10.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a, b\}$	$\{a\}$	$\{a, d\}$	$\{a\}$
c	$\{a\}$	$\{a, e\}$	$\{a, c\}$	$\{a, c\}$	$\{a, e\}$
d	$\{a\}$	$\{a, b\}$	$\{a, d\}$	$\{a, d\}$	$\{a, b\}$
e	$\{a\}$	$\{a, e\}$	$\{a\}$	$\{a, c\}$	$\{a\}$

Figure 4. The order of Example 2.10.

Table 12. The ordered semigroup of Example 2.10.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

Remark 2.11 *Table 11 with Figure 5 also define an ordered hypersemigroup. The same holds for every ordered hypersemigroup. If we keep the same hyperoperation and turn the figure upside down we still have an ordered hypersemigroup. Clearly, if $(S, \circ, \leq)$ is an ordered hypersemigroup, then $(S, \circ, \geq)$ is an ordered hypersemigroup.*
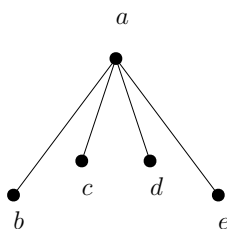
Figure 5. The order in Remark 2.11.

Example 2.12 *According to Example 7 by Pibaljommee and Davvaz [31], "one can show that Table 13 and Figure 6 define a strongly regular ordered hypersemigroup." There is a right reference to this example [19], but it is very difficult (not to say impossible) to check that this is really a strongly regular ordered hypersemigroup. Besides, there is not any reason to check it as it holds for the ordered semigroup from which has been constructed (see [25]).*

Table 13. The hyperoperation of Example 2.12.

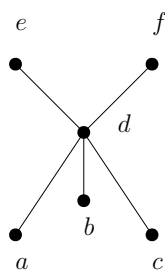| $\circ$	a	b	c	d	e	f
a	$\{b\}$	$\{c\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
b	$\{c\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
c	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
d	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
e	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$
f	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$

Figure 6. The order of Example 2.12.

The next proposition shows that from every ordered set having incomparable elements, an hypersemigroup can be obtained. The obtained hypersemigroup may be but it is not always ordered hypersemigroup.

Proposition 2.13 *Let $(S, \leq)$ be an ordered set. Fix two incomparable elements a and b of S (if such elements exist), and define an hyperoperation on S as follows:*

$x \circ x = \{x\}$ *for every* $x \in S$;

$x \circ y = \{a,b\}$ *for every* $x, y \in S$, $x \neq y$.

Then the set $(S, \circ, \leq)$ *is an hypersemigroup.*

Proof Let $z, t, h \in S$. Then

$$(z \circ t) * \{h\} = \{z\} * (t \circ h) \tag{2.3}$$

Indeed: We consider the cases:

(1) $z = t = h$. Then

$z \circ t = h \circ h = \{h\}$, $(z \circ t) * \{h\} = \{h\} * \{h\} = h \circ h = \{h\}$;

$t \circ h = h \circ h = \{h\}$, $\{z\} * (t \circ h) = \{h\} * \{h\} = \{h\}$

and property (2.3) holds.

(2) $z = t$, $t \neq h$. Then

$z \circ t = t \circ t = \{t\}$, $(z \circ t) * \{h\} = \{t\} * \{h\} = t \circ h = \{a, b\}$;

$\{z\} * (t \circ h) = \{z\} * \{a, b\} = (z \circ a) \cup (z \circ b)$.

On the other hand,

$$(z \circ a) \cup (z \circ b) = \{a, b\} \tag{2.4}$$

Indeed:

(A) If $z = a$, then $z \circ a = a \circ a = \{a\}$. Since $z = a \neq b$, we have $z \circ b = \{a, b\}$. Then $(z \circ a) \cup (z \circ b) = \{a, b\}$ and property (2.4) holds.

(B) Let $z \neq a$. Then $z \circ a = \{a, b\}$.

If $z = b$, then $z \circ b = b \circ b = \{b\}$. Since $z = b \neq a$, we have $z \circ a = \{a, b\}$, then $(z \circ a) \cup (z \circ b) = \{a, b\}$ and property (2.4) holds.

If $z \neq b$, then $z \circ b = \{a, b\}$, then $(z \circ a) \cup (z \circ b) = \{a, b\}$ and again property (2.4) holds.

(3) $z \neq t$, $t = h$. Then

$(z \circ t) * \{h\} = \{a, b\} * \{h\} = (a \circ h) \cup (b \circ h)$;

$\{z\} * (t \circ h) = \{z\} * \{h\} = \{a, b\}$ (since $h \neq z$).

On the other hand,

$$(a \circ h) \cup (b \circ h) = \{a, b\} \tag{2.5}$$

Indeed:

(A) If $h = a$, then $a \circ h = \{a\}$. Since $h = a \neq b$, we have $h \neq b$ and so $b \circ h = \{a, b\}$ and property (2.5) holds.

(B) Let $h \neq a$. Then $a \circ h = \{a, b\}$.

If $b = h$, then $b \circ h = \{b\}$ and property (2.5) holds.

If $b \neq h$, then $b \circ h = \{a, b\}$ and again (2.5) holds.

(4) $z \neq t$, $t \neq h$. Then

$z \circ t = \{a, b\}$, $t \circ h = \{a, b\}$,

$(z \circ t) * \{h\} = \{a, b\} * \{h\} = (a \circ h) \cup (b \circ h)$;

$\{z\} * (t \circ h) = \{z\} * \{a, b\} = (z \circ a) \cup (z \circ b)$.

On the other hand,

$$(a \circ h) \cup (b \circ h) = (z \circ a) \cup (z \circ b) \tag{2.6}$$

Indeed:

(A) If $h = a$, then $a \circ h = \{a\}$. Since $h = a \neq b$, we have $h \neq b$ and so $b \circ h = \{a, b\}$, then $(a \circ h) \cup (b \circ h) = \{a, b\}$. We have

$$(z \circ a) \cup (z \circ b) = \{a, b\} \tag{2.7}$$

Indeed:

If $z = a$, then $z \circ a = \{a\}$. Since $z = a \neq b$, we have $z \circ b = \{a, b\}$ and property (2.7) holds.

Let $z \neq a$. Then $z \circ a = \{a, b\}$.

If $z = b$, then $z \circ b = \{b\}$ and (2.7) holds. If $z \neq b$, then $z \circ b = \{a, b\}$ and again (2.7) holds.

(B) Let $h \neq a$. Then $a \circ h = \{a, b\}$.

If $b = h$, then $b \circ h = \{b\}$, then $(a \circ h) \cup (b \circ h) = \{a, b\}$. As in the previous case, we prove that $(z \circ a) \cup (z \circ b) = \{a, b\}$ and property (2.6) is satisfied.

If $b \neq h$, then $b \circ h = \{a, b\}$. Then $(a \circ h) \cup (b \circ h) = \{a, b\}$. Again $(z \circ a) \cup (z \circ b) = \{a, b\}$ and property (2.6) holds. □

The hypersemigroup defined by Proposition 2.13 is not an ordered hypersemigroup in general. We prove it by the following example.

Example 2.14 *Consider the ordered set given by Figure 7 and the incomparable elements a and d. According to Proposition 2.13, the Table 14 with the order given by Figure 7 is an hypersemigroup. This is not an ordered hypersemigroup as $a \leq b$ but $a \circ b \nleq b \circ b$.*
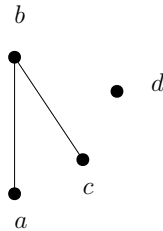

Figure 7. The order of Example 2.14.

Table 14. The hyperoperation of Example 2.14 regarding the a, d.

$\circ$	a	b	c	d
a	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$
b	$\{a, d\}$	$\{b\}$	$\{a, d\}$	$\{a, d\}$
c	$\{a, d\}$	$\{a, d\}$	$\{c\}$	$\{a, d\}$
d	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$

If we consider the incomparable elements a and c, according to Proposition 2.13, we obtain the Table 15; this table with Figure 7 defines an hypersemigroup. One can check that this is an ordered hypersemigroup.

We also have $c//d$, $b//d$, so if we replace the $\{a, d\}$ of Table 14 with $\{c, d\}$ or $\{b, d\}$ we obtain two tables, each of them with the same Figure 7 defines an hypersemigroup. The hypersemigroup defined by the incomparable elements c and d is not an ordered hypersemigroup as $a \leq b$ while $a \circ a \nleq b \circ a$ ($a \circ a = \{a\}$ and $b \circ a = \{c, d\}$). The hypersemigroup defined by the incomparable elements b and d is not an ordered hypersemigroup as well as $a \leq b$ while $a \circ b \nleq b \circ b$ ($a \circ b = \{b, d\}$ and $b \circ b = \{b\}$).

Table 15. The hyperoperation of Example 2.14 regarding the a, c.

∘	a	b	c	d
a	$\{a\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$
b	$\{a,c\}$	$\{b\}$	$\{a,c\}$	$\{a,c\}$
c	$\{a,c\}$	$\{a,c\}$	$\{c\}$	$\{a,c\}$
d	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{d\}$

It is very difficult (not to say impossible) to make examples of ordered hypersemigroups by hand, to write programs or even to check if they are correct or not. Besides, we do not need to do so since we can get whatever we like from ordered semigroups. So for the examples on ordered hypersemigroups reference to ordered semigroups should be given (even an example of 5 or 6 elements it is difficult to be checked by hand). If we use an algorithm to do that, then still reference is needed to be able someone to check their validity. There are wrong examples in the bibliography, that is a further indication than the referee was not able to check them during the refereeing process or he/she just believed they are correct.

3. Regular, intra-regular ordered hypersemigroups

An ordered semigroup $(S, \cdot, \leq)$ is called regular [18] if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. If we want to extend it to an ordered hypersemigroup, then we have to write "∘" instead of the "·". But as the $a \circ x \circ a$ has no sense, we have to write it as $(a \circ x) * \{a\}$ or $\{a\} * (x \circ a)$ (they are equal). Then, the $a \circ x$ is a subset of H, $x \circ a$ is a subset of H, $\{a\}$ is a subset of H and "$*$" is an operation between sets. So the axa part of the regularity has been written in a suitable way as $(a \circ x) * \{a\}$. As this is a set, we have to write "$\{a\}$" (instead of the "a" in $a \leq axa$). As the "$\leq$" is a relation between elements in ordered semigroups, to transfer it to an ordered hypersemigroup we have to use a relation between sets. Let us show it by $\preceq$ and write $\{a\} \preceq (a \circ x) * \{a\}$. It is natural now to define the $\{a\} \preceq (a \circ x) * \{a\}$ as follows: $a \leq t$ for some $t \in (a \circ x) * \{a\}$.

So the concept of regularity of ordered semigroups $(S, \cdot, \leq)$ (for every $a \in S$ there exists $x \in S$ such that $a \leq axa$) can be transferred to an ordered hypersemigroup S as follows:

Definition 3.1 *[26] An ordered hypersemigroup $(S, \circ, \leq)$ is called regular if for every $a \in S$ there exists $x \in S$ such that $\{a\} \preceq (a \circ x) * \{a\}$; in the sense that for every $a \in S$ there exist $x, t \in S$ such that $t \in (a \circ x) * \{a\}$ and $a \leq t$.*

Let us come now to the concept of intra-regularity.

An ordered semigroup $(S, \cdot, \leq)$ is called intra-regular [13, 14] if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$.

According to the Abstract in [3], the concept of intra-regular ordered hypersemigroups has been introduced. According to p. 502, l. 6–8 in [3], the concept of intra-regular ordered hypersemigroup generalizes the concept of an intra-regular ordered semigroup.

This is the Definition 3.14 by Changphas and Davvaz in [3]: An ordered hypersemigroup $(S, \circ, \leq)$ is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq x \circ a^2 \circ y$ where a^2 means $a \circ a$.

As we see, this is exactly the definition of regular ordered semigroups in which "·" has been changed to

$\circ$. But what is the $x \circ a \circ a \circ y$ in this structure and how the order $\leq$ can be used in this definition? This definition does not have any meaning.

To correct it, first we get the definition of the ordered semigroup and put $\circ$ instead of "$\cdot$" and we have $a \leq x \circ a \circ a \circ y$. Notation $x \circ a \circ a \circ y$ does not have any sense so should be written as $(x \circ a) * (a \circ y)$. This being a set, the a should be replaced by $\{a\}$. The order $\leq$ being a relation between elements should be changed to a relation between sets, say $\preceq$, and we have $\{a\} \preceq (x \circ a) * (a \circ y)$. What is the $\{a\} \preceq (x \circ a) * (a \circ y)$ now? A natural definition is the following: $a \leq t$ for some $t \in (x \circ a) * (a \circ y)$. So the correct definition of intra-regularity should be as follows:

Definition 3.2 *[23] An ordered hypersemigroup $(S, \circ, \leq)$ is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $\{a\} \preceq (x \circ a) * (a \circ y)$; that is, for every $a \in S$ there exist $x, y, t \in S$ such $t \in (x \circ a) * (a \circ y)$ and $a \leq t$.*

After Definition 3.14 in the above mentioned article, the authors give the following example.

This is copy of the Example 7 by Changphas and Davvaz in [3]: We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

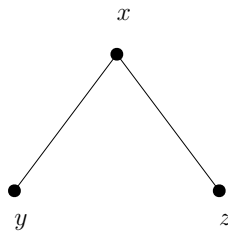Table 16. The multiplication table of the Example 7 in [3].

$\circ$	x	y	z
x	x	$\{x,y\}$	$\{x,z\}$
y	x	$\{x,y\}$	$\{x,y\}$
z	x	$\{x,y\}$	z

$$\leq = \{(x,x), (y,y), (z,z), (y,x), (z,x)\}$$

and Table 16. The covering relation and the figure of S are given by:

$$\prec = \{(y,x), (z,x)\}$$

and Figure 8.

Figure 8. The order of the Example 7 in [3].

Now, we have

(A) $x \leq x \circ x^2 \circ x = x \circ x \circ x = x$.

(B) $y \leq x \circ y^2 \circ x = x \circ \{x,y\} \circ x = \{x,y\} \circ x = x$.

(C) $z \le z \circ z^2 \circ z = z \circ z \circ z = z$.

Therefore, $(S, \circ, \le)$ is intra-regular.

Apparently, the elements x and z in Table 16 are in fact the singletons $\{x\}$ and $\{z\}$, respectively (written, for short as x and y -otherwise Table 16 does not define an hypersemigroup). But in property (A), the authors write: $x \le x \circ x^2 \circ x$. Looking at the table, the x^2 is the element x. So they write $x \circ x^2 \circ x = x \circ x \circ x = x$ and $x \le x$. But x^2 is not the element x but the set $\{x\}$, so the $x \circ \{x^2\}$ in (A) has no sense. According to the definition given in the paper, the properties (A)–(C) have no sense as well and should be corrected as follows:

(A) $\{x\} \preceq (x \circ x) * (x \circ x)$ as

$(x \circ x) * (x \circ x) = \{x\} * \{x\} = x \circ x = \{x\}$, $x \in (x \circ x) * (x \circ x)$ and $x \le x$.

(B) $\{y\} \preceq (y \circ y) * (y \circ y)$ as

$$(y \circ y) * (y \circ y) = \{x, y\} * \{x, y\} = \bigcup_{u \in \{x,y\}, v \in \{x,y\}} u \circ v = \{x, x\} \cup \{x, y\} \cup \{y, x\} \cup \{y, y\}$$

$$= \{x\} \cup \{x, y\} \cup \{x, y\} \cup \{y\} = \{x, y\},$$

$y \in (y \circ y) * (y \circ y)$ and $y \le y$.

(C) $\{z\} \preceq (z \circ z) * (z \circ z)$ as $z \in (z \circ z) * (z \circ z)$ and $z \le z$.

This example is wrong, that is a further indication that the definition of intra-regularity needs correction.

In addition, there is a mistake in the example, and this is not an hypersemigroup, but this does not affect what we said about the definition of intra-regularity (always given at the same way in the literature). We have

$(x \circ y) * \{z\} = \{x, y\} * \{z\} = (x \circ z) \cup (y \circ z) = \{x, z\} \cup \{x, y\} = \{x, y, z\}$,

$\{x\} * (y \circ z) = \{x\} * \{x, y\} = (x \circ x) \cup (x \circ y) = \{x\} \cup \{x, y\} = \{x, y\}$ but

$(x \circ y) * \{z\} \ne \{x\} * (y \circ z)$, so this is not an hypersemigroup.

Proposition 3.3 *If $(S, \cdot, \le)$ is a regular (resp. intra-regular) ordered semigroup and $(S, \circ, \le)$ the ordered hypersemigroup constructed by $(S, \cdot, \le)$ in the way indicated in Proposition 2.6, then $(S, \circ, \le)$ is regular (resp. intra-regular).*

Proposition 3.4 *(see also the Example 5 in [3]). If $(S, \le)$ is an ordered set and $\circ$ an hyperoperation on S defined by $a \circ b = \{a, b\}$, then $(S, \circ, \le)$ is a regular and intra-regular ordered hypersemigroup.*

Proof We have $(a \circ b) * \{c\} = (a \circ c) \cup (b \circ c) = \{a, c\} \cup \{b, c\} = \{a, b, c\}$ and $\{a\} * (b \circ c) = (a \circ b) \cup (a \circ c) = \{a, b\} \cup \{a, c\} = \{a, b, c\}$; thus we have $(a \circ b) * \{c\} = \{a\} * (b \circ c)$. Let now $a \le b$ and $c \in S$. Since $a \le b$ and $c \le c$, we have $\{a, c\} \preceq \{b, c\}$. Since $\{a, c\} = a \circ c$ and $\{b, c\} = b \circ c$, we have $a \circ c \preceq b \circ c$. Since $a \circ c = c \circ a$ and $b \circ c = c \circ b$, we also have $c \circ a \preceq c \circ b$ and so S is an ordered hypersemigoup. Let now $a \in S$. Then $\{a\} \preceq (a \circ x) * \{a\}$ for some $x \in S$. Indeed, for the element $t = x = a$, we have $t \in (a \circ a) * \{a\}$ and $t \le a$. We also have $\{a\} \preceq (x \circ a) * (a \circ y)$ for some $x, y \in S$. Indeed, for the element $t = x = y = a$, we have $t \in (a \circ a) * (a \circ a)$ and $t \le a$. Thus S is regular and intra-regular. □

According to the next example, if we get the order of any ordered hypersemigroup S and apply Proposition 3.4, we get an ordered hypersemigroup that is different from the ordered hypersemigroup S.

Example 3.5 *The order of the (previous) Example 2.12 with the hyperoperation given in Proposition 3.4 defines the ordered hypersemigroup given by Table 17 and the same Figure 6 that is different than the ordered hypersemigroup of the Example 2.12.*

Table 17. The hyperoperation of Example 3.5.

∘	a	b	c	d	e	f
a	$\{a)$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{a,e\}$	$\{a,f\}$
b	$\{b,a\}$	$\{b\}$	$\{b,c\}$	$\{b,d\}$	$\{b,e\}$	$\{b,f\}$
c	$\{c,a\}$	$\{c,b\}$	$\{c\}$	$\{c,d\}$	$\{c,e\}$	$\{c,f\}$
d	$\{d,a\}$	$\{d,b\}$	$\{d,c\}$	$\{d\}$	$\{d,e\}$	$\{d,f\}$
e	$\{e,a\}$	$\{e,b\}$	$\{e,c\}$	$\{e,d\}$	$\{e\}$	$\{e,f\}$
f	$\{f,a\}$	$\{f,b\}$	$\{f,c\}$	$\{f,d\}$	$\{f,e\}$	$\{f\}$

4. Every result on an ordered hypersemigroup is a consequence of a corresponding result of an ordered semigroup

Most of the results on ordered semigroups can be transferred to ordered hypersemigroups. The published papers on ordered hypersemigroups consist from results transferred from ordered semigroups. Whenever we have a look at any published result on an ordered hypersemigroup, we immediately know the theorem of ordered semigroup from which has been transferred. There is no any result in published papers on ordered hypersemigroups that does not come from an ordered semigroup.

Let us give the following proposition on *poe*-groupoids. As corollary, the corresponding result on hypergroupoids can be obtained. The analogous proposition in case of an ordered groupoid is given. Then the corresponding result on ordered hypergroupoid can be obtained just by replacing the "·" of the ordered groupoid by ∘.

A *poe*-groupoid is a groupoid S at the same time an ordered set having a greatest element e ($e \geq a$ $\forall a \in S$) such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for every $c \in S$. An element a of a *poe*-groupoid is called a left (resp. right) ideal element if $ea \leq a$ (resp. $ae \leq a$); an ideal element if it is both a right and a left ideal element. It is called a quasi-ideal element if $ae \wedge ea$ exists and $ae \wedge ea \leq a$ [11].

Proposition 4.1 *[poe-groupoid] [11] Let S be a poe-groupoid at the same time semilattice under $\wedge$, a a right ideal element and b a left ideal element of S. Then the $a \wedge b$ is a quasi-ideal element of S.*

Proof Indeed, $(a \wedge b)e \wedge e(a \wedge b) \leq ae \wedge eb \leq a \wedge b$ and so $a \wedge b$ is a quasi-ideal element of S. □

An ordered groupoid is a groupoid with an order that is compatible with the ordering. A nonempty subset A of an ordered groupoid $(S, \cdot, \leq)$ is called a right (resp. left) ideal of S if (1) $AS \subseteq A$ (resp. $SA \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$; that is $(A] = A$. It is called ideal if it is both right and left ideal. For a subset A of S, we denote by $(A]$ the subset of S defined by $\{t \in S$ such that $t \leq a$ for some $a \in A\}$. A nonempty subset Q of an ordered groupoid S is called a quasi-ideal if (1) $(QS] \cap (SQ] \subseteq Q$ and (2) if $a \in Q$ and $S \ni b \leq a$, then $b \in Q$; that is $(Q] = Q$ [16]. If the multiplication of an ordered groupoid is associative, then it is called an ordered semigroup. By putting $*$ instead of "·" we get the corresponding definitions for the hyper case.

Corollary 4.2 *[hypergroupoid] If $(S, \circ)$ is an hypergroupoid, A is a right ideal and B is a left ideal of S, then $A \cap B$ is a quasi-ideal of S.*

Proof Let A be a right ideal and B a left ideal of S. Then $(\mathcal{P}^*(S), \cdot, \subseteq)$ is a *poe*-groupoid, A is a right ideal element and B is a left ideal element of S. By Proposition 4.1, $A \cap B$ is a quasi-ideal element of $(\mathcal{P}^*(S), \cdot, \subseteq)$ and so an ideal of S. □

Proposition 4.3 *[ord. groupoid] If $(S, \cdot, \leq)$ is an ordered groupoid, A is a right ideal and B is a left ideal of S, then the intersection $A \cap B$ is a quasi-ideal of S.*

Proof Let A be a right ideal and B a left ideal of S. Take $a \in A$, $b \in B$ $(A, B \neq \emptyset)$. Then $ab \in AS \cap SB \subseteq A \cap B$ and so $A \cap B \neq \emptyset$. In addition,

$$\Big[(A \cap B)S\Big] \cap \Big[S(A \cap B)\Big] \subseteq (AS] \cap (SB] \subseteq (A] \cap (B] = A \cap B$$

and $x \in A \cap B$ and $S \ni y \leq x$ implies $y \in A \cap B$ and so $A \cap B$ is a quasi-ideal of S. □

Proposition 4.4 *[ord. hypergr.] Let $(S, \circ, \leq)$ be an ordered hypergroupoid, A a right ideal and B a left ideal of S. Then the intersection $A \cap B$ is a quasi-ideal of S.*

Proof Exactly as in an ordered semigroup, $A \cap B$ is a nonempty set. Moreover,

$$\Big[(A \cap B) * S\Big] \cap \Big[S * (A \cap B)\Big] \subseteq (A * S] \cap (S * B] \subseteq (A] \cap (B] = A \cap B.$$

and $(A \cap B] = A \cap B$. Thus $A \cap B$ is a quasi-ideal of S. □

As we see, the proof of Proposition 4.4 is exactly the same with the proof of Proposition 4.3. Proposition 4.4 is the Theorem 3.4 by Changphas and Davvaz in [3] but its proof should be corrected (the definition of the quasi-ideal is wrong in it -though correct in other parts of the paper). In addition, the theorem holds for ordered hypergroupoids in general.

This is from the Abstract of the above mentioned article by Changphas and Davvaz: "We introduce the concepts of bi-hyperideals and quasi-hyperideals and present several examples of them. In particular, we introduce the concepts of intra-regular ordered semihypergroups and give their characterizations in terms of bi-hyperideals and quasi-hyperideals."

The Theorem 3.16 by Changphas and Davvaz in [3]: Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then we have the following:

(1) S is intra-regular if and only if for any bi-hyperideal B and any quasi-hyperideal Q of S, we have $B \cap Q \subseteq (S \circ B \circ Q \circ S]$.

(2) S is intra-regular if and only if for any bi-hyperideal B and any quasi-hyperideal Q of S, we have $B \cap Q \subseteq (S \circ Q \circ B \circ S]$.

The authors consider that the bi-hyperideals are subsemigroups at the same time; we do not consider them subsemigroups. Besides, in the proof of (1) the subsemigroup is not necessary, and considering the bi-hyperideals not subsemigroups in (2), we have a little change in the proof of (2); $B(a) = (a \cup aSa]$ instead of $B(a) = (a \cup a^2 \cup aSa]$ that is more technical. This does not affect what we would like to say.

We will rewrite the proof by Changphas and Davvaz to be able to compare it with the proof given in the present paper. The following is copy from [3].

(1) $\Rightarrow$. Assume that S is intra-regular. Let $a \in B \cap Q$. Since S is intra-regular, there exist $x, y \in S$ such that $a \leq x \circ a^2 \circ y$. We have

$$
\begin{aligned}
a \;\; &\leq \;\; x \circ a^2 \circ y \leq x \circ a \circ (x \circ a^2 \circ y) \circ y \\
&= \;\; x \circ (a \circ x \circ a) \circ a \circ y^2 \\
&\subseteq \;\; S \circ (B \circ S \circ B) \circ Q \circ S \\
&\subseteq \;\; S \circ B \circ Q \circ S.
\end{aligned}
$$

Then $B \cap Q \subseteq (S \circ B \circ Q \circ S]$.

$\Leftarrow$. Let $a \in S$. We consider the bi-hyperideal $B(a)$ and a quasi-hyperideal $Q(a)$. Then,

$$
\begin{aligned}
a \in B(a) \cap Q(a) \;\; &\subseteq \;\; (S \circ B(a) \circ Q(a) \circ S] \\
&= \;\; (S \circ (a \cup a^2 \cup a \circ S \circ a] \circ (a \cup ((a \circ S] \cap (S \circ a])] \circ S] \\
&\subseteq \;\; ((S \circ a \cup S \circ a^2 \cup S \circ a] \circ (a \cup a \circ S]] \circ S] \\
&\subseteq \;\; ((S \circ a] \circ (a \cup (a \circ S]] \circ S] \\
&\subseteq \;\; ((S \circ a] \circ (a \circ S \cup (a \circ S^2]] \\
&\subseteq \;\; ((S \circ a^2 \circ S] \cup (S \circ a^2 \circ S^2]] \\
&\subseteq \;\; ((S \circ a^2 \circ S]] \\
&= \;\; (S \circ a^2 \circ S].
\end{aligned}
$$

Thus, S is intra-regular.

(2) $\Rightarrow$. Assume that S is intra-regular. Let $a \in B \cap Q$. Since S is intra-regular, there exist $x, y \in S$ such that $a \leq x \circ a^2 \circ y$. By

$$
\begin{aligned}
a \;\; &\leq \;\; x \circ a^2 \circ y \leq x \circ (x \circ a^2 \circ y) \circ a \circ y \\
&= \;\; x^2 \circ a \circ (a \circ y \circ a) \circ y \\
&\subseteq \;\; S \circ Q \circ (B] \circ S \circ B) \circ S \\
&\subseteq \;\; S \circ Q \circ B \circ S,
\end{aligned}
$$

it follows $B \cap Q \subseteq (S \circ Q \circ B \circ S]$.

$\Leftarrow$. Let $a \in S$. We consider a bi-hyperideal $B(a)$ and a quasi-hyperideal $Q(a)$ of S; then

$$
\begin{aligned}
a \in B(a) \cap Q(a) \;\; &\subseteq \;\; (S \circ Q(a) \circ B(a) \circ S] \\
&= \;\; (S \circ (a \cup ((a \circ S] \cap (S \circ a])] \circ (a \cup a^2 \cup a \circ S \circ a] \circ S] \\
&\subseteq \;\; (S \circ (a \cup (S \circ a]] \circ (a \circ S \cup a^2 \circ S \cup a \circ S \circ a \circ S]] \\
&\subseteq \;\; ((Sa \cup (S^\circ a]] \circ (a \circ S]] \\
&\subseteq \;\; ((S \circ a](a \circ S]] \\
&\subseteq \;\; ((S \circ a^2 \circ S]] \\
&= \;\; (S \circ a^2 \circ S].
\end{aligned}
$$

Thus, S is intra-regular.

This is the proof for an ordered semigroup in which the "·" has been replaced by $\circ$.

Let us first give the proof of the $\Rightarrow$-part of (1) for an ordered semigroup on the line of the proof given by Changphas and Davvaz in [3] and then, based on the proof of ordered semigroups, we will correct the proof in [3].

Later, we will see that the theorem by Changphas and Davvaz can be obtained using methods of the le-semigroups.

Regarding the ordered semigroups:

Let $a \in B \cap Q$. Since S is intra-regular, there exist $x, y \in S$ such that

$$\begin{aligned}
a &\leq& xa^2y \leq xa(xa^2y)y \leq xax(xa^2y)ay^2 \\
&=& x(ax^2a)a(yay^2) \in S(BSB)QS \subseteq SBQS
\end{aligned}$$

and so $a \in (SBQS]$.

The proof in case of an ordered hypersemigroup is as follows:

Let $a \in B \cap Q$. Since S is intra-regular, there exist $x, y \in S$ such that

$$\begin{aligned}
\{a\} &\preceq& \{x\} * \{a\} * \{a\} * \{y\} \\
&\preceq& \{x\} * \{a\} * \Big(\{x\} * \{a\} * \{a\} * \{y\}\Big) * \{y\} \\
&\preceq& \Big(\{x\} * \{a\} * \{x\}\Big) * \Big(\{x\} * \{a\} * \{a\} * \{y\}\Big) * \{a\} * \{y\} * \{y\} \\
&=& \{x\} * \Big(\{a\} * \{x\} * \{x\} * \{a\}\Big) * \{a\} * \Big(\{y\} * \{a\} * \{y\} * \{y\}\Big) \\
&\subseteq& S * (B * S * B) * Q * S \\
&\subseteq& S * B * Q * S.
\end{aligned}$$

We have

$$\{a\} \preceq \{x\} * \{a\} * \Big(\{x\} * \{a\} * \{a\} * \{y\}\Big) * \{y\} \subseteq S * B * Q * S,$$

then $\{a\} \subseteq (S * B * Q * S]$ and so $a \in (S * B * Q * S]$.

The proof is essentially the same with the ordered semigroup case, but some basic properties in the proof for ordered semigroups, should be proved for ordered hypersemigroups.

In the proof we used the following properties:

(A) $A \preceq B$ implies $A * C \preceq B * C$ and $C * A \preceq C * B$. Indeed: Let $u \in A * C$. Then there exist $a \in A$, $c \in C$ such that $u \in a \circ c$. Since $a \in A$ and $A \preceq B$, there exists $b \in B$ such that $a \leq b$. Since $a \leq b$, we have $a \circ c \preceq b \circ c$. Since $u \in a \circ c$, there exists $v \in b \circ c \subseteq B * C$ such that $u \leq v$. Thus we have $A * C \preceq B * C$. The proof of $C * A \preceq C * B$ is similar.

(B) If $A \preceq B \subseteq C$, then $A \subseteq (C]$. Indeed: Let $a \in A$. Since $A \preceq B$, there exists $b \in B$ such that $a \leq b$. Since $B \subseteq C$, we have $a \leq b \in C$ and so $a \in (C]$.

The following should ne noted:

Whenever we look at Theorem 3.16 by Changphas and Davvaz in [3], we immediately see that it comes from the following Proposition 4.5 on le-semigroups or from Proposition 4.6 on ordered semigroups. A second proof of the Theorem 3.16 in [3] has been given in Proposition 4.7 below.

A *poe*-semigroup S is called intra-regular if $a \le ea^2e$ for every $a \in S$. An element b of S is called a bi-ideal element if $beb \le b$ [11].

Proposition 4.5 *[le-sem.] If S is an le-semigroup, then the following statements hold:*

(1) *S is intra-regular if and only if for any bi-ideal element b and any quasi-ideal element q of S, we have $b \wedge q \le ebqe$.*

(2) *S is intra-regular if and only if for any bi-ideal element b and any quasi-ideal element q of S, we have $b \wedge q \le eqbe$.*

Proof A *poe*-semigroup S is intra-regular if and only if for every right ideal element a and every left ideal element b of S, we have $a \wedge b \le ba$ [11].

(1) $\Rightarrow$. Let b be a bi-ideal element and q a quasi-ideal element of S. Since S is intra-regular, we have $b \wedge q \le e(b \wedge q)(b \wedge q)e \le ebqe$.
$\Leftarrow$. Let a be a right ideal element and b a left ideal element of S. Then a is a quasi-ideal element and b a bi-ideal element of S as well. By hypothesis, we have $b \wedge a \le ebae \le ba$, so S is intra-regular.

(2) $\Rightarrow$. Let b be a bi-ideal element and q a quasi-ideal element of S. Since S is intra-regular, we have $b \wedge q \le e(b \wedge q)(b \wedge q)e \le eqbe$.
$\Leftarrow$. Let a be a right ideal element and b a left ideal element of S. Then a is a bi-ideal element and b a quasi-ideal element of S as well. By hypothesis, we have $a \wedge b \le ebae \le ba$, so S is intra-regular. $\square$

The corresponding result for an ordered semigroup is the following.

A nonempty subset B of an ordered semigroup S is called a bi-ideal of S if (1) $BSB \subseteq B$ and (2) if $a \in B$ and $S \ni b \le a$, then $b \in B$ [12]. A nonempty subset B of an ordered hypersemigroup S is called a bi-ideal of S if (1) $B * S * B \subseteq B$ and (2) if $a \in B$ and $S \ni b \le a$, then $b \in B$.

An ordered semigroup S is intra-regular if and only if $A \subseteq (SA^2S]$ for every nonempty subset A of S [16]. An ordered hypersemigroup S is intra-regular if and only if, for every nonempty subset A of S, we have $A \subseteq (S * A * A * A]$.

Proposition 4.6 *[ord. sem.] Let $(S, \cdot, \le)$ be an ordered semigroup. Then we have the following:*

(1) *S is intra-regular if and only if for any bi-ideal B and any quasi-ideal Q of S, we have $B \cap Q \subseteq (SBQS]$.*

(2) *S is intra-regular if and only if for any bi-ideal B and any quasi-ideal Q of S, we have $B \cap Q \subseteq (SQBS]$.*

Proof An ordered semigroup S is intra-regular if and only if for any right ideal A and any left ideal B of S we have $A \cap B \subseteq (BA]$ [16].

(1) $\Rightarrow$. Let B be a bi-ideal and Q a quasi-ideal of S. Since S is intra-regular, we have

$$B \cap Q \subseteq \Big(S(B \cap Q)(B \cap Q)S \Big] \subseteq (SBQS].$$

$\Leftarrow$ Let A be a right ideal and B a left ideal of S. Then A is a quasi-ideal and B is a bi-ideal of S as well. By hypothesis, we have $B \cap A \subseteq (SBAS] \subseteq (BA]$ and so S is intra-regular.

$(2) \Rightarrow$. Let B be a bi-ideal and Q a quasi-ideal of S. Since S is intra-regular, we have

$$B \cap Q \subseteq \Big(S(B \cap Q)(B \cap Q)S \Big] \subseteq (SQBS].$$

$\Leftarrow$. Let A be a right ideal and B a left ideal of S. Then A is a bi-ideal and B a quasi-ideal of S as well. By hypothesis, we have $A \cap B \subseteq (SBAS] \subseteq (BA]$ and so S is intra-regular. $\qquad\square$

Proposition 4.7 *[ord. hypersem.] Let $(S, \circ, \leq)$ be an ordered hypersemigroup. Then we have the following:*

(1) S is intra-regular if and only if for any bi-ideal B and any quasi-ideal Q of S, we have $B \cap Q \subseteq (S*B*Q*S]$.

(2) S is intra-regular if and only if for any bi-ideal B and any quasi-ideal Q of S, we have $B \cap Q \subseteq (S*Q*B*S]$.

Proof We get the proof of ordered semigroups given in Propososition 4.6 and replace the "$\cdot$" by "$*$". $\qquad\square$
So we have a second proof of [3, Theorem 3.16].

The Theorem 3.17 by Changphas and Davvaz in [3]: The following statements hold for an ordered semihypergroup $(S, \circ, \leq)$:

(1) S is intra-regular if and only if for a left hyperideal L and a bi-hyperideal B of S, we have $L \cap B \subseteq (L \circ B \circ S]$.

(2) S is intra-regular if and only if for a right hyperideal R and a bi-hyperideal B of S, $B \cap R \subseteq (S \circ B \circ R]$.

When we look at it, we immediately know that it comes from Proposition 4.8 or Proposition 4.9 given below. Its proof in the Italian journal is like that one of Theorem 3.16 and it is wrong as the concept of intra-regularity is wrong in it.

Proposition 4.8 *Let S be a poe-semigroup at the same time semilattice under $\wedge$. Then the following statements are equivalent:*

(1) S is intra-regular if and only if for any left ideal element a and any bi-ideal element b of S, we have $a \wedge b \leq abe$.

(2) S is intra-regular if and only if for any right ideal element a and any bi-ideal element b of S, we have $a \wedge b \leq eba$.

Proof $(1) \Rightarrow$. Let a be a left ideal element and b a bi-ideal element of S. Since S is intra-regular, we have

$$a \wedge b \leq e(a \wedge b)(a \wedge b)e \leq (ea)be \leq abe.$$

$\Leftarrow$. Let a be a right ideal element and b a left ideal element of S. Since b is a left ideal element and a is a bi-ideal element of S, by hypothesis, we have $b \wedge a \leq b(ae) \leq ba$ and so S is intra-regular.

(2) $\Rightarrow$. Let a be a right ideal element and b a bi-ideal element of S. Since S is intra-regular, we have

$$b \wedge a \leq e(b \wedge a)(b \wedge a)e \leq eb(ae) \leq eba.$$

$\Leftarrow$. Let a be a right ideal element and b a left ideal element of S. Since a is a right ideal element and b a bi-ideal element of S, by hypothesis, we have $a \wedge b \leq (eb)a \leq ba$ and so S is intra-regular. $\square$

For ordered semigroups, the following proposition holds.

Proposition 4.9 *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then we have the following:*

(1) S is intra-regular if and only if for any left ideal A and any bi-ideal B of S, we have $A \cap B \subseteq (ABS]$.

(2) S is intra-regular if and only if for any right ideal A and any bi-ideal B of S, we have $A \cap B \subseteq (SBA]$.

Proof (1) $\Rightarrow$. Let A be a left ideal and B a bi-ideal of S. Since S is intra-regular, we have

$$A \cap B \subseteq \Big(S(A \cap B)(A \cap B)S] \subseteq \Big((SA)BS\Big] \subseteq (ABS].$$

$\Leftarrow$. Let A be a left ideal and B a bi-ideal of S. Since B is a left ideal and A a bi-ideal of S, by hypothesis, we have $A \cap B \subseteq (SBA] \subseteq (BA]$ and so S is intra-regular.

(2) $\Rightarrow$. Let A be a right ideal and B a bi-ideal of S. Since S is intra-regular, we have

$$A \cap B \subseteq \Big(S(A \cap B)(A \cap B)S\Big] \subseteq \Big(SB(AS)\Big] \subseteq (SBA].$$

$\Leftarrow$. Let A be a right ideal and B a left ideal of S. Then A is a right ideal and B a bi-ideal of S. By hypothesis, we have $A \cap B \subseteq \big((SB)A\big] \subseteq (BA]$ and so S is intra-regular. $\square$

Proposition 4.10 *(ord. hypers.) Let $(S, \circ, \leq)$ be an ordered hypersemigroup. Then we have the following:*

(1) S is intra-regular if and only if for any left ideal A and any bi-ideal B of S, we have $A \cap B \subseteq (A * B * S]$.

(2) S is intra-regular if and only if for any right ideal A and any bi-ideal B of S, we have $A \cap B \subseteq (S * B * A]$.

Proof We get the proof of Proposition 4.9 and put $*$ instead of "$\cdot$". $\square$

Proposition 4.10 corrects the proof of Theorem 3.17 in [3].

We apply the above results to the following example.

A *poe*-semigroup S is called regular if $a \leq aea$ for every $a \in S$ [11].

Example 4.11 *The ordered semigroup defined by Table 18 and Figure 9 is an example of an intra-regular (not regular) le-semigroup. Propositions 4.5, 4.6, 4.8, and 4.9 can be applied.*

The quasi-ideal elements of S are the elements a, b, c, and e and coincide with the bi-ideal elements of S. The bi-ideals of S are the sets $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, and S and coincide with the quasi-ideals and with the right ideals of S. The set S is the only left ideal of S. According to [25], to this ordered semigroup corresponds the ordered hypersemigroup given by Table 19 and the same Figure 9 and it has the same bi-ideals, quasi-ideals, right and left ideals with the ordered semigroup.

Table 18. The ordered semigroup of Example 4.11.

·	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	a	a	a
d	e	e	e	e	e
e	e	e	e	e	e
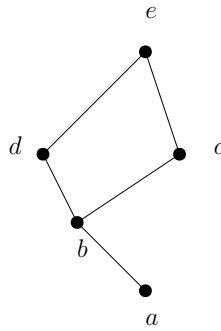
Figure 9. The order of Example 4.11.

Table 19. The hyperoperation of Example 4.11.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	S	S	S	S	S
e	S	S	S	S	S

This is the Theorem 4.13 by Kamali Ardekani and Davvaz in [10]: Let $(S, \circ, \leq)$ be an ordered semihypergroup such that $A \cap B = (B \circ A]$ for every right hyperideal A and every left hyperideal B of S. Then S is duo and $x \in (S \circ x \circ S]$ for all $x \in S$.

When we look at it, we immediately know that this comes from the following proposition.

We denote by F_r the set of right ideal elements and by F_l the set of left ideal elements of S; by $r(x)$ (resp. $l(x)$) the right (resp. left) ideal element of S generated by x i.e. the least (with respect to the order relation) ideal element of S containing x. S is called a duo if $F_r = F_l$.

Proposition 4.12 *Let S be an le-semigroup such that $a \wedge b = ba$ for every right ideal element a and every left ideal element b of S. Then S is duo and $x \leq exe$ for all $x \in S$.*

Proof Let $a \in F_r$. Since $e \in F_l$, by hypothesis, we have $a = a \wedge e = ea$, then $ea \leq a$, that is $a \in F_l$. If $a \in F_l$ then, since $e \in F_r$, by hypothesis, we have $a = e \wedge a = ae$, then $ae \leq a$ and so $a \in F_r$. Thus S is a

duo. Let now $x \in S$. Since $xe \in F_r$ and $ex \in F_l$, by hypothesis, we have

$$x \le r(x) \wedge l(x) \quad \le \quad l(x)r(x) = (x \vee ex)(x \vee xe) = x^2 \vee ex^2 \vee x^2e \vee ex^2e \le x^2 \vee exe.$$

Then $x^2 \le x^3 \vee exex \le exe$ and so $x \le exe$. □

An ordered groupoid S is called a duo if the sets of right ideals and left ideals of S coincide. Denote by $R(X)$ (resp. $L(X)$) the right (resp. left) ideal generated by X.

The analogous of Proposition 4.12 in case of ordered semigroups is the following:

Proposition 4.13 *Let $(S, \cdot, \le)$ be an ordered semigroup such that $A \cap B = (BA]$ for every right ideal A and every left ideal B of S. Then S is a duo and $X \subseteq (SXS]$ for every nonempty subset X of S.*

Proof Let A be a right ideal of S. Since S is a left ideal of S, by hypothesis, we have $A = A \cap S = (SA]$. Then $SA \subseteq A$ and so A is a left ideal of S. If A is a left ideal of S then, since S is a right ideal of S, by hypothesis, we have $A = S \cap A = (AS]$, then $AS \subseteq A$ and so A is a right ideal of S. Thus S is a duo. Let now X be a nonempty subset of S. By hypothesis, we have

$$\begin{aligned}
X \quad &\subseteq \quad R(X) \cap L(X) = \Big(L(X)(R(X) \Big] = \Big((X \cup SX](X \cup XS] \Big] \\
&= \quad \Big((X \cup SX)(X \cup XS) \Big] = \Big(X^2 \cup SX^2 \cup X^2S \cup SX^2S \Big] \\
&\subseteq \quad (X^2 \cup SXS]
\end{aligned}$$

Then $X^2 \subseteq (X^2 \cup SXS](X] \subseteq (X^3 \cup SXSX] \subseteq (SXS]$, then $X \subseteq \Big((SXS] \cup SXS \Big] = \Big((SXS] \Big] = (SXS]$ and so $X \subseteq (SXS]$. □

From the ordered semigroup $(S, \cdot, \le)$ to pass to ordered hypersemigroup $(S, \circ, \le)$ we just have to prove that $(A] * (B] \subseteq (A * B]$ for every nonempty subsets A, B of S. Then we replace the multiplication "$\cdot$" of semigroup by $*$ and this is the Theorem 4.13 in [10]. For this purpose, let $x \in (A] * (B]$. Then $x \in u \circ v$ for some $u \in (A]$, $v \in (B]$. We have $u \le a$ for some $a \in A$ and $v \le b$ for some $b \in B$. Since $(S, \circ, \le)$ is an ordered hypersemigroup, we have $u \circ v \preceq a \circ b$. Since $x \in u \circ v$, there exists $y \in a \circ b$ such that $x \le y \in a \circ b \subseteq A * B$ and so $x \in (A * B]$.

For a *poe*-semigroup S and an element a of S, we denote by $\beta(a)$ the bi-ideal element of S generated by a; that is, the least (with respect to the order relation) bi-ideal element of S containing a.

Proposition 4.14 *If S is a poe-semigroup at the same time semilattice under $\vee$ then, for every $a \in S$, we have*

$$\beta(a) = a \vee aea.$$

Proof Let $a \in S$. The element $a \vee aea$ is a bi-ideal element of S. Indeed:

$$\begin{aligned}
(a \vee aea)e(a \vee aea) \quad &= \quad (ae \vee aeae)(a \vee aea) = aea \vee aeaea \vee aeaeaea \\
&= \quad aea \le a \vee aea
\end{aligned}$$

and $a \vee aea \ge a$. If now t be a bi-ideal element of S such that $t \ge a$, then $a \vee aea \le t \vee tet = t$. □

For a nonempty subset A of an ordered semigroup or an ordered hypersemigroup S, we denote by $B(A)$ the bi-ideal of S generated by A; that is, the least (with respect to the inclusion relation) bi-ideal of S containing A.

Proposition 4.15 *Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then $B(A) = (A \cup ASA]$.*

Proof The nonempty set $(A \cup ASA]$ is a bi-ideal of S. Indeed:

$$
\begin{aligned}
(A \cup ASA]S(A \cup ASA] &= (A \cup ASA](S](A \cup ASA] \subseteq (AS \cup ASAS](A \cup ASA] \\
&\subseteq \Big((AS \cup ASAS)(A \cup ASA)\Big] \subseteq (ASA \cup ASASA \cup ASASASA] \\
&= (ASA] \subseteq (A \cup ASA],
\end{aligned}
$$

$\big((ASA]\big] = (ASA]$ (as it holds for any $\emptyset \neq X \subseteq S$) and $(A \cup ASA] \supseteq A$. If now T is a bi-ideal of S such that $T \supseteq A$, then $(A \cup ASA] \subseteq (T \cup TST] = (T] = T$. $\qquad \square$

Proposition 4.16 *Let $(S, \cdot, \leq)$ be an ordered hypersemigroup and $\emptyset \neq A \subseteq S$. Then $B(A) = (A \cup A * S * A]$.*

Proof We get the proof of Proposition 4.15 and put $*$ instead of "$\cdot$" and this is the proof of Lemma 3.15(1) by Changphas and Davvaz in [3]. $\qquad \square$

In a similar way, the Lemmas 2.3, 2.4, 2.6 on ordered hypersemigroups in [3], can been transferred from ordered semigroups.

This is Theorem 2.7, the main result of section 2 in [3]: An hypersemigroup $(S, \circ, \leq)$ is left and right simple if and only if does not contain proper bi-ideals. Its proof, though not written in the best way, is on the line of the Proposition in [28]; but [28] is not cited among the 30 (most of them not related) papers in References of [3].

This is Lemma 27 by Farooq, Khan and Davvaz in [5]: Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then the following are equivalent:

(1) S is regular.

(2) $B \cap L \subseteq (B \circ L]$ for every generalized bi-hyperideal B and every left hyperideal L of S.

(3) $B(a) \cap L(a) \subseteq \big(B(a) \circ L(a)\big]$ for every $a \in S$.

($B(a)$, $L(a)$ is the bi-ideal, left ideal of S, respectively, generated by a).

The proof of $(1) \Rightarrow (2)$ in [5] is as follows:

Let S be regular, B a bi-ideal (the authors call it generalized bi-ideal) and L a left ideal of S then $B \cap L \subseteq (B \circ L]$. In fact, if $a \in B \cap L$, then $a \in B$ and $a \in L$. Since S is regular, there exists $x \in S$ such that $a \leq a \circ (x \circ a) \subseteq B \circ (S \circ L) \subseteq B \circ L$. Then $a \in (B \circ L]$.

In the above proof if we delete the $\circ$ and write "$\cdot$" instead, then this is the proof for an ordered semigroup.

The correct proof of the implication $(1) \Rightarrow (2)$ of Lemma 27 in [5] is the following: Let $a \in B \cap L$. Since S is regular, we have $\{a\} \preceq (a \circ x) * \{a\} = \{a\} * \{x\} * \{a\} \subseteq B * S * L$, then $\{a\} \subseteq (B * S * L]$ and so $a \in (B * S * L]$.

When we look at [5, Lemma 27], we immediately know that it comes from the following proposition and that in Proposition 4.18 the implication $(1) \Rightarrow (2)$ holds for any nonempty subset A and not only for bi-ideals of S.

Proposition 4.17 *Let S be an le-semigroup. The following are equivalent:*

(1) S is regular.

(2) $b \wedge y \leq by$ for every bi-ideal element b and every left ideal element y of S.

(3) $\beta(a) \wedge l(a) \leq \beta(a)l(a)$ for every $a \in S$.

Proof $(1) \Longrightarrow (2)$. Let b be a bi-ideal element and y a left ideal element of S. Since S is regular, we have

$$b \wedge y \leq (b \wedge y)e(b \wedge y) \leq bey \leq by.$$

The implication $(2) \Rightarrow (3)$ is obvious.

$(3) \Longrightarrow (1)$. Let $a \in S$. By hypothesis, we have

$$
\begin{aligned}
a &\leq \beta(a) \wedge l(a) \leq \beta(a)l(a) = (a \vee aea)(a \vee ea) \\
&= a^2 \vee aea^2 \vee aea \vee aeaea = a^2 \vee aea.
\end{aligned}
$$

Then $a^2 \leq a^3 \vee aea^2 \leq aea$, thus $a \leq aea$ and so S is regular. □

The corresponding result for an ordered semigroup is as follows:

Proposition 4.18 *Let S be an ordered semigroup. The following are equivalent:*

(1) S is regular.

(2) $B \cap Y \subseteq (BY]$ for every bi-ideal B and every left ideal Y of S.

(3) $B(A) \cap L(A) \subseteq \big(B(A)L(A)\big]$ for every nonempty subset A of S.

If we replace the multiplication "$\cdot$" in the proof of the above proposition by $*$, then we have a second proof of the Lemma 27 in [5] for ordered hypersemigroup.

We apply the results on regularity to the following example.

Example 4.19 *The ordered semigroup given by Table 20 and Figure 10 is a regular (at the same time intra-regular) le-semigroup. The bi-ideal elements of S are the elements b, c, and e and coincide with the left ideal elements of S. The set S is the only bi-ideal of S and this is the only left ideal of S as well.*

Every result on an ordered hypersemigroup comes from a corresponding result of a *poe* (*le*)-semigroup or ordered semigroup. Let us give some further results to justify what we say.

This is the Definition 34 by Farooq, Khan, and Davvaz in [5]: Let $(S, \circ, \leq)$ be an ordered semihyper-group. Then S is called left weakly regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq x \circ a \circ y \circ a$.

This is the Lemma 38 by Farooq, Khan, and Davvaz in [5]: Let $(S, \circ, \leq)$ be an ordered semihypergroup. The following are equivalent:

(1) S is left weakly regular.

Table 20. The ordered semigroup of Example 4.19.

·	a	b	c	d	e
a	e	b	c	e	e
b	b	b	b	b	b
c	c	b	c	c	c
d	e	b	c	e	e
e	e	b	c	e	e
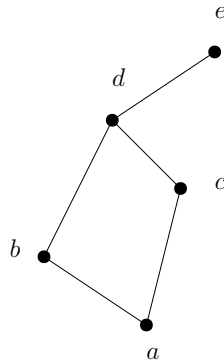
Figure 10. The order of Example 4.19.

(2) $B \cap I \subseteq (I \circ B]$ for every bi-ideal B and every ideal I of S.

(3) $B(a) \cap I(a) \subseteq (I(a) \circ B(a)]$ for every $a \in S$.

($I(a)$ is the ideal of S generated by a).

This lemma also comes from the le-semigroups.

A poe-semigroup S is called weakly regular if $a \leq eaea$ for every $a \in S$. For an le-semigroup, the following are equivalent:

(1) S is left weakly regular.

(2) $b \wedge a \leq ab$ for every bi-ideal element b and every ideal element a of S.

(3) $\beta(a) \wedge i(a) \leq i(a)\beta(a)$ for every $a \in S$.

($i(a)$ is the ideal element of S generated by a).

$(1) \Rightarrow (2)$. In fact, $b \wedge a \leq e(b \wedge a)e(b \wedge a) \leq (eae)b \leq ab$. As we see, b can be any element of S and not only bi-ideal element of S.

We give the implication $(1) \Rightarrow (2)$ for an ordered semigroup.

An ordered semigroup $(S, \cdot, \leq)$ is called left weakly regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xaya$. This is equivalent to saying that $A \subseteq (SASA]$ for any nonempty subset A of S. On the line of the proof by Farooq, Khan, and Davvaz: Let $a \in B \cap I$. Then there exist $x, y \in S$ such that $a \leq (xay)a \in (SIS)B \subseteq IB$ and so $a \in (IB]$.

If $(S, \cdot, \leq)$ is an ordered semigroup then, for every bi-ideal B and every ideal I of S, we have $B \cap I \subseteq (I \cap B]$. Indeed, since S is weakly regular, we have $B \cap I \subseteq \left(S(B \cap I)S(B \cap I) \right] \subseteq \left((SIS)B \right] \subseteq (IB]$ (again B can be any nonempty subset of S).

If we get the proof of ordered semigroup, delete the "$\cdot$" and put $*$ instead, then this is the corresponding result for an ordered hypersemigroup.

Definition 4.20 *A poe-semigroup S is said to be quasi-simple if e is the only quasi-ideal element of S.*

Proposition 4.21 *Let S be poe-semigroup at the same time semilattice as well. Then S is quasi-simple if and only if, for any $x \in S$, we have $xe \wedge ex = e$.*

Proof $\implies$. Let $x \in S$. The element $xe \wedge ex$ is a quasi-ideal element of S. Indeed,

$$(xe \wedge ex)e \wedge e(xe \wedge ex) \le xe^2 \wedge e^2x \le xe \wedge ex.$$

Since S is quasi-simple, we have $xe \wedge ex = e$.
$\impliedby$. If q be a quasi-ideal element of S, then $qe \wedge eq \le q$. By hypothesis, we have $qe \wedge eq = e$, thus we have $e \le q$ and so $q = e$. $\qquad\square$

We give the proposition on ordered semigroups that corresponds to Proposition 4.21.

We begin with the following definition.

Definition 4.22 *An ordered semigroup S is said to be quasi-simple if S is the only quasi-ideal of S.*

Proposition 4.23 *An ordered semigroup $(S, \cdot, \le)$ is quasi-simple if and only if, for any nonempty subset A of S, we have $(AS] \cap (SA] = S$.*

Proof $\implies$. Let $\emptyset \ne A \subseteq S$. Since $(AS]$ is a right ideal and $(SA]$ a left ideal of S, by Proposition 4.3, the set $(AS] \cap (SA]$ is a quasi-ideal of S. Since S is quasi-simple, we have $(AS] \cap (SA] = S$.
$\impliedby$. If Q be a quasi-ideal of S, then $(QS] \cap (SQ] \subseteq Q$. By hypothesis, we have $(QS] \cap (SQ] = S$. Thus we have $S \subseteq Q$ and so $S = Q$; that is S is quasi-simple. $\qquad\square$

If we get Proposition 4.23 and replace (in its proof) the multiplication by $*$, then this is the Theorem 3.11 on ordered hypersemigroups in [3] in a more general form.

This is Definition 3.1 by Pibaljommee and Davvaz [31]: An ordered semihypergroup (we call it hypersemigroup) $(H, \circ, \le)$ is called regular if for every $a \in H$ there exists $x \in H$ such that $a \le a \circ x \circ a$.

This is Lemma 3.2 by Pibaljommee and Davvaz in [31]: An ordered hypersemigroup $(H, \circ, \le)$ is regular if and only if $a \in (a \circ H \circ a]$ for all $a \in H$, equivalently if $A \subseteq (A \circ H \circ A]$ for every $A \subseteq H$; and its proof is given as follows: Let $a \in H$. If H is regular, there exists $x \in H$ such that $a \le a \circ x \circ a \subseteq a \circ H \circ a \subseteq (a \circ H \circ a]$. Then $a \le y$ for some $y \in a \circ x \circ a$ and so $a \in (a \circ H \circ a]$. For the converse statement, let $a \in H$. By hypothesis, we have $\{a\} \subseteq (\{a\} \circ H \circ \{a\}]$. Then there exists $x \in H$ such that $a \le a \circ x \circ a$ i.e. H is regular.

The correct proof is the following: $\Rightarrow$. Let $a \in H$. Since H is regular, there exists $x \in H$ such that $\{a\} \preceq (a \circ x) * \{a\}$; that is there exist $x, t \in H$ such that $t \in (a \circ x) * \{a\}$ and $a \le t$. We have $a \le t \in (a \circ x) * \{a\} \subseteq a * H * a$ and so $a \in (a * H * a]$. $\Leftarrow$. Let $a \in H$. By hypothesis, we have $a \in (a * H * a]$. Then $a \le t$ for some $t \in a * H * a$. Since $t \in a * H * a$, we have $t \in u \circ a$ for some $u \in a * H$. Since $u \in a * H$, we have $u \in a \circ x$ for some $x \in H$. Thus we have $t \in \{u\} * \{a\} \subseteq (a \circ x) * \{a\}$ and $a \le t$ and so H is regular.

It is not enough to get a result of an ordered semigroup, put $\circ$ instead of the multiplication of the semigroup and present it as a result of ordered hypersemigroup. Let us give one more very typical example.

This is Proposition 24 by Farooq, Khan, and Davvaz in [5]: Let $(S, \circ, \leq)$ be a regular ordered semihypergroup and B be a generalized bi-ideal of S. Then B is a bi-hyperideal of S.

This is copy of the proof: Let S be a regular ordered semihypergroup. Let B be a generalized bi-hyperideal of S and $a, b \in B$. Since S is regular, there exists $x \in S$ such that $b \leq b \circ x \circ b$, Then $a \circ b \leq a \circ (b \circ x \circ b) = a \circ (b \circ x) \circ b \subseteq B \circ S \circ B \subseteq B$, then $a \circ b \subseteq (B] = B$. Thus, B is a bi-ideal of S.

An element a of a *poe*-semigroup is called subidempotent if $a^2 \leq a$.

Proposition 4.24 *Let S be a regular poe-semigroup and b a bi-ideal element of S. Then $b^2 \leq b$ (i.e. b is a subidempotent element of S).*

Proof Since b is a bi-ideal element, we have $beb \leq b$. Since S is regular, we have $b \leq beb$ and so $b = beb$. Thus $b^2 = (beb)b \leq beb = b$ and so $b^2 \leq b$. □

The analogous of Proposition 4.24 in case of an ordered semigroup is as follows:

We call bi-ideal what the authors call generalized bi-ideal and subidempotent bi-ideal (or bi-ideal that is also a subsemigroup of S) what the authors call bi-ideal. Only for the following proposition we will use the concept given by the authors.

Proposition 4.25 *Let $(S, \cdot, \leq)$ be a regular ordered semigroup and B a generalized bi-ideal of S. Then $B^2 \subseteq B$ (i.e. B is a subsemigroup of S).*

Proof Since B is a bi-ideal of S, we have $BSB \subseteq B$. Since S is regular, we have $B \subseteq (BSB]$, then $(BSB] \subseteq (B] = B$ and so $B = (BSB]$. Then we have

$$B^2 = (BSB]B \subseteq (BSB](B] \subseteq (BSB^2] \subseteq (BSB] = B$$

and so $B^2 \subseteq B$. □

Proposition 4.26 *Let $(S, \circ, \leq)$ be a regular ordered hypersemigroup and B a generalized bi-ideal of S. Then $B * B \subseteq B$ (i.e. B is a subsemigroup of S).*

Proof Since B is a bi-ideal of S, we have $B * S * B \subseteq B$. Since S is regular, we have $B \subseteq (B * S * B]$, then $(B * S * B] \subseteq (B] = B$ and so $B = (B * S * B]$. Then we have

$$B * B = (B * S * B] * B \subseteq (B * S * B] * (B] \subseteq (B * S * B * B] \subseteq (B * S * B] = B$$

and so $B * B \subseteq B$. □

As we see, the proof of Proposition 4.26 is exactly the same with the proof of Proposition 4.25 we just have to put $*$ instead of the multiplication of the semigroup and observe that, for an ordered hypersemigroup, we have (a) $(A] * (B] \subseteq (A * B]$ and (b) $S * B \subseteq S$.

The proof of Proposition 24 by Farooq, Khan, and Davvaz in [5] needs correction. If we want to get it using elements as Farooq et al. did, then we have a second proof of Proposition 4.26, which is as follows:

We have $B * B \subseteq B$; that is, if $a, b \in B$, then $a \circ b \subseteq B$. Indeed: Since S is regular and $b \in S$, we have $\{b\} \preceq (b \circ x) * \{b\}$ for some $x \in S$. That is, there exist $x, t \in S$ such that $t \in (b \circ x) * \{b\}$ and $b \leq t$. Since

$b \le t$, we have

$$a \circ b \quad \preceq \quad a \circ t = \{a\} * \{t\} \subseteq \{a\} * \{b\} * \{x\} * \{b\} \subseteq B * S * B \subseteq B.$$

We have $a \circ b \preceq a \circ t \subseteq B$ and so $a \circ b \subseteq (B] = B$.

According to Theorem 3.6 by Changphas and Davvaz in [3], every quasi-hyperideal of an ordered semihypergroup S is the intersection of a right hyperideal and a left hyperideal of S. As soon as we look at it, we immediately see that it comes from the following proposition:

Proposition 4.27 *If S is an le-semigroup at the same time a distributive lattice and q a quasi-ideal element of S, then there is a right ideal element a and a left ideal element b of S such that $q = a \wedge b$.*

Proof If q is a quasi-ideal element of S, then $qe \wedge eq \le q$ from which

$$
\begin{aligned}
q &= q \vee (qe \wedge eq) = (q \vee qe) \wedge (q \vee eq) \text{ (since S is a distr. latt.)} \\
&= r(q) \wedge l(q),
\end{aligned}
$$

where $r(q)$ is a right ideal element and $l(q)$ is a left ideal element of S. □

The analogous of Proposition 4.27 in case of ordered semigroup is the following.

Proposition 4.28 *If $(S, \cdot, \le)$ is an ordered semigroup and Q is a quasi-ideal of S, then Q is the intersection of a right ideal and a left ideal of S.*

Proof Let Q be a quasi-ideal of S. Then $(QS] \cap (SQ] \subseteq Q$ from which

$$
\begin{aligned}
Q &= Q \cup \left((QS] \cap (SQ] \right) = \left(Q \cup (QS] \right) \cap \left(Q \cup (SQ] \right) \\
&= R(Q) \cap L(Q),
\end{aligned}
$$

where $R(Q)$ is a right ideal and $L(Q)$ is a left ideal of S. □

The analogous of Proposition 4.28 in case of ordered hypersemigroups is the following.

Proposition 4.29 *If $(S, \circ, \le)$ is an ordered hypersemigroup and Q is a quasi-ideal of S, then Q is the intersection of a right ideal and a left ideal of S.*

Proof Let Q be a quasi-ideal of S. Then $(Q * S] \cap (S * Q] \subseteq Q$ from which

$$
\begin{aligned}
Q &= Q \cup \left((Q * S] \cap (S * Q] \right) = \left(Q \cup (Q * S] \right) \cap \left(Q \cup (S * Q] \right) \\
&= R(Q) \cap L(Q),
\end{aligned}
$$

where $R(Q)$ is a right ideal and $L(Q)$ is a left ideal of S. □

As we see, the proof of Proposition 4.29 is the same with the Proposition 4.28 with the usual change.

This is Theorem 5.22 by Kamali Ardekani and Davvaz in [10]: Let $(S, \circ, \le)$ be an ordered semihypergroup. If for all $a \in S$, we have $R(a) \cap Q(a) \cap L(a) \subseteq \left(L(a) \circ Q(a) \circ R(a) \right]$, then S is intra-regular. ($L(a)$, $R(a)$, and $Q(a)$ is the left ideal, right ideal and quasi-ideal of $(S, \circ, \le)$, respectively, generated by a).

As soon as we look at this theorem, we immediately know that it comes from the following proposition and holds for any nonempty subset A of S.

Denote by $r(a)$, $l(a)$, $q(a)$, the right ideal element, left ideal element, and the quasi-ideal element of S, respectively, generated by a.

For the sake of completeness, we give the following proposition (see also [20]).

Proposition 4.30 *Let S be an le-semigroup. Suppose for every $a \in S$, we have*

$$r(a) \wedge q(a) \wedge l(a) \le l(a)q(a)r(a).$$

Then S is intra-regular.

Proof Let $a \in S$. By hypothesis, we have

$$
\begin{aligned}
a \ &\le\ r(a) \wedge q(a) \wedge l(a) \le l(a)q(a)r(a) = (a \vee ea)\Big(a \vee (ae \wedge ea)\Big)(a \vee ae) \\
&\le\ (a \vee ea)(a \vee ae)(a \vee ae) = (a^2 \vee ea^2 \vee a^2 e \vee ea^2 e)(a \vee ae) \\
&=\ a^3 \vee ea^3 \vee a^2 ea \vee ea^2 ea \vee a^3 e \vee ea^3 e \vee a^2 eae \vee ea^2 eae \\
&\le\ a^3 \vee ea^3 \vee a^2 e \vee ea^2 e \\
&=\ a^2 e \vee ea^2 e.
\end{aligned}
$$

Then $a^2 \le a^3 e \vee aea^2 e \le ea^2 e$, $a^2 e \le ea^2 e$. Thus we get $a \le ea^2 e$ and so S is intra-regular. □

The analogous of Proposition 4.30 in case of ordered semigroups is the following.

We denote by $R(A)$, $L(A)$, $Q(A)$ the right ideal, left ideal, and the quasi-ideal of S generated by A.

We have $Q(A) = \left(A \cup \Big((AS] \cap (SA] \Big) \right]$ [20].

Proposition 4.31 *(see also [20, Lemma 2.1]) Let S be an ordered semigroup. Suppose for every $\emptyset \ne A \subseteq S$, we have*

$$R(A) \cap Q(A) \cap L(A) \subseteq \big(L(A)Q(A)R(A) \big].$$

Then S is intra-regular.

If we get the Proposition 4.31 and put $*$ instead of "$\cdot$" in its proof, then this is the Theorem 5.22 by Kamali and Davvaz in a more general form.

This is the Lemma 29 by Farooq, Khan, and Davvaz in [5]: For an ordered semihypergroup $(S, \circ, \le)$, the following are equivalent: (1) S is regular; (2) $B \cap I = (B \circ I \circ B]$ for every generalized bi-hyperideal (we call it bi-ideal) B and every hyperideal I of S; (3) $B(a) \cap I(a) = \big(B(a) \circ I(a) \circ B(a) \big]$ for every $a \in S$. ($B(a)$, $I(a)$ is the bi-ideal and the ideal of S, respectively, generated by a).

Their proof is exactly the proof for an ordered semigroup in which the "$\cdot$" has been replaced by $\circ$; even the order relation is the same. There are notations like

$$
\begin{aligned}
&a^3 \cup a^3 \circ S \circ a \cup a \circ S \circ a^2 \cup a \circ S \circ a^2 \circ S \circ a \cup a^2 \circ S \circ a \cup a^2 \circ S \circ a \circ S \circ a \cup \\
&a \circ S \circ a \circ S \circ a \cup a \circ S \circ a \circ S \circ a \circ S \circ a
\end{aligned}
$$

in the proof of $(3) \Rightarrow (1)$, without any explanation what these notations (obvious for semigroups) in case of an hypersemigroup mean.

On the other hand, this lemma also comes from the le-semigroups. In fact, the following proposition holds.

Proposition 4.32 *Let S be an le-semigroup. The following are equivalent:*

(1) S is regular.

(2) $b \wedge a = bab$ for every bi-ideal element b and every ideal element a of S.

(3) $\beta(a) \wedge i(a) = \beta(a)i(a)\beta(a)$ for every $a \in S$.

($i(a)$ is the ideal element of S generated by a).

Proof $(1) \Longrightarrow (2)$. Let b be a bi-ideal element and a an ideal element of S. Since S is regular, we have

$$b \wedge a \le (b \wedge a)e(b \wedge a) \le (b \wedge a)e(b \wedge a)e(b \wedge a) \le b(eae)b \le bab \le beb \wedge eae \le b \wedge a$$

and so $b \wedge a = bab$.

The implication $(2) \Rightarrow (3)$ is obvious.

$(3) \Longrightarrow (1)$. Let $a \in S$. By hypothesis, we have

$$a \le \beta(a) \wedge i(a) \le \beta(a)i(a)\beta(a) = (a \vee aea)(a \vee ea \vee ae \vee eae)(a \vee aea) = a^3 \vee aea.$$

Then we have $a^3 \le a^5 \vee aea^3 \le aea$, thus $a \le aea$ and so S is regular. $\square$

Most of the results By Kamali Ardekani and Davvaz in [10] are well known published results in le-semigroups or poe-semigroups (see, for example, [13, 22]). All the results in [10] can be written for le-semigroups.

Note In the paper by Changphas and Davvaz in Ital J (mentioned above) [3] (and not only) the authors wrote: "We try to use sets instead of elements in the proof of our results similar to Kapp [17] and Leoreanu [25] (the citations are the References in [3] and they are wrong). However, if the authors of [3] knew the meaning of this statement, then they should know that the publish results of ordered hypersemigroups in the bibliography come from the le-semigroups. Every result in their paper can be proved using sets (thought they didn't). This has been said several times by the author of the present paper in an attempt to show the relation of some of the results of ordered semigroups with the le-semigroups (see, for example [16, 17]).

Conclusion In the present paper we gave some results on le-semigroups and the analogous results for ordered semigroups. The results for ordered hypersemigroups are the results on ordered semigroups (with some explanations). The question is: Why is it so? The answer is the following: The correct definition of the hypersemigroup is as follows: It is an hypergroupoid S such that $\{x\} * (y \circ z) = (x \circ y) * \{z\}$ for all $x, y, z \in S$. But, on the other hand, $\{x\} * \{y\} = x \circ y$ for all $x, y \in S$. Therefore, the definition of the hypersemigroup can be written in the following way as well: $\{x\} * (\{y\} * \{z\}) = (\{x\} * \{y\}) * \{z\}$ for all $x, y, z \in S$. If now, we identify the singletons with the element they contain, then we can write it as $x * (y * z) = (x * y) * z$; and if we write $\circ$ instead of $*$ then we can write it as $x \circ (y \circ z) = (x \circ y) \circ z$. We are then in a semigroup and we do not have anything to prove as everything has been proved for semigroups. But certainly this is NOT the correct way to work. The investigation on ordered hypersemigroups is based (1) on the correct definitions. (2) On the associativity of $*$ (operation between sets). (3) On some Lemmas for which correct proofs are needed.

(4) Examples that also come from ordered semigroups appeared many years ago. So at least, the definitions, the necessary few lemmas, and references should be given in a correct way. In addition, we have not seen any independent result on ordered hypersemigroups that cannot come from ordered semigroups. According to the bibliography, the hyperstructure is a very important subject having applications in many fields of pure and applied mathematics. If we search for applications of the theory of ordered hypersemigroups, then we have to search them among the *le*-semigroups (that is a much more simpler structure). But it is usual to say such things to attract the interest of referees, editors, and authors. As an example, the commutative AG-groupoids (Abel Grasmann's groupoids) are semigroups while some authors say that the AG-groupoid is a generalization of a semigroup! According to Amjad, Hila and Yousafsai [Generalized hyperideals in locally associative left almost semihypergroups. New York J Math 20 (2014) 1063–1076], "an LA-semigroup (we call it AG-groupoid) *is a generalization of a semigroup and it has vast applications in semigroups, as well as in other branches of mathematics*."

Acknowledgments

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