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Clairaut semi-invariant Riemannian maps from almost Hermitian manifolds

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Abstract: In this article, we define Clairaut semi-invariant Riemannian maps (CSIR Maps, In short) from almost Hermitian manifolds onto Riemannian manifolds and investigate fundamental results on such maps. We also obtain conditions for totally geodesicness on distributions defined in the introduced notion. Moreover, we provide an explicit example of CSIR map.

Key words: Kähler manifold, Riemannian map, Clairaut semi-invariant Riemannian map.

1. Introduction

In the study of the geodesic upon a surface of revolution, a well known Clairaut's theorem [5] tells that for any geodesic $c(c : I_1 \subset R \to M$ on $M)$ on the revolution surface M the product $rsin\varphi$ is constant with along c, where $\varphi(s)$ be the angle between $c(s)$ and the meridian curve through $c(s)$, $s \in I_1$. It means, it is independent of s. Bishop introduced and studied Riemannian submersions which satisfy a generalization of Clairaut's theorem. He showed the concept of Clairaut submersion in the following way: a submersion $\pi : M \to N$ is said to be a Clairaut submersion if there is a function $r : M \to R^+$ such that for every geodesic, making an angle φ with the horizontal subspaces, $rsin\varphi$ is constant [5]. Moreover, he gave a characterization of Clairaut submersion, studied the behaviour of geodesic, and further obtain a generalization of Clairaut's theorem. This notion has been studied in Lorentzian spaces, timelike and spacelike spaces [8] (see also [19], [20], [21], [22]). In [1], Allison has shown that such submersions have their applications in static spacetimes. In [7], the author also showed that the notion of Clairaut submersion is a useful tool for obtaining decomposition theorems on Riemannian manifolds. Moreover, Clairaut submersions have been further generalized in [2]. Lee et al. [8] investigated new conditions for anti-invariant Riemannian submersions [14] to be Clairaut when the total manifolds are Kahlerian.

On the other hand, Fischer [6] introduced the notion of Riemannian map between Riemannian manifolds, which generalizes and unifies the notions of an isometric immersion, a Riemannian submersion and an isometry. Fischer defined the concept of Riemannian map in the following way: Let $\pi : (N_1, g_1) \to (N_2, g_2)$ be a differentiable map between Riemannian manifolds such that $0 < rank\ \pi_* < \min\{m, n\}$. If we denote the *kernel* space of π_* by $\ker \pi_*$ and the orthogonal complementary space of $\ker \pi_*$ by $(\ker \pi_*)^\perp$ in TN_1, then

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the TN_1 has the following orthogonal decomposition:

$$TN_1 = \ker \pi_* \oplus (\ker \pi_*)^{\perp}. \tag{1.1}$$

Here, if we denote the *range* of π_* by $range\pi_*$ and for a point $q \in N_1$ the orthogonal complementary space of $range\pi_{*\pi(q)}$ by $(range\pi_{*\pi(q)})^{\perp}$ in $T_{\pi(q)}N_2$. Then the tangent space $T_{\pi(q)}N_2$ has the following orthogonal decomposition:

$$T_{\pi(q)}N_2 = (range\pi_{*\pi(q)}) \oplus (range\pi_{*\pi(q)})^{\perp}.$$

A differentiable map $\pi : (N_1, g_1) \to (N_2, g_2)$ is called a Riemannian map at $q \in N_1$ if the horizontal restriction $\pi^h_{*q} : (\ker \pi_{*q})^{\perp} \to (range\pi_{*\pi(q)})$ is linear isometry between the inner product space $((\ker \pi_{*q})^{\perp}, (g_1)_{(q)}|_{(\ker \pi_{*q})^{\perp}})$ and $(range\pi_{*\pi_{(q)}},\ (g_2)_{(\pi(q))}|_{(range\pi_{*q})})$. Fischer showed a conspicuous property of this map is that it satisfies the generalized eikonal equation $\| \pi_* \|^2 = rank\pi$. The eikonal equation is a bridge between geometric optics and physical optics. In [6], the author also proposed an approach to build a quantum model and he pointed out the success of such a program of building a quantum model of nature using Riemannian maps that would provide an interesting relationship between Riemannian maps, harmonic maps, and Lagrangian field theory. Further, the notion of Riemannian map is being studied continuously from different perspectives, as semi-invariant Riemannian maps [15], slant Riemannian maps [16], semi-slant Riemannian maps [10], hemi-slant Riemannian maps [18] (see also [11], [12] etc).

In [17], Sahin introduced Clairaut Riemannian maps, in which he obtained necessary and sufficient conditions for Riemannian maps to be Clairaut Riemannian maps. In this paper, we are interested in studying the above idea in the setting of CSIR maps. The article is organized as follows. Section 2, we gather some concepts, which are needed in the following parts. In section 3, we define CSIR map from almost Hermitian manifold onto Riemannian manifold and study the geometry of leaves of distributions. In section 4, we present an example of the CSIR map.

2. Preliminaries

Let N_1 be an even-dimensional differentiable manifold. Let J be a $(1,1)$ tensor field on N_1 such that

$$J^2 = -I, \tag{2.1}$$

where I is identity operator. Then J is called an almost complex structure on N_1. The manifold N_1 with an almost complex structure J is called an almost complex manifold [14]. It is well known that an almost complex manifold is necessarily orientable. Nijenhuis tensor N of an almost complex structure is defined as:

$$N(V_1, V_2) = [JV_1, JV_2] - [V_1, V_2] - J[JV_1, V_2] - J[V_1, JV_2], \tag{2.2}$$

for all $V_1, V_2 \in \Gamma(TN_1)$.

If Nijenhuis tensor field N vanishes on an almost complex manifold N_1, then the almost complex manifold N_1 is called a complex manifold.

Let g_1 be a Riemannian metric on N_1 such that

$$g_1(JZ_1, JZ_2) = g_1(Z_1, Z_2), \text{ for all } Z_1, Z_2 \in \Gamma(TN_1). \tag{2.3}$$

This metric g_1 is called a Hermitian metric on N_1 and manifold N_1 with Hermitian metric g_1 is called an almost Hermitian manifold. The Riemannian connection ∇ of an almost Hermitian manifold N_1 can be extended to the whole tensor algebra on N_1. Tensor fields $(\nabla_{Y_1}J)$ areis defined as

$$(\nabla_{Y_1}J)Y_2 = \nabla_{Y_1}JY_2 - J\nabla_{Y_1}Y_2, \tag{2.4}$$

for all $Y_1, Y_2 \in \Gamma(TN_1)$.

An almost Hermitian manifold (N_1, g_1, J) is called a Kähler manifold [4] if

$$(\nabla_{Y_1}J)Y_2 = 0, \tag{2.5}$$

for all $Y_1, Y_2 \in \Gamma(TN_1)$.

Define O'Neill's tensors [9] $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2, \tag{2.6}$$

$$\mathcal{T}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2, \tag{2.7}$$

for any vector fields E_1, E_2 on N_1, where ∇ is the Levi-Civita connection of g_1. It is easy to see that $\mathcal{T}_{E_1}$ and $\mathcal{A}_{E_1}$ are skew-symmetric operators on the tangent bundle of N_1 reversing the vertical and the horizontal distributions.

From equations (2.6) and (2.7), we have

$$\nabla_{Z_1}Z_2 = \mathcal{T}_{Z_1}Z_2 + \mathcal{V}\nabla_{Z_1}Z_2, \tag{2.8}$$

$$\nabla_{Z_1}V_1 = \mathcal{T}_{Z_1}V_1 + \mathcal{H}\nabla_{Z_1}V_1, \tag{2.9}$$

$$\nabla_{V_1}Z_1 = \mathcal{A}_{V_1}Z_1 + \mathcal{V}\nabla_{V_1}Z_1, \tag{2.10}$$

$$\nabla_{V_1}V_2 = \mathcal{H}\nabla_{V_1}V_2 + \mathcal{A}_{V_1}V_2, \tag{2.11}$$

for all $Z_1, Z_2 \in \Gamma(\ker\pi_*)$ and $V_1, V_2 \in \Gamma(\ker\pi_*)^{\perp}$, where $\mathcal{H}\nabla_{Z_1}V_1 = \mathcal{A}_{V_1}Z_1$, if V_1 is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

It is seen that for $q \in N_1$, $Y_1 \in \mathcal{V}_q$ and $V_1 \in \mathcal{H}_q$ the linear operators

$$\mathcal{A}_{V_1},\ \mathcal{T}_{Y_1} : T_qN_1 \to T_qN_1,$$

are skew-symmetric, i.e.

$$g_1(\mathcal{A}_{V_1}E_1, E_2) = -g_1(E_1, \mathcal{A}_{V_1}E_2) \text{ and } g_1(\mathcal{T}_{Y_1}E_1, E_2) = -g_1(E_1, \mathcal{T}_{Y_1}E_2), \tag{2.12}$$

for each $E_1, E_2 \in T_qN_1$. Since $\mathcal{T}_{Y_1}$ is skew-symmetric, we observe that π has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

The differentiable map π between two Riemannian manifolds is totally geodesic if

$$(\nabla\pi_*)(Z_1, Z_2) = 0, \text{ for all } Z_1, Z_2 \in \Gamma(TN_1).$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths. A Riemannian map is a Riemannian map with totally umbilical fibers if [15]

$$\mathcal{T}_{X_1}X_2 = g_1(X_1, X_2)H, \tag{2.13}$$

for all $X_1, X_2 \in \Gamma(\ker \pi_*)$, where H is the mean curvature vector field of fibers.

Let $\pi : (N_1, g_1) \to (N_2, g_2)$ is a smooth map between Riemannian manifolds. Then the differential map π_* of π can be observed a section of the bundle $Hom(TN_1, \pi^{-1}TN_2) \to N_1$, where $\pi^{-1}TN_2$ is the bundle which has fibers $(\pi^{-1}TN_2)_x = T_{\pi(x)}N_2$ and has a connection ∇ induced from the Riemannian connection ∇^{N_1} and the pullback connection. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(Z_1, Z_2) = \nabla^{\pi}_{Z_1}\pi_*(Z_2) - \pi_*(\nabla^{N_1}_{Z_1}Z_2), \tag{2.14}$$

for vector field $Z_1, Z_2 \in \Gamma(TN_1)$, where ∇^{π} is the pullback connection. We know that the second fundamental form is symmetric.

Now we have the following [13]:

Lemma 2.1 *Let $\pi : (N_1, g_1) \to (N_2, g_2)$ be a Riemannian map between Riemannian manifolds. Then*

$$g_2((\nabla\pi_*)(V_1, V_2), \pi_*(V_3)) = 0, \quad \textit{for all } V_1, V_2, V_3 \in \Gamma(\ker \pi_*)^{\perp}. \tag{2.15}$$

As a result of above Lemma, we get

$$(\nabla\pi_*)(V_1, V_2) \in (\Gamma(range\pi_*)^{\perp}), \quad \textit{for all } V_1, V_2 \in \Gamma(\ker \pi_*)^{\perp}. \tag{2.16}$$

Lemma 2.2 *[3] Let (N_1, g_1) and (N_2, g_2) are two Riemannian manifolds. If $\pi : N_1 \to N_2$ Riemannian map between Riemannian manifolds, then for any horizontal vector fields X_1, X_2 and vertical vector fields Y_1, Y_2, we have*

(i) $(\nabla\pi_*)(X_1, X_2) = 0$,

(ii) $(\nabla\pi_*)(Y_1, Y_2) = -\pi_*(\mathcal{T}_{Y_1}Y_2) = -\pi_*(\nabla^{N_1}_{Y_1}Y_2)$,

(iii) $(\nabla\pi_*)(X_1, Y_1) = -\pi_*(\nabla^{N1}_{X_1}Y_1) = -\pi_*((\mathcal{A}_{X_1}Y_1)$.

Now, we recall following definitions for later use:

Definition 2.3 *[14] Let π be a Riemannian map from an almost Hermitian manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2). Then, we say that π is an invariant Riemannian map if the vertical distribution is invariant with respect to the complex structure J, i.e.,*

$$J(\ker \pi_*) = \ker \pi_*.$$

Definition 2.4 *[14] Let N_1 be an almost Hermitian manifold with Hermitian metric g_1 and almost complex structure J and N_2 be a Riemannian manifold with Riemannian metric g_2. Suppose that there exists a Riemannian map $\pi : (N_1, g_1, J) \to (N_2, g_2)$ such that $J(\ker \pi_*) \subseteq (\ker \pi_*)^{\perp}$. Then we say that π is an anti-invariant Riemannian map.*

Definition 2.5 *[15] Let π be a Riemannian map from an almost Hermitian manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then we say that π is a semi-invariant Riemannian map if there is a distribution $\mathfrak{D}_1 \subseteq \ker \pi_*$ such that*

$$\ker \pi_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2, J\mathfrak{D}_1 = \mathfrak{D}_1, J\mathfrak{D}_2 \subseteq (\ker \pi_*)^{\perp},$$

where $\mathfrak{D}_2$ is orthogonal complementary to $\mathfrak{D}_1$ in $\ker \pi_*$.

Let μ denotes the complementary orthogonal subbundle to $J(\ker \pi_*)$ in $(\ker \pi_*)^{\perp}$. Then, we have

$$(\ker \pi_*)^{\perp} = J\mathfrak{D}_2 \oplus \mu.$$

Obviously μ is an invariant subbundle of $(\ker \pi_*)^{\perp}$ with respect to the complex structure J.

3. CSIR maps

The notion of Clairaut Riemannian map was defined by Sahin in [17]. According to the definition, a Riemannian map $\pi : (N_1, g_1) \to (N_2, g_2)$ is called a Clairaut Riemannian map if there exists a positive function r on N_1, such that for any geodesic α on N_1, the function $(r \circ \alpha) \sin \theta$ is constant, where for any t, $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$. He also gave the following necessary and sufficient condition for a Riemannian map to be a Clairaut Riemannian map as follows:

Theorem 3.1 *[17] Let $\pi : (N_1, g_1) \to (N_2, g_2)$ be a Riemannian map with connected fibers. Then, π is a Clairaut Riemannian map with $r = e^f$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla f$ is the gradient of the function f with respect to g_1.*

We now present the notion of Clairaut semi-invariant Riemannian maps (CSIR map) as follows:

Definition 3.2 *A semi-invariant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2) is called Clairaut semi-invariant Riemannian map if it satisfies the condition of Clairaut Riemannian map.*

We denote the complementary distribution to JD_2 in $(\ker \pi_*)^{\perp}$ by μ. Then for $X_1 \in (\ker \pi_*)$, we get

$$JX_1 = \varphi X_1 + \omega X_1, \tag{3.1}$$

where $\varphi X_1 \in \Gamma(D_1)$ and $\omega X_1 \in \Gamma(JD_2)$. Also for $X_2 \in \Gamma(\ker \pi_*)^{\perp}$, we have

$$JX_2 = BX_2 + CX_2, \tag{3.2}$$

where $BX_2 \in \Gamma(D_2)$ and $CX_2 \in \Gamma(\mu)$.

Lemma 3.3 *Let π be a semi-invariant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). If $\alpha : I_2 \subset R \to M$ is a regular curve and $X_1(t)$ and $X_2(t)$ are the vertical and horizontal components of the tangent vector field $\dot{\alpha} = E$ of $\alpha(t)$, respectively, then α is a geodesic if and only if along α the following equations hold:*

$$\mathcal{V}\nabla_{\dot{\alpha}} BX_2 + \mathcal{V}\nabla_{\dot{\alpha}} \varphi X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})CX_2 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})\omega X_1 = 0, \tag{3.3}$$

$$\mathcal{H}\nabla_{\dot{\alpha}} CX_2 + \mathcal{H}\nabla_{\dot{\alpha}} \omega X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})BX_2 + (\mathcal{A}_{X_2} + \mathcal{T}_{X_1})\varphi X_1 = 0. \tag{3.4}$$

Proof Let $\alpha : I_2 \rightarrow N_1$ be a regular curve on N_1. Since $X_1(t)$ and $X_2(t)$ are the vertical and horizontal parts of the tangent vector field $\dot{\alpha}(t)$, i.e., $\dot{\alpha}(t) = X_1(t)+X_2(t)$. From equations (2.1), (2.3), (2.8), (2.9), (2.10), (2.11), (3.1) and (3.2), we get

$$\begin{aligned}\nabla_{\dot{\alpha}}\dot{\alpha} &= -J(\nabla_{\dot{\alpha}}J\dot{\alpha}),\\ &= -J(\nabla_{X_1}\varphi X_1 + \nabla_{X_1}\omega X_1 + \nabla_{X_2}\varphi X_1 + \nabla_{X_2}\omega X_1 +\\ &\quad \nabla_{X_1}BX_2 + \nabla_{X_1}CX_2 + \nabla_{X_2}BX_2 + \nabla_{X_2}CX_2),\\ &= -J(\mathcal{V}\nabla_{\dot{\alpha}}BX_2 + \mathcal{V}\nabla_{\dot{\alpha}}\varphi X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})CX_2 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})\omega X_1 +\\ &\quad \mathcal{H}\nabla_{\dot{\alpha}}CX_2 + \mathcal{H}\nabla_{\dot{\alpha}}\omega X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})BX_2 + (\mathcal{A}_{X_2} + \mathcal{T}_{X_1})\varphi X_1).\end{aligned}$$

Taking the vertical and horizontal components in above equation, we have

$$\mathcal{V}J\nabla_{\dot{\alpha}}\dot{\alpha} = \mathcal{V}\nabla_{\dot{\alpha}}BX_2 + \mathcal{V}\nabla_{\dot{\alpha}}\varphi X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})CX_2 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})\omega X_1,$$

$$\mathcal{H}J\nabla_{\dot{\alpha}}\dot{\alpha} = \mathcal{H}\nabla_{\dot{\alpha}}CX_2 + \mathcal{H}\nabla_{\dot{\alpha}}\omega X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})BX_2 + (\mathcal{A}_{X_2} + \mathcal{T}_{X_1})\varphi X_1,$$

Now, α is a geodesic on N_1 if and only if $\mathcal{V}J\nabla_{\dot{\alpha}}\dot{\alpha} = 0$ and $\mathcal{H}J\nabla_{\dot{\alpha}}\dot{\alpha} = 0$, which completes the proof. □

Theorem 3.4 *Let π be a semi-invariant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then π is a CSIR map with $r = e^f$ if and only if*

$$\begin{aligned}&g_1(\mathcal{V}\nabla_{\dot{\alpha}}\varphi X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})CX_2 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})\omega X_1, BX_2) +\\ &g_1(\mathcal{H}\nabla_{\dot{\alpha}}\omega X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})BX_2 + (\mathcal{A}_{X_2} + \mathcal{T}_{X_1})\varphi X_1, CX_2) + g_1(X_1, X_1)\frac{df}{dt} = 0\end{aligned}$$

where $\alpha : I_2 \rightarrow N_1$ is a geodesic on N_1 and X_1, X_2 are vertical and horizontal components of $\dot{\alpha}(t)$.

Proof Let $\alpha : I_2 \rightarrow N_1$ be a geodesic on N_1 with $X_1(t) = \mathcal{V}\dot{\alpha}(t)$ and $X_2(t) = \mathcal{H}\dot{\alpha}(t)$. Let $\theta(t)$ denote the angle in $[0, \pi]$ between $\dot{\alpha}(t)$ and $X_2(t)$. Assuming $\upsilon = ||\dot{\alpha}(t)||,^2$ then we get

$$g_1(X_1(t), X_1(t)) = \upsilon \sin^2\theta(t), \tag{3.5}$$

$$g_1(X_2(t), X_2(t)) = \upsilon \cos^2\theta(t). \tag{3.6}$$

Now, differentiating (3.6), we get

$$\frac{d}{dt}g_1(X_2(t), X_2(t)) = -2\upsilon\cos\theta(t)\sin\theta(t)\frac{d\theta}{dt}. \tag{3.7}$$

On the other hand, using equation (2.3), we get

$$\frac{d}{dt}g_1(X_2, X_2) = \frac{d}{dt}g_1(JX_2, JX_2). \tag{3.8}$$

Since π is a semi-invariant Riemannian map and using equation (3.2) in (3.8), we have

$$\frac{d}{dt}g_1(X_2, X_2) = 2g_1(\nabla_{\dot{\alpha}}BX_2, BX_2) + 2g_2(\pi_*(\nabla^{N_1}_{\dot{\alpha}}CX_2), \pi_*(CX_2)). \tag{3.9}$$

By using equation (2.14) in (3.9), we obtain

$$\frac{d}{dt}g_1(X_2,X_2) = 2g_1(\nabla_{\dot{\alpha}}BX_2,BX_2) - 2g_2((\nabla\pi_*)(\dot{\alpha},CX_2),\pi_*(CX_2)) + g_2(\overset{\pi}{\nabla}_{\dot{\alpha}}\pi_*(CX_2),(CX_2)).$$

Since second fundamental form of π is linear, therefore from above equation, we get

$$\begin{aligned} &\frac{d}{dt}g_1(X_2,X_2) \\ =\ & 2g_1(\nabla_{\dot{\alpha}}BX_2,BX_2) - 2g_2((\nabla\pi_*)(X_1,CX_2),\pi_*(CX_2)) - \\ & 2g_2((\nabla\pi_*)(X_2,CX_2),\pi_*(CX_2)) + g_2(\overset{\pi}{\nabla}_{\dot{\alpha}}\pi_*(CX_2),(CX_2)). \end{aligned} \tag{3.10}$$

Now, using equations (2.14), (2.15) and (3.10), we have

$$\begin{aligned} &\frac{d}{dt}g_1(X_2,X_2) \\ =\ & 2g_1(\nabla_{\dot{\alpha}}BX_2,BX_2) - 2g_2(\overset{\pi}{\nabla}_{X_1}\pi_*(CX_2),\pi_*(CX_2)) + \\ & 2g_2(\pi_*(\nabla^{N_1}_{X_1}CX_2),\pi_*(CX_2)) + g_2(\overset{\pi}{\nabla}_{X_1}\pi_*(CX_2),(CX_2)) + \\ & g_2(\overset{\pi}{\nabla}_{X_2}\pi_*(CX_2),(CX_2)). \end{aligned} \tag{3.11}$$

From equation (2.14), we get

$$\frac{d}{dt}g_1(X_2,X_2) = 2g_1(\mathcal{V}\nabla_{\dot{\alpha}}BX_2,BX_2) + 2g_1(\mathcal{H}\nabla_{\dot{\alpha}}CX_2,CX_2). \tag{3.12}$$

From (3.7) and (3.12), we have

$$g_1(\mathcal{V}\nabla_{\dot{\alpha}}BX_2,BX_2) + g_1(\ \mathcal{H}\nabla_{\dot{\alpha}}CX_2,CX_2) = -\upsilon\cos\theta(t)\sin\theta(t)\frac{d\theta}{dt}. \tag{3.13}$$

Also, using equations (3.3) and (3.4) in (3.13), we get

$$\begin{aligned} & g_1(\mathcal{V}\nabla_{\dot{\alpha}}\varphi X_1 + (\mathcal{T}_{X_1}+\mathcal{A}_{X_2})CX_2 + (\mathcal{T}_{X_1}+\mathcal{A}_{X_2})\omega X_1, BX_2) + \\ & g_1(\mathcal{H}\nabla_{\dot{\alpha}}\omega X_1 + (\mathcal{T}_{X_1}+\mathcal{A}_{X_2})BX_2 + (\mathcal{A}_{X_2}+\mathcal{T}_{X_1})\varphi X_1, CX_2) \\ =\ & \upsilon\cos\theta(t)\sin\theta(t)\frac{d\theta}{dt}. \end{aligned} \tag{3.14}$$

Moreover, π is a CSIR map with $r = e^f$ if and only if $\frac{d}{dt}(e^{f\circ\alpha}\sin\theta) = 0$, i.e., $e^{f\circ\alpha}(\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{df}{dt}) = 0$. By multiplying this with nonzero factor $\upsilon\sin\theta$, we have

$$-\upsilon\cos\theta\sin\theta\frac{d\theta}{dt} = \upsilon\sin^2\theta\frac{df}{dt}. \tag{3.15}$$

Thus, from equations (3.6), (3.14), and (3.15), we have

$$\begin{aligned} & g_1(\mathcal{V}\nabla_{\dot\alpha}\varphi X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})CX_2 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})\omega X_1, BX_2) + \\ & g_1(\mathcal{H}\nabla_{\dot\alpha}\omega X_1 + (\mathcal{T}_{X_1} + \mathcal{A}_{X_2})BX_2 + (\mathcal{A}_{X_2} + \mathcal{T}_{X_1})\varphi X_1, CX_2) \\ = \; & -g_1(X_1, X_1)\frac{df}{dt}. \end{aligned}$$

Hence the theorem 3.4 is proved. □

Theorem 3.5 *Let π be a CSIR map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2) with $r = e^f$, then at least one of the following statement is true:*

(i) f is constant on $J(D_2)$,

(ii) the fibers are one-dimensional,

(iii) $\overset{\pi}{\nabla}_{JX_1}\pi_*(Z_1) = -Z_1(f)\pi_*(JX_1)$, for all $X_1 \in \Gamma(D_2)$ and $Z_1 \in \Gamma(\mu)$.

Proof Let π be CSIR map from a Kähler manifold to a Riemannian manifold. For $Y_1, Y_2 \in \Gamma(D_2)$, using equation (2.13) and Theorem 3.1, we get

$$\mathcal{T}_{Y_1}Y_2 = -g_1(Y_1, Y_2)gradf. \tag{3.16}$$

Taking inner product in equation (3.16) with JX_1, we have

$$g_1(\mathcal{T}_{Y_1}Y_2, JX_1) = -g_1(Y_1, Y_2)g_1(gradf, JX_1), \tag{3.17}$$

for all $X_1 \in \Gamma(D_2)$.

From equations (2.3), (2.8), and (3.17), we obtain

$$g_1(\nabla_{Y_1}JY_2, X_1) = g_1(Y_1, Y_2)g_1(gradf, JX_1).$$

Since ∇ is metric connection, using equations (2.9) and (3.16) in above equation, we get

$$g_1(Y_1, X_1)g_1(gradf, JY_2) = g_1(Y_1, Y_2)g_1(gradf, JX_1). \tag{3.18}$$

Taking $X_1 = Y_2$ and interchanging the role of Y_1 and Y_2, we obtain

$$g_1(Y_2, Y_2)g_1(gradf, JY_1) = g_1(Y_1, Y_2)g_1(gradf, JY_2). \tag{3.19}$$

Using equation (3.18) with $X_1 = Y_1$ in (3.19), we have

$$g_1(gradf, JY_1) = \frac{(g_1(Y_1, Y_2))^2}{||Y_1||^2||Y_2||^2}g_1(gradf, JY_1). \tag{3.20}$$

If $gradf \in \Gamma(J(D_2))$, then equation (3.20) and the condition of equality in the Schwarz inequality implyies that either f is constant on $J(D_2)$ or the fibers are one dimensional. This implies the proof of (i) and (ii).

Now, from equations (2.8) and (3.16), we get

$$g_1(\nabla_{Y_1}X_1, Z_1) = -g_1(Y_1, X_1)g_1(gradf, Z_1), \tag{3.21}$$

for all $Z_1 \in \Gamma(\mu)$. Using equations (2.3) and (3.21), we have

$$g_1(\nabla_{Y_1}JX_1, JZ_1) = -g_1(Y_1, X_1)g_1(gradf, Z_1),$$

which implies

$$g_1(\nabla_{JX_1}Y_1, JZ_1) = -g_1(Y_1, X_1)g_1(gradf, Z_1). \tag{3.22}$$

Since ∇ is metric connection and using equations (3.18) and (3.22), we have

$$g_1(\mathcal{H}\nabla_{JX_1}Z_1, JY_1) = -g_1(JY_1, JX_1)g_1(\ gradf, Z_1).$$

Also, for Riemannian map π, we have

$$g_2(\pi_*(\nabla^{N_1}_{JX_1}Z_1), \pi_*(JY_1)) = -g_2(\pi_*(JY_1), \pi_*(JX_1))g_1(gradf, Z_1). \tag{3.23}$$

Again, using equations (2.14), (2.16), and (3.23), we obtain

$$g_2(\overset{\pi}{\nabla}_{JX_1}\pi_*(Z_1), \pi_*(JY_1)) = -g_2(\pi_*(JY_1), \pi_*(JX_1))g_1(gradf, Z_1),$$

which implies.

$$\overset{\pi}{\nabla}_{JX_1}\pi_*(Z_1) = -Z_1(f)\pi_*(JX_1). \tag{3.24}$$

If $gradf \in \Gamma(\mu)$, then (3.24) implies (iii). This completes the proof. □

Lemma 3.6 *Let π be a CSIR map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2) with $r = e^f$ and $\dim(\upsilon) > 1$. Then $\overset{\pi}{\nabla}_{V_1}\pi_*(JY_1) = V_1(f)\pi_*(JY_1)$, for all $Y_1 \in \Gamma(D_2)$ and $V_1 \in \Gamma(\ker \pi_*)^{\perp}$.*

Proof Let π be a CSIR map from a Kähler manifold to a Riemannian manifold. From Theorem 3.1, fibers are totally umbilical with mean curvature vector field $H = -gradf$, then we have

$$\begin{aligned} -g_1(\nabla_{Y_1}V_1, Y_2) &= g_1(\nabla_{Y_1}Y_2, V_1), \\ -g_1(\nabla_{Y_1}V_1, Y_2) &= -g_1(Y_1, Y_2)g_1(gradf, V_1), \end{aligned}$$

for all $Y_1, Y_2 \in \Gamma(D_2)$ and $V_1 \in \Gamma(\ker \pi_*)^{\perp}$.

Using equation (2.3) in above equation, we get

$$g_1(\nabla_{V_1}JY_1, JY_2) = g_1(JY_1, JY_2)g_1(gradf, V_1). \tag{3.25}$$

Since π is semi-invariant Riemannian map and using equation (3.25), we have

$$g_2(\nabla^{\pi}_{V_1}\pi_*(JY_1), \pi_*(JY_2)) = g_2(\pi_*(JY_1), \pi_*(JY_2))g_1(gradf, V_1). \tag{3.26}$$

From (2.14) in (3.26), we obtain

$$g_2(\overset{\pi}{\nabla}_{V_1}\pi_*(JY_1), \pi_*(JY_2)) = g_2(\pi_*(JY_1), \pi_*(JY_2))g_1(gradf, V_1), \tag{3.27}$$

which implies $\overset{\pi}{\nabla}_{V_1}\pi_*(JY_1) = V_1(f)\pi_*(JY_1)$, for all $Y_1 \in \Gamma(D_2)$ and $V_1 \in \Gamma(\ker \pi_*)^{\perp}$. □

Theorem 3.7 *Let π be a CSIR map with $r = e^f$ from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). If $\mathcal{T}$ is not equal to zero identically, then the invariant distribution D_1 cannot defined a totally geodesic foliation on N_1.*

Proof For $X_1, X_2 \in \Gamma(D_1)$, and $Y_1 \in \Gamma(D_2)$, using equations (2.3), (2.8), and (2.13), we get

$$\begin{aligned} g_1(\nabla_{X_1} X_2, Y_1) &= g_1(\nabla_{X_1} JX_2, JY_1), \\ &= g_1(\mathcal{T}_{X_1} JX_2, JY_1), \\ &= -g_1(X_1, JX_2) g_1(gradf, JY_1). \end{aligned}$$

Thus, the assertion can be seen from the above equation and the fact that $gradf \in J(D_2)$. □

Theorem 3.8 *Let π be a CSIR map with $r = e^f$ from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, the fibers of π are totally geodesic or the anti-invariant distribution D_2 one-dimensional.*

Proof If the fibers of π are totally geodesic, it is obvious. For second one, since π is a Clairaut proper semi-invariant Riemannian map, then either $\dim(D_2) = 1$ or $\dim(D_2) > 1$. If $\dim(D_2) > 1$, then we can choose $Z_1, Z_2 \in \Gamma(D_2)$ such that $\{Z_1, Z_2\}$ is orthonormal. From equations (2.9), (3.1), and (3.2), we get

$$\begin{aligned} \mathcal{T}_{Z_1} JZ_2 + \mathcal{H}\nabla_{Z_1} JZ_2 &= \nabla_{Z_1} JZ_2, \\ \mathcal{T}_{Z_1} JZ_2 + \mathcal{H}\nabla_{Z_1} JZ_2 &= B\mathcal{T}_{Z_1} Z_2 + C\mathcal{T}_{Z_1} Z_2 + \varphi\mathcal{V}\nabla_{Z_1} Z_2 + \omega\mathcal{V}\nabla_{Z_1} Z_2. \end{aligned}$$

Taking inner product above equation with Z_1, we obtain

$$g_1(\mathcal{T}_{Z_1} JZ_2, Z_1) = g_1(B\mathcal{T}_{Z_1} Z_2, Z_1) + g_1(\varphi\mathcal{V}\nabla_{Z_1} Z_2, Z_1). \tag{3.28}$$

From equation (2.5), we have

$$g_1(\mathcal{T}_{Z_1} Z_1, JZ_2) = -g_1(\mathcal{T}_{Z_1} JZ_2, Z_1) = g_1(\mathcal{T}_{Z_1} Z_2, JZ_1). \tag{3.29}$$

Now, using equations (2.13) and (3.29), we get

$$g_1(\mathcal{T}_{Z_1} Z_1, JZ_2) = g_1(gradf, JZ_2). \tag{3.30}$$

From equations (2.13), (3.29), and (3.30), we obtain

$$g_1(gradf, JZ_2) = g_1(\mathcal{T}_{Z_1} Z_1, JZ_2) = -g_1(\mathcal{T}_{Z_1} JZ_2, Z_1) = g_1(\mathcal{T}_{Z_1} Z_2, JZ_1). \tag{3.31}$$

From above equation, we get

$$\begin{aligned} g_1(gradf, JZ_2) &= g_1(\mathcal{T}_{Z_1} Z_2, JZ_1), \\ g_1(gradf, JZ_2) &= g_1(Z_1, Z_2) g_1(gradf, JZ_1), \\ g_1(gradf, JZ_2) &= 0. \end{aligned}$$

So, we get

$$gradf \perp J(D_2).$$

Therefore, the dimension of D_2 must be one. □

Theorem 3.9 *Let π be a CSIR map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2) with $r = e^f$ and $\dim(\upsilon) > 1$. Then, we get*

$$\sum_{\kappa=1}^{\omega} g_1(\mathcal{A}_{Z_1} u_\kappa, \mathcal{A}_{Z_1} u_\kappa) = \sum_{\kappa=1}^{\omega} g_2(\nabla^{\pi}_{Z_1} \pi_*(J u_\kappa), \nabla^{\pi}_{Z_1} \pi_*(J u_\kappa)), \tag{3.32}$$

$$\sum_{i=1}^{\beta+\int} g_2((\nabla \pi_*)(E_i, Z_1), (\nabla \pi_*)(Z_1, E_i)) = \sum_{l=1}^{\int} g_2((\nabla \pi_*)(\mu_l, Z_1), (\nabla \pi_*)(Z_1, \mu_l)), \tag{3.33}$$

$$\sum_{j=1}^{\beta} g_1(\mathcal{A}_{Z_1} v_j, \mathcal{A}_{Z_1} v_j) = (Z_1(f))^2 \sum_{j=1}^{\beta} g_1(v_j, v_j), \tag{3.34}$$

for all $Z_1 \in \Gamma(\ker \pi_*)^\perp$, where $\{u_1, u_2, ..., u_\omega\}, \{v_1, v_2, ..., v_\beta\}, \{E_1, E_2, ..., E_{\beta+\int}\}$ and $\{\mu_1, \mu_2, ...\mu_\int\}$ are orthonormal frames of $D_1, D_2, J(D_2)^\perp \oplus \mu$ and μ, respectively.

Proof Let $\pi : (N_1, g_1, J) \to (N_2, g_2)$ be a CSIR map. For all $Z_1 \in \Gamma(\ker \pi_*)^\perp$, we have

$$\sum_{\kappa=1}^{\omega} g_1(\mathcal{A}_{Z_1} u_\kappa, \mathcal{A}_{Z_1} u_\kappa) = \sum_{\kappa=1}^{\omega} g_1(\mathcal{H}\nabla_{Z_1 J} u_\kappa, \mathcal{H}\nabla_{Z_1} J u_\kappa). \tag{3.35}$$

Since π is a Riemannian map and using equation (2.14) in above equation (3.35), we have

$$\begin{aligned} \sum_{\kappa=1}^{\omega} g_1(\mathcal{A}_{Z_1} u_\kappa, \mathcal{A}_{Z_1} u_\kappa) &= \sum_{\kappa=1}^{\omega} g_2(\pi_*(\nabla^{N_1}_{Z_1} J u_\kappa), \pi_*(\nabla^{N_1}_{Z_1} J u_\kappa)), \\ &= \sum_{\kappa=1}^{\omega} g_2(\nabla^{\pi}_{Z_1} \pi_*(J u_\kappa), \nabla^{\pi}_{Z_1} \pi_*(J u_\kappa)). \end{aligned}$$

Now, for all $Z_1, \in \Gamma(\ker \pi_*)^\perp$, we get

$$\begin{aligned} &\sum_{i=1}^{\beta+\int} g_2((\nabla \pi_*)(E_i, Z_1), (\nabla \pi_*)(Z_1, E_i)) \\ &= \sum_{j=1}^{\beta} \sum_{l=1}^{\int} g_2((\nabla \pi_*)(J v_j + \mu_l, Z_1), (\nabla \pi_*)(Z_1, J v_j + \mu_l)). \end{aligned}$$

Since $J v_j \in \Gamma(\ker \pi_*)^\perp$ and $(\nabla \pi_*)$ is linear then from above equation, we have

$$\sum_{i=1}^{\beta+\int} g_2((\nabla\pi_*)(E_i,Z_1),(\nabla\pi_*)(E_i,Z_1)) \tag{3.36}$$

$$= \sum_{j=1}^{\beta} g_2((\nabla\pi_*)(Jv_j,Z_1),(\nabla\pi_*)(Z_1,Jv_j)) +$$

$$\sum_{j=1}^{\beta}\sum_{l=1}^{\int} g_2((\nabla\pi_*)(\mu_l,Z_1),(\nabla\pi_*)(Z_1,Jv_j)) +$$

$$\sum_{j=1}^{\beta}\sum_{l=1}^{\int} g_2((\nabla\pi_*)(Jv_j,Z_1),(\nabla\pi_*)(Z_1,\mu_l)) +$$

$$\sum_{l=1}^{\int} g_2((\nabla\pi_*)(\mu_l,Z_1),(\nabla\pi_*)(Z_1,\mu_l)).$$

Thus, (3.32) holds.

On the other hand, using (2.14) in first term of (3.36) in right hand side, we have

$$\sum_{j=1}^{\beta} g_2((\nabla\pi_*)(Jv_j,Z_1),(\nabla\pi_*)(Z_1,Jv_j))$$

$$= \sum_{j=1}^{\beta} g_2((\nabla\pi_*)(Jv_j,Z_1),\overset{\pi}{\nabla}_{Z_1}\pi_*(Jv_j)-\pi_*(\nabla^{N_1}_{Z_1}Jv_j)).$$

Now, from equations (2.4), (2.5), and (3.36), we get

$$\sum_{j=1}^{\beta} g_2((\nabla\pi_*)(Jv_j,Z_1),(\nabla\pi_*)(Z_1,Jv_j)) \tag{3.37}$$

$$= \sum_{j=1}^{\beta} g_2((\nabla\pi_*)(Jv_j,Z_1),\overset{\pi}{\nabla}_{Z_1}\pi_*(Jv_j)).$$

Also, using Lemma 3.6 in equation (3.37), we obtain $\sum_{j=1}^{\beta} g_2((\nabla\pi_*)(Jv_j,Z_1),(\nabla\pi_*)(Z_1,Jv_j))$ $= \sum_{j=1}^{\beta} g_2((\nabla\pi_*)(Jv_j,Z_1),Z_1(f)\pi_*(Jv_j))$, which implies $\sum_{j=1}^{\beta} g_2((\nabla\pi_*)(Jv_j,Z_1),(\nabla\pi_*)(Z_1,Jv_j))$ $= \sum_{j=1}^{\beta} Z_1(f)g_2((\nabla\pi_*)(Jv_j,Z_1),\pi_*(Jv_j))$.

By using equation (2.15) in above equation, we have

$$\sum_{j=1}^{\beta} g_2((\nabla\pi_*)(Jv_j,Z_1),(\nabla\pi_*)(Z_1,Jv_j)) = 0, \tag{3.38}$$

similarly

$$\sum_{j=1}^{\beta}\sum_{l=1}^{\int} g_2((\nabla\pi_*)(\mu_l,Z_1),(\nabla\pi_*)(Z_1,Jv_j)) = 0, \tag{3.39}$$

$$\sum_{j=1}^{\beta}\sum_{l=1}^{\int} g_2((\nabla\pi_*)(Jv_j, Z_1), (\nabla\pi_*)(Z_1, \mu_l)) = 0. \tag{3.40}$$

Thus, by using equations (3.38), (3.39), and (3.40) in equation (3.36), then we obtain (3.33). Now, for $Z_1 \in \Gamma(\ker\pi_*)^\perp$, we get

$$\begin{aligned}\sum_{j=1}^{\beta} g_1(\mathcal{A}_{Z_1}v_j, \mathcal{A}_{Z_1}v_j) &= \sum_{j=1}^{\beta} g_1(\mathcal{H}\nabla_{Z_1}v_j, \mathcal{H}\nabla_{Z_1}v_j),\\ &= \sum_{j=1}^{\beta} g_1(\mathcal{H}\nabla_{Z_1}Jv_j, \mathcal{H}\nabla_{Z_1}Jv_j).\end{aligned}$$

Since π is a Riemannian map and using equation (2.14) in above equation, we get

$$\begin{aligned}&\sum_{j=1}^{\beta} g_1(\mathcal{A}_{Z_1}v_j, \mathcal{A}_{Z_1}v_j) \\ = &\sum_{j=1}^{\beta}\{g_2((\nabla\pi_*)(Z_1, Jv_j), (\nabla\pi_*)(Z_1, Jv_j)) - 2g_2((\nabla\pi_*)(Z_1, Jv_j), \overset{\pi}{\nabla}_{Z_1}\pi_*(jv_j)) + \\ &g_2(\overset{\pi}{\nabla}_{Z_1}\pi_*(Jv_j), \overset{\pi}{\nabla}_{Z_1}\pi_*(Jv_j))\}.\end{aligned} \tag{3.41}$$

Using Lemma 3.6 and equations (2.16), (3.38) in (3.41), we get

$$\begin{aligned}\sum_{j=1}^{\beta} g_1(\mathcal{A}_{Z_1}v_j, \mathcal{A}_{Z_1}v_j) &= \sum_{j=1}^{\beta} g_2(Z_1(f)\pi_*(Jv_j), Z_1(f)\pi_*(Jv_j)),\\ &= (Z_1(f))^2\sum_{j=1}^{\beta} g_2(\pi_*(Jv_j), \pi_*(Jv_j)).\end{aligned} \tag{3.42}$$

Since $Jv_j \in \Gamma(\ker\pi_*)^\perp$ and π is a Riemannian map then from (3.42), we obtain

$$\sum_{j=1}^{\beta} g_1(\mathcal{A}_{Z_1}v_j, \mathcal{A}_{Z_1}v_j) = (Z_1(f))^2\sum_{j=1}^{\beta} g_1(v_j, v_j). \tag{3.43}$$

From equations (2.3) and (3.43), we obtain (3.34), which completes the proof. □

4. Example

Example 4.1 *Let N_1 be an Euclidean space given by $N_1 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in R^6 : (x_1, x_2, x_3, x_4, x_5, x_6) \neq (0, 0, 0, 0, 0, 0)\}$. We define the Riemannian metric g_1 on N_1 given by $g_1 = e^{2x_6}dx_1^2 + e^{2x_6}dx_2^2 + e^{2x_6}dx_3^2 + e^{2x_6}dx_4^2 + e^{2x_6}dx_5^2 + dx_6^2$ and the complex structure on J and N_1 defined as*

$$J(x_1, x_2, x_3, x_4, x_5, x_6) = (-x_2, x_1, -x_4, x_3, -x_6, x_5).$$

Let $N_2 = \{(v_1, v_2, v_3, v_4) \in R^4\}$ *be a Riemannian manifold with Riemannian metric* g_2 *on* N_2 *given by* $g_2 = e^{2x_6}dv_1^2 + e^{2x_6}dv_2^2 + e^{2x_6}dv_3^2 + dv_4^2$.

Define a map $\pi : R^6 \to R^4$ *by*

$$\pi(x_1, x_2, x_3, x_4, x_5, x_6) = (\frac{x_3 - x_4}{\sqrt{2}}, x_5, x_6, 0).$$

Then, we have

$$\ker \pi_* = D_1 \oplus D_2,$$

where

$$D_1 =< X_1 = e_1, X_2 = e_2 >, D_2 =< X_3 = e_3 + e_4 >,$$

and

$$(\ker \pi_*)^\perp =< V_1 = e_3 - e_4, V_2 = e_5, V_3 = e_6 >,$$

where $\{e_1 = e^{-x_6}\frac{\partial}{\partial x_1}, e_2 = e^{-x_6}\frac{\partial}{\partial x_2}, e_3 = e^{-x_6}\frac{\partial}{\partial x_3}, e_4 = e^{-x_6}\frac{\partial}{\partial x_4}, e_5 = e^{-x_6}\frac{\partial}{\partial x_5}, e_6 = \frac{\partial}{\partial x_6}\}$, $\{e_1^* = \frac{\partial}{\partial v_1}, e_2^* = \frac{\partial}{\partial v_2}, e_3^* = \frac{\partial}{\partial v_3}, e_4^* = \frac{\partial}{\partial v_4}\}$ *are bases on* T_qN_1 *and* $T_{\pi(q)}N_2$*, respectively, for all* $q \in N_1$*. By direct computations, we can see that* $\pi_*(V_1) = \sqrt{2}e^{-x_6}e_1^*, \pi_*(V_2) = e^{-x_6}e_2^*, \pi_*(V_3) = e^{-x_6}e_3^*$ *and* $g_1(V_i, V_j) = g_2(\pi_*V_i, \pi_*V_j)$ *for all* $V_i, V_j \in \Gamma(\ker \pi_*)^\perp$*,* $i, j = 1, 2, 3$*. Thus* π *is Riemannian map with* $(range\pi_*)^\perp =< e_4^* >$*. Moreover it is easy to see that* $JX_3 = V_1$*. Therefore* π *is a semi-invariant Riemannian map.*

Now, we will find smooth function f on N_1 satisfying $T_XX = g_1(X, X)\nabla f$, for all $X \in \Gamma(\ker \pi_*)$. Since covariant derivative for vector fields $E = E_i\frac{\partial}{\partial x_i}, F = F_j\frac{\partial}{\partial x_j}$ on N_1 is defined as

$$\nabla_E F = E_iF_j\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} + E_i\frac{\partial F_j}{\partial x_i}\frac{\partial}{\partial x_j}, \tag{4.1}$$

where the covariant derivative of basis vector fields $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial x_i}$ is defined by

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \Gamma_{ij}^k\frac{\partial}{\partial x_k}, \tag{4.2}$$

and Christoffel symbols are defined by

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\frac{\partial g_{1jl}}{\partial x_i} + \frac{\partial g_{1il}}{\partial x_j} - \frac{\partial g_{1ij}}{\partial x_l}). \tag{4.3}$$

Now, we get

$$g_{1ij} = \begin{bmatrix} e^{2x_6} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2x_6} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2x_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2x_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2x_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, g_1^{ij} = \begin{bmatrix} e^{-2x_6} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-2x_6} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-2x_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2x_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-2x_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.4}$$

By using equations (4.3) and (4.4), we get

$$\begin{aligned}
\Gamma^1_{11} &= 0, \Gamma^2_{11} = 0, \Gamma^3_{11} = 0, \Gamma^4_{11} = 0, \Gamma^5_{11} = 0, \Gamma^6_{11} = -e^{2x_6}, \\
\Gamma^1_{22} &= 0, \Gamma^2_{22} = 0, \Gamma^3_{22} = 0, \Gamma^4_{22} = 0, \Gamma^5_{22} = 0, \Gamma^6_{22} = -e^{2x_6}, \\
\Gamma^1_{33} &= 0, \Gamma^2_{33} = 0, \Gamma^3_{33} = 0, \Gamma^4_{33} = 0, \Gamma^5_{33} = 0, \Gamma^6_{33} = -e^{2x_6}, \\
\Gamma^1_{44} &= 0, \Gamma^2_{44} = 0, \Gamma^3_{44} = 0, \Gamma^4_{44} = 0, \Gamma^5_{44} = 0, \Gamma^6_{44} = -e^{2x_6}, \\
\Gamma^1_{12} &= \Gamma^2_{12} = \Gamma^3_{12} = \Gamma^4_{12} = \Gamma^5_{12} = \Gamma^4_{12} = \Gamma^5_{12} = \Gamma^6_{12} = 0, \\
\Gamma^1_{21} &= \Gamma^2_{21} = \Gamma^3_{21} = \Gamma^4_{21} = \Gamma^5_{21} = \Gamma^4_{21} = \Gamma^5_{21} = \Gamma^6_{21} = 0, \\
\Gamma^1_{13} &= \Gamma^2_{13} = \Gamma^3_{13} = \Gamma^4_{13} = \Gamma^5_{13} = \Gamma^4_{13} = \Gamma^5_{13} = \Gamma^6_{13} = 0, \\
\Gamma^1_{31} &= \Gamma^2_{31} = \Gamma^3_{31} = \Gamma^4_{31} = \Gamma^5_{31} = \Gamma^4_{31} = \Gamma^5_{31} = \Gamma^6_{31} = 0, \\
\Gamma^1_{14} &= \Gamma^2_{14} = \Gamma^3_{14} = \Gamma^4_{14} = \Gamma^5_{14} = \Gamma^4_{14} = \Gamma^5_{14} = \Gamma^6_{14} = 0, \\
\Gamma^1_{41} &= \Gamma^2_{41} = \Gamma^3_{41} = \Gamma^4_{41} = \Gamma^5_{41} = \Gamma^4_{41} = \Gamma^5_{41} = \Gamma^6_{41} = 0, \\
\Gamma^1_{23} &= \Gamma^2_{23} = \Gamma^3_{23} = \Gamma^4_{23} = \Gamma^5_{23} = \Gamma^4_{23} = \Gamma^5_{23} = \Gamma^6_{23} = 0, \\
\Gamma^1_{32} &= \Gamma^2_{32} = \Gamma^3_{32} = \Gamma^4_{32} = \Gamma^5_{32} = \Gamma^4_{32} = \Gamma^5_{32} = \Gamma^6_{32} = 0 \\
\Gamma^1_{24} &= \Gamma^2_{24} = \Gamma^3_{24} = \Gamma^4_{24} = \Gamma^5_{24} = \Gamma^4_{24} = \Gamma^5_{24} = \Gamma^6_{24} = 0, \\
\Gamma^1_{42} &= \Gamma^2_{42} = \Gamma^3_{42} = \Gamma^4_{42} = \Gamma^5_{42} = \Gamma^4_{42} = \Gamma^5_{42} = \Gamma^6_{42} = 0, \\
\Gamma^1_{34} &= \Gamma^2_{34} = \Gamma^3_{34} = \Gamma^4_{34} = \Gamma^5_{34} = \Gamma^4_{34} = \Gamma^5_{34} = \Gamma^6_{34} = 0, \\
\Gamma^1_{43} &= \Gamma^2_{43} = \Gamma^3_{43} = \Gamma^4_{43} = \Gamma^5_{43} = \Gamma^4_{43} = \Gamma^5_{43} = \Gamma^6_{43} = 0.
\end{aligned} \tag{4.5}$$

Using equations (4.1), (4.2), and (4.5), we obtain

$$\begin{aligned}
\nabla_{e_1} e_1 &= \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = -\frac{\partial}{\partial x_6}, \\
\nabla_{e_1} e_2 &= \nabla_{e_1} e_3 = \nabla_{e_1} e_4 = \nabla_{e_2} e_1 = \nabla_{e_2} e_3 = \nabla_{e_2} e_4 = 0, \\
\nabla_{e_3} e_1 &= \nabla_{e_3} e_2 = \nabla_{e_3} e_4 = \nabla_{e_4} e_1 = \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = 0.
\end{aligned} \tag{4.6}$$

Therefore,

$$\begin{aligned}
\nabla_{X_1} X_1 &= \nabla_{e_1} e_1 = -\frac{\partial}{\partial x_6}, \nabla_{X_2} X_2 = \nabla_{e_2} e_2 = -\frac{\partial}{\partial x_6}, \\
\nabla_{X_3} X_3 &= \nabla_{e_3+e_4} e_3 + e_4 = -2\frac{\partial}{\partial x_6}.
\end{aligned} \tag{4.7}$$

Now, we have

$$T_X X = T_{\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3} \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \lambda_1, \lambda_2, \lambda_3 \in R,$$

$$\begin{aligned}
T_X X &= \lambda_1^2 T_{X_1} X_1 + \lambda_2^2 T_{X_2} X_2 + \lambda_3^2 T_{X_3} X_3 + 2\lambda_1\lambda_2 T_{X_1} X_2 + \\
&\quad 2\lambda_1\lambda_3 T_{X_1} X_3 + 2\lambda_2\lambda_3 T_{X_2} X_3.
\end{aligned} \tag{4.8}$$

Using equations (2.8) and (4.7), we obtain

$$\begin{aligned} T_{X_1}X_1 &= -\frac{\partial}{\partial x_6}, T_{X_2}X_2 = -\frac{\partial}{\partial x_6}, T_{X_3}X_3 = -2\frac{\partial}{\partial x_6}, \\ T_{X_1}X_2 &= 0, T_{X_1}X_3 = 0, T_{X_2}X_3 = 0. \end{aligned} \tag{4.9}$$

Next, using equations (4.8) and (4.9), we get

$$T_X X = -(\lambda_1^2 + \lambda_2^2 + 2\lambda_3^2)\frac{\partial}{\partial x_6}. \tag{4.10}$$

Since $X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$, so $g_1(\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3) = \lambda_1^2 + \lambda_2^2 + 2\lambda_3^2$. For any smooth function f on R^6, the gradient of f with respect to the metric g_1 is given by $\nabla f = \sum_{i,j=1}^{6} g_1^{ij} \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}$. Hence $\nabla f = e^{-2x_6}\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + e^{-2x_6}\frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_2} + e^{-2x_6}\frac{\partial f}{\partial x_3}\frac{\partial}{\partial x_3} + e^{-2x_6}\frac{\partial f}{\partial x_4}\frac{\partial}{\partial x_4} + e^{-2x_6}\frac{\partial f}{\partial x_5}\frac{\partial}{\partial x_5} + \frac{\partial f}{\partial x_6}\frac{\partial}{\partial x_6}$. Hence $\nabla f = \frac{\partial}{\partial x_6}$ for the function $f = x_6$. Then it is easy to see that $T_X X = -g_1(X, X)\nabla f$, thus by Theorem (3.1), π is CSIR map.

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