Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1210 – 1223
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doi:10.3906/mat-2111-106

Product-type operators on weak vector valued α-Besov spaces

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Received: 26.03.2021 • **Accepted/Published Online:** 02.03.2022 • **Final Version:** 05.05.2022

Abstract: Let ψ_1 and ψ_2 be analytic functions on the open unit disk $\mathbb{D}$ and φ an analytic self map on $\mathbb{D}$. Let M_ψ, C_φ and D denote the multiplication, composition and differentiation operators. We consider operators $M_{\psi_1}C_\varphi$, $M_{\psi_2}C_\varphi D$ and the Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}(f) = M_{\psi_1}C_\varphi(f)+M_{\psi_2}C_\varphi D(f)$ on α-Besov space $\mathcal{B}_{p,\alpha}$ and weak vector valued α-Besov space $w\mathcal{B}_{p,\alpha}(X)$ for complex Banach space X and find some equivalent statements for boundedness of these operators. Also, boundedness and compactness of composition operator C_φ on $\mathcal{B}_{p,\alpha}(\mathbb{D})$ and $w\mathcal{B}_{p,\alpha}(\mathbb{D})$ are given.

Key words: Product of composition multiplication and differentiation, α-Besov spaces, Carleson measure, weak vector valued α-Besov spaces, boundedness

1. Introduction

Let X be a complex Banach space and $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. The Lebesgue area measure on $\mathbb{D}$ is defined by $dA(z) = rdrd\theta = dxdy$. Denote by $H(X)$ the class of all analytic functions $f : \mathbb{D} \to X$. For $p \geq 1$, the vector valued weighted Bergman space $A^p_\alpha(X)$ consists of all functions $f \in H(X)$ for which

$$||f||^p_{A^p_\alpha(X)} = \int_{\mathbb{D}} ||f(z)||^p(1-|z|^2)^\alpha dA(z) < \infty.$$

Note that $A^p_\alpha(X)$ is Banach space for $p \geq 1$, see [2, 3, 11] for the theory of these spaces.

Let $1 \leq p < \infty$, $-1 \leq \alpha < \infty$, the vector valued α-Besov space $\mathcal{B}_{p,\alpha}(X)$ is the space of all functions $f \in H(X)$ such that

$$||f||^p_{\mathcal{B}_{p,\alpha}(X)} = \int_{\mathbb{D}} ||f'(z)||^p_X(1-|z|^2)^\alpha dA(z) < \infty.$$

Note that for $X = \mathbb{D}$, we have the α-Besov space $\mathcal{B}_{p,\alpha}(\mathbb{D})$ and for $X = \mathbb{D}$, $p = 2$ and $\alpha = 0$ we have the classic Dirichlet space $\mathcal{D}$.

The weak vector valued α-Besov space $w\mathcal{B}_{p,\alpha}(X)$ consists of all analytic functions $f : \mathbb{D} \to X$, for which

$$||f||_{w\mathcal{B}_{p,\alpha}(X)} = \sup_{||x^*||\leq 1} ||x^* \circ f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}$$

is finite. Here $x^* \in X^*$, the dual space of X. In fact, such kind of weak version spaces $wE(X)$ can be introduced under more general conditions on any Banach spaces E consists of analytic functions $f : \mathbb{D} \to X$, see

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2010 *AMS Mathematics Subject Classification:* 47B38, 46E40, 30H25.

[13]. Some strong and weak version spaces are completely different such as Hardy spaces $H^2(X)$ and $wH^2(X)$. Also, Dirichlet spaces $w\mathcal{D}_\alpha(X)$ and $\mathcal{D}_\alpha(X)$ are different for any infinite dimensional complex Banach space X, [20]. Some others are the same such as Bloch spaces $\mathcal{B}(X)$ and $w\mathcal{B}(X)$ (refer to [1]).

Given analytic functions φ and ψ in the unit disc $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, the weighted composition operator ψC_φ on $H(\mathbb{D})$ is defined by $W_{\psi,\varphi} f(z) = \psi(z) f(\varphi(z))$, for $z \in \mathbb{D}$. If $\psi = 1$, it becomes the composition operator C_φ and if $\varphi(z) = z$, it becomes the multiplication operator M_φ. Since $W_{\psi,\varphi} = M_\psi C_\varphi$, it is a product-type operator.

When studying an operator on a space, the first question is about the properties of the operator such as boundedness, compactness, adjoint, normality and so on. For composition and the other operators on spaces of analytic functions we used here, the question is the relation between operator-theoretic properties of the operator with the function and geometric properties of the inducing functions ψ and φ. For example, the well known Littlewood Subordination Theorem in the excellent book by Cowen and Maccluer [5] computed the norm of the operator and related the results to the boundedness. Beside this, Shapiro in [18] studied the compact composition operators widely in many aspects. For example, he proved that if $||\varphi||_\infty < 1$ the C_φ is a compact operator on Hilbert Hardy space H^2.

Weighted composition operators as well as the operators studied here are all generalization of well known composition operators which paly an important role in operator theory. Some applications of (weighted) composition operators are for example isometries of H^p, Hardy space, $p \neq 2$ and $p > 1$ are weighted composition operators [7]. Also, backward shifts of all multiplicities can be represented as composition operators. Composition operators have arisen in the study of commutates of multiplication operators and more general operators, Cowen and Maccluer [5], and play a role in theory of dynamical systems. De Branges' proof of the Bieberbach conjecture depended on composition operators on a spaces of analytic functions [4].

The action of composition operators and weighted composition operators on analytic function spaces such as Bergman, Hardy, Dirichlet and Dirichlet type spaces has been studied by many authors, see for example [8–10, 12, 21, 22].

(Weighted) composition operators can be generalized in some manners. One of the important generalizations is the following so-called Stević-Sharma operator:

$$T_{\psi_1,\psi_2,\varphi} f(z) = \psi_1 f(\varphi(z)) + \psi_2 f'(\varphi(z)),\ f \in H(\mathbb{D}), \tag{1.1}$$

which includes many operators. Other operators related to the weighted composition operators are

$$\begin{aligned} M_\psi D C_\varphi(f) &= \psi(f \circ \varphi)' = \psi\varphi'(f' \circ \varphi), \\ M_\psi C_\varphi D(f) &= \psi(f' \circ \varphi), \end{aligned} \tag{1.2}$$

where D is differentiation operator on $H(\mathbb{D})$ and defined by

$$Df = f'.$$

In this paper, we characterize boundedness of weighted composition operators $W_{\psi,\varphi} = M_\psi C_\varphi$ and the product-type operators of (1.2) on α-Besov space $\mathcal{B}_{p,\alpha}(\mathbb{D})$ and weak vector valued α-Besov space $w\mathcal{B}_{p,\alpha}(X)$. Then we find equivalent statements for boundedness of the Stević-Sharma operator on these spaces. Also, boundedness and compactness of composition operator C_φ on $\mathcal{B}_{p,\alpha}(\mathbb{D})$ and $w\mathcal{B}_{p,\alpha}(X)$ are given. The most interesting point of the results is that for all operators mentioned above, the boundednes and compactness on the arbitrary Banach space X is equivalent to the study on the unit ball $\mathbb{D}$. As the applications of our main results, readers can obtain some characterization for the boundedness of all operators contained in Stević-Sharma operator and also the differences of those operators on $w\mathcal{B}_{p,\alpha}(X)$ and $\mathcal{B}_{p,\alpha}(\mathbb{D})$.

Let μ be a finite positive Borel measure on $\mathbb{D}$. Then μ is said to be Carleson if there exists a constant C such that $\mu(S(\xi,h) \leq Ch^2$ for all ξ and h, when $|\xi| = 1$ and $0 < h < 2$. The measure is said to be compact Carleson if $\lim_{h\to 0}\sup_{|\xi|=1}\frac{\mu(S(\xi,h))}{h^2} = 0$. Carleson measures have been useful in the study of composition operators in several settings (see for example [12, 14–17, 22]). For $w \in \mathbb{D}$, let $N_2(\varphi, w)$ denote the number of zeros (counting multiplicities) of $\varphi(z) - w$. For $1 \leq p < \infty$ and $w \in \mathbb{D}$ and analytic map ψ on $\mathbb{D}$, we define modified counting function

$$N_{p,\alpha,\psi}(\varphi, w) = \sum \frac{(1-|z|^2)^{\alpha}|\psi(z)|^p}{|\varphi'(z)|^{2-p}}$$

where the sum extends over the zeros of $\varphi - w$, repeated by multiplicity. In particular, $N_{p,\alpha,\psi}(\varphi, w) = 0$ for $w \notin \varphi(\mathbb{D})$. Clearly with $\psi = 1$, $\alpha = 0$ and $p = 2$ we have $N_2(\varphi, w)$.
Let $\mu_{p,\alpha,\psi}$ be the measure defined on $\mathbb{D}$ by $d\mu_{p,\alpha,\psi}(w) = N_{p,\alpha,\psi}(\varphi, w)dA(w)$, $1 \leq p < \infty$.

A nonnegative measure μ on $\mathbb{D}$ is called a Carleson measure for $\mathcal{B}_{p,\alpha}(X)$ if there is a constant $C > 0$ such that

$$\int_{\mathbb{D}} ||f(z)||_X^p d\mu(z) \leq C||f||^p_{\mathcal{B}_{p,\alpha}(X)},$$

for all $f \in \mathcal{B}_{p,\alpha}(X)$. That is, the inclusion operator i from $\mathcal{B}_{p,\alpha}(X)$ into $L^p(X,\mu)$ is bounded. We call the Carleson measure μ, a compact Carleson measure for $\mathcal{B}_{p,\alpha}(X)$ if the inclusion operator i from $\mathcal{B}_{p,\alpha}(X)$ into $L^p(X,\mu)$ is compact.
Through these facts, one can have the following theorem (as a definition) that characterizes Carleson measure for $A^p_\alpha(X)$.

Definition 1.1 *Take $1 \leq p < \infty$. Let μ be a positive Borel measure on $\mathbb{D}$. Then*
(a) μ is said to be a Carleson measure for $A^p_\alpha(X)$ if and only if $A^p_\alpha(X) \subset L^p(\mu, X)$ and the inclusion operator

$$I : A^p_\alpha(X) \to L^p(\mu, X)$$

is a bounded operator.
(b) μ is said to be a compact Carleson measure for $A^p_\alpha(X)$ if and only if $A^p_\alpha(X) \subset L^p(\mu, X)$ and the inclusion operator I from $A^p_\alpha(X)$ into $L^p(\mu, X)$ is compact.

Remark 1.2 *Part (a) of the above definition is equivalent with the following statement:*
There exists a constant C such that

$$\int_{\mathbb{D}} ||f(z)||_X^p d\mu(z) \leq C||f||^p_{A^p_\alpha(X)},$$

for all$f \in A^p_\alpha(X)$.

Throughout this paper, constants are denoted by C, they are positive and not necessarily the same as each occurrence.

2. Product-type operators on weak vector valued α-Besov spaces

The following lemma from [11] will help us prove our next results.

Lemma 2.1 *For any $\alpha > -1$ and $p > 0$, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) \leq C\big[|f(0)|^p + \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^\alpha dA(z)\big]$$

and

$$|f(0)|^p + \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^\alpha dA(z) \leq C \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z)$$

for all analytic functions $f \in H(\mathbb{D})$, where

$$g(z) = (1-|z|^2) f'(z) \qquad z \in \mathbb{D}.$$

Notation 2.2 *As a result of the above lemma, we can see that $f \in A^p_\alpha(\mathbb{D})$ if and only if $f' \in A^p_{\alpha+p}(\mathbb{D})$.*

Lemma 2.3 *Suppose that $-1 < \alpha$ and $2+\alpha < p < \infty$. Then for any $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$, there exists a constant M such that $||f||_\infty \leq M||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}$.*

Proof Let $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$, then $||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = ||f'||_{A^p_\alpha(\mathbb{D})} + |f(0)| < \infty$. So $f' \in A^p_\alpha(\mathbb{D})$ and according to [11], we have that $|f'(z)| \leq \frac{||f'||_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}$. However, $2+\alpha < p$,; therefore;

$$|f(z) - f(0)| = |\int_0^z f'(w)dw| = |\int_0^1 zf'(az)da| \leq \int_0^1 |z||f'(az)|da \leq \int_0^1 \frac{||f'||_{A^p_\alpha}|z|}{(1-|az|^2)^{\frac{2+\alpha}{p}}}da$$

$$\leq ||f'||_{A^p_\alpha} \int_0^1 \frac{|z|}{(1-|az|)^{\frac{2+\alpha}{p}}}da \leq M||f'||_{A^p_\alpha(\mathbb{D})},$$

for a constant M. Hence, for any $z \in \mathbb{D}$,

$$|f(z)| \leq M||f'||_{A^p_\alpha(\mathbb{D})} + |f(0)| \leq M||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}.$$

It follows that

$$\sup_{z\in\mathbb{D}} |f(z)| = ||f||_\infty \leq M||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}.$$

□

Now our first plan is to obtain conditions for boundedness of the operator $M_\psi C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to w\mathcal{B}^p_\alpha(X)$. Since $(M_\psi C_\varphi(f))' = M_{\psi'} C_\varphi(f) + M_\psi DC_\varphi(f)$, in the next two theorems, we characterize boundedness of the operators $M_\psi C_\varphi$ and $M_\psi DC_\varphi$ from $w\mathcal{B}_{p,\alpha}(X)$ into $wA^p_\alpha(X)$.

Theorem 2.4 *Let $\alpha \geq -1$, $p > \alpha + 2$ and φ be an analytic self map on $\mathbb{D}$. Then the following statements are equivalent:*

a) Operator $M_\psi C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to wA^p_\alpha(X)$ is bounded.

b) Operator $M_\psi C_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is bounded.

c) $\psi \in A^p_\alpha(\mathbb{D})$.

Proof (a) $\Rightarrow$ (b). Suppose that $M_\psi C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to wA^p_\alpha(X)$ is bounded and $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$. If $x \in X$ with $||x|| = 1$ and consider the function $g : \mathbb{D} \to X$, $g(z) = xf(z)$ for $z \in \mathbb{D}$ then we have

$$(x^*og)'(z) = (x^*oxf)'(z) = \lim_{w\to z} \frac{x^*(xf(w)) - x^*(xf(z))}{w-z}$$
$$= \lim_{w\to z} \frac{f(w)x^*(x) - f(z)x^*(x)}{w-z} = f'(z)x^*(x).$$

It follows that

$$||g||^p_{w\mathcal{B}_{p,\alpha}(X)} = \sup_{||x^*||_{X^*}\le 1} ||x^*og||^p_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = \sup_{||x^*||_{X^*}\le 1} \int_{\mathbb{D}} (|(x^*og)'(z)|^p(1-|z|^2)^\alpha dA(z) + |(x^*og)(0)|)$$
$$= \sup_{||x^*||_{X^*}\le 1} (\int_{\mathbb{D}} |f'(z)x^*(x)|^p(1-|z|^2)^\alpha dA(z) + |x^*(x)f(0)|)$$
$$= \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^\alpha dA(z) + |f(0)| = ||f||^p_{\mathcal{B}_{p,\alpha}(\mathbb{D})} < \infty.$$

So $g \in w\mathcal{B}_{p,\alpha}(X)$ and we have

$$||M_\psi C_\varphi g||_{wA^p_\alpha(X)} \le C||g||_{w\mathcal{B}_{p,\alpha}(X)} = C||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}, \tag{2.1}$$

for some constant C. On the other hand,

$$||M_\psi C_\varphi g||^p_{wA^p_\alpha(X)} = \sup_{||x^*||_{X^*}\le 1} (\int_{\mathbb{D}} |(x^*\psi C_\varphi g)(z)|^p(1-|z|^2)^\alpha dA(z))$$
$$= \sup_{||x^*||_{X^*}\le 1} (\int_{\mathbb{D}} |(x^*\psi C_\varphi(xf))(z)|^p(1-|z|^2)^\alpha dA(z)$$
$$= \sup_{||x^*||_{X^*}\le 1} (\int_{\mathbb{D}} |x^*(x)(\psi C_\varphi f)(z)|^p(1-|z|^2)^\alpha dA(z)$$
$$= \int_{\mathbb{D}} |(\psi C_\varphi f)(z)|^p(1-|z|^2)^\alpha dA(z) = ||M_\psi C_\varphi f||^p_{A^p_\alpha(\mathbb{D})}. \tag{2.2}$$

Hence, from (2.1) and (2.2) we obtain

$$||M_\psi C_\varphi f||_{A^p_\alpha(\mathbb{D})} \le C||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}.$$

This implies boundedness of $M_\psi C_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$.
(b)$\Rightarrow$ (c). If $M_\psi C_\varphi$ is a bounded operator from $\mathcal{B}_{p,\alpha}(\mathbb{D})$ to $A^p_\alpha(\mathbb{D})$, then by choosing $f = 1$, we get that

$$||\psi||_{A^p_\alpha(\mathbb{D})} \le C||1||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} < \infty,$$

for a constant C. Thus, $\psi \in A^p_\alpha(\mathbb{D})$.
(c)$\Rightarrow$ (a). Suppose that $\psi \in A^p_\alpha(\mathbb{D})$. Then by Lemma 2.3 , for $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$, we have

$$||M_\psi C_\varphi f||_{A^p_\alpha(\mathbb{D})} \le ||f||_\infty \int_{\mathbb{D}} |\psi(z)|^p(1-|z|^2)^\alpha dA(z) \le C||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}, \tag{2.3}$$

for a constant C. On the other hand, for any $f \in w\mathcal{B}_{p,\alpha}(X)$ and $x^* \in X^*$ such that $||x^*|| \leq 1$, we have that $x^*of \in \mathcal{B}_{p,\alpha}(\mathbb{D})$. So (2.3) gives us

$$||M_\psi C_\varphi(x^*of)||_{A^p_\alpha(\mathbb{D})} \leq C||x^*of||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq C||f||_{w\mathcal{B}_{p,\alpha}(X)},$$

for constant C. Hence,

$$||M_\psi C_\varphi f||_{wA^p_\alpha(X)} = \sup_{||x^*||\leq 1} ||x^*o(M_\psi C_\varphi(f)||_{A^p_\alpha(\mathbb{D})} = \sup_{||x^*||\leq 1} ||M_\psi C_\varphi(x^*of)||_{A^p_\alpha(\mathbb{D})} \leq C||f||_{w\mathcal{B}_{p,\alpha}(X)},$$

for constant C. This completes the proof. □

Theorem 2.5 *Let $1 \leq p < \infty$ and φ be an analytic self map on $\mathbb{D}$. Then the following statements are equivalent:*
a) Operator $M_\psi DC_\varphi : w\mathcal{B}_{p,\alpha}(X) \to wA^p_\alpha(X)$ is bounded.
b) Operator $M_\psi DC_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is bounded.
c) $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_\alpha(\mathbb{D})$.

Proof (a) ⇒ (b). Suppose that $M_\psi DC_\varphi : w\mathcal{B}_{p,\alpha}(X) \to wA^p_\alpha(X)$ is bounded. Then for any $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ and $x \in X$ with $||x|| = 1$, we consider the function $g : \mathbb{D} \to X$ such that $g(z) = xf(z)$ for $z \in \mathbb{D}$. Then similar to the proof of Theorem 2.4, we have that $g \in w\mathcal{B}_{p,\alpha}(X)$ and $||g||_{w\mathcal{B}_{p,\alpha}(X)} = ||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}$. So from the boundedness of $M_\psi DC_\varphi : w\mathcal{B}_{p,\alpha}(X) \to wA^p_\alpha(X)$,

$$||M_\psi DC_\varphi g||_{wA^p_\alpha(X)} \leq C||g||_{w\mathcal{B}_{p,\alpha}(X)} = C||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}, \tag{2.4}$$

for some constant C. Also,

$$\begin{aligned}||M_\psi DC_\varphi g||^p_{wA^p_\alpha(X)} &= \sup_{||x^*||\leq 1} ||x^*o\psi(go\varphi)'||^p_{A^p_\alpha(\mathbb{D})} = \sup_{||x^*||\leq 1} \int |\psi(z)|^p|x^*(xfo\varphi)'|^p(1-|z|^2)^\alpha dA(z) \\ &= \sup_{||x^*||\leq 1} \int |\psi(z)|^p|x^*(x)(fo\varphi)'|^p(1-|z|^2)^\alpha dA(z) = ||M_\psi DC_\varphi f||^p_{A^p_\alpha(\mathbb{D})}.\end{aligned} \tag{2.5}$$

Therefore, inequalities (2.4) and (2.5) imply the boundedness of $M_\psi DC_\varphi$ form $\mathcal{B}_{p,\alpha}(\mathbb{D})$ to $A^p_\alpha(\mathbb{D})$.
(b)⇒ (c). Suppose that $M_\psi DC_\varphi$ is bounded on $\mathcal{B}_{p,\alpha}(\mathbb{D})$. Then for $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ with the property $f(0) = 0$, there exists a constant C such that $||\psi(fo\varphi)'||_{A^p_\alpha(\mathbb{D})} \leq C||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}$. So

$$\int_{\mathbb{D}} |\psi|^p|f'(\varphi(z)|^p|\varphi'(z)|^p(1-|z|^2)^\alpha dA(z) \leq C(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^\alpha dA(z)). \tag{2.6}$$

By the usual change of variable formula, if $w = \varphi(z)$ then $dA(w) = |\varphi'(z)|^2 dA(z)$, then

$$\int_{\mathbb{D}} |\psi|^p|f'(\varphi(z))||^p|\varphi'(z)|^p(1-|z|^2)^\alpha dA(z) = \int_{\mathbb{D}} |f'(w)|^p N_{p,\alpha,\psi}(\varphi,w)dA(w) = \int_{\mathbb{D}} |f'(w)|^p d\mu_{p,\alpha,\psi}(w). \tag{2.7}$$

Let $g \in A^p_\alpha(\mathbb{D})$ and define $f(z) = \int_0^z g(t)dt$. Then $f'(z) \in A^p_\alpha(\mathbb{D})$, and $f(0) = 0$. By using (2.6) and (2.7), we get

$$\int_{\mathbb{D}} |g(z)|^p d\mu_{p,\alpha,\psi}(z) \leq C\int_{\mathbb{D}} |g(z)|^p(1-|z|^2)^\alpha dA(z),$$

and by Remark 1, $\mu_{p,\alpha,\psi}$ is a Carleson measure.
(c)$\Rightarrow$ (a). Suppose that $\mu_{p,\alpha,\psi}$ is a Carleson measure. Then for $g \in A^p_\alpha(\mathbb{D})$, we have

$$\int_{\mathbb{D}} |g(z)|^p d\mu_{p,\alpha,\psi}(z) \leq C \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^\alpha dA(z),$$

for a Constant C. However, for $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$, we have $f' \in A^p_\alpha(\mathbb{D})$ and by using (2.7), we get

$$||\psi(fo\varphi)'||^p_{A^p_\alpha(\mathbb{D})} = \int_{\mathbb{D}} |\psi|^p |(fo\varphi)'(z)|^p (1-|z|^2)^p dA(z) = \int_{\mathbb{D}} |f'(w)|^p d\mu_{p,\alpha,\psi}(w)$$

$$\leq C \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^\alpha dA(w) \leq C||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}, \tag{2.8}$$

for some constant C. However, for any $f \in w\mathcal{B}_{p,\alpha}(X)$ and $x^* \in X^*$ such that $||x^*|| \leq 1$, we have that $x^*of \in \mathcal{B}_{p,\alpha}(\mathbb{D})$. So (2.8) gives us

$$||M_\psi DC_\varphi(x^*of)||_{A^p_\alpha(\mathbb{D})} \leq C||x^*of||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq C||f||_{w\mathcal{B}_{p,\alpha}(X)},$$

for constant C. Hence,

$$||M_\psi DC_\varphi f||_{wA^p_\alpha(X)} = \sup_{||x^*||\leq 1} ||x^*o(M_\psi DC_\varphi(f)||_{A^p_\alpha(\mathbb{D})} = \sup_{||x^*||\leq 1} ||M_\psi DC_\varphi(x^*of)||_{A^p_\alpha(\mathbb{D})} \leq C||f||_{w\mathcal{B}_{p,\alpha}(X)},$$

for constant C. This completes the proof. □

Theorem 2.6 *If $\alpha+2<p$ and φ be an analytic self-map on $\mathbb{D}$. Then the following statements are equivalent:*
a) Operator $M_\psi C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to w\mathcal{B}_{p,\alpha}(X)$ is bounded.
b) Operator $M_\psi C_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded.
c) $\psi \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ and $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_\alpha(\mathbb{D})$.
d) Operator $M_\psi C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to \mathcal{B}_{p,\alpha}(X)$ is bounded.

Proof (a) $\Rightarrow$ (b). It is similar to the proof of Theorem 2.4.
(b) $\Rightarrow$ (c). Suppose that $M_\psi C_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded. By choosing $f = 1$, we get that $||\psi||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} < \infty$. Also, for any $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$, with the boundedness of $M_\psi C_\varphi$ on $\mathcal{B}_{p,\alpha}(\mathbb{D})$, we have that $||\psi f \circ \varphi||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq C||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}$. However, we show that $\psi \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ and then $\psi' \in A^p_\alpha(\mathbb{D})$. Now Lemma 2.3, gives us

$$||\psi(fo\varphi)'||_{A^p_\alpha(\mathbb{D})} \leq ||(\psi fo\varphi)'||_{A^p_\alpha(\mathbb{D})} + ||\psi'(fo\varphi)||_{A^p_\alpha(\mathbb{D})}$$
$$\leq ||(\psi fo\varphi)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} + ||\psi'||_{A^p_\alpha(\mathbb{D})}||f||_\infty \leq C||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})},$$

for a constant C. Thus, $M_\psi DC_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is bounded and according to part (c) of Theorem 2.5, $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_\alpha(\mathbb{D})$.
(c)$\Rightarrow$ (d). For any $f \in w\mathcal{B}_{p,\alpha}(X)$ and $x^* \in X^*$, we have that $x^*of \in \mathcal{B}_{p,\alpha}(\mathbb{D})$. By the pointwise estimate of the derivative of Bergman space functions, we get

$$||f'(z)||^p_X = \sup_{||x^*||\leq 1} |x^*(f'(z))|^p = \sup_{||x^*||\leq 1} |(x^*of)'(z)|^p. \tag{2.9}$$

On the other hand, evaluation at $\varphi(0)$ is a bounded linear operator on $\mathcal{B}_{p,\alpha}(\mathbb{D})$. So the hypothesis, (2.9) and application of Theorems 2.4 and 2.5, give us

$$\begin{aligned}
||M_\psi C_\varphi f||_{\mathcal{B}_{p,\alpha}(X)} &= \int ||(\psi f o\varphi)'||_X^p(1-|z|^2)^\alpha dA(z) + ||(\psi f o\varphi)(0)||_X \\
&\leq \sup_{||x^*||\leq 1} \int |(x^* \circ \psi f o\varphi)'|^p(1-|z|^2)^\alpha dA(z) + ||(\psi f o\varphi)(0)||_X \\
&\leq \sup_{||x^*||\leq 1} \int |\psi'(z)x^* o f o\varphi(z)|^p(1-|z|^2)^\alpha dA(z) \\
&+ \sup_{||x^*||\leq 1} \int |\psi(z)(x^* o f o\varphi)'(z)|^p(1-|z|^2)^\alpha dA(z) + \sup_{||x^*||\leq 1} |\psi(0)||x^* o f(\varphi(0))| \\
&= \sup_{||x^*||\leq 1} ||M_{\psi'} C_\varphi(x^* o f)||_{A^p_\alpha(\mathbb{D})} + \sup_{||x^*||\leq 1} ||M_\psi D C_\varphi(x^* o f)||_{A^p_\alpha(\mathbb{D})} + C \sup_{||x^*||\leq 1} |x^* o f(\varphi(0))| \\
&\leq C_1 \sup_{||x^*||\leq 1} ||x^* o f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} + C_2 \sup_{||x^*||\leq 1} ||x^* o f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} + C \sup_{||x^*||\leq 1} ||x^* o f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq M||f||_{w\mathcal{B}_{p,\alpha}(X)},
\end{aligned}$$

for constants C_1, C_2, C and M.

(d) $\Rightarrow$ (a). For any $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ and $x \in X$ with $||x|| = 1$, let $g(z) = xf(z)$. Then similar to the proof of Theorem 2.4, we have that $||g||_{w\mathcal{B}_{p,\alpha}(X)} = ||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}$ and $||\psi C_\varphi g||_{\mathcal{B}_{p,\alpha}(X)} = ||\psi C_\varphi f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}$. So the boundedness of $M_\psi C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to \mathcal{B}_{p,\alpha}(X)$ gives us the boundedness of $M_\psi C_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$. However, for any $f \in w\mathcal{B}_{p,\alpha}(X)$ and $x^* \in X^*$, we have that $x^* o f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$. So,

$$\begin{aligned}
||\psi C_\varphi f||_{w\mathcal{B}_{p,\alpha}(X)} &= \sup_{||x^*||\leq 1} ||x^* \psi C_\varphi f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = \sup_{||x^*||\leq 1} ||\psi C_\varphi x^* o f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \\
&\leq C \sup_{||x^*||\leq 1} ||x^* o f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = C||f||_{w\mathcal{B}_{p,\alpha}(X)},
\end{aligned}$$

for a constant C. This completes the proof. □

The following lemma gives us a characterization for the boundedness of operator $M_\psi D C_\varphi$ on weighted Bergman space $A^p_\alpha(\mathbb{D})$.

Lemma 2.7 *Let $1 \leq p < \infty$, φ be an analytic self map on $\mathbb{D}$. Then the operator $M_\psi D C_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is bounded if and only if $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$.*

Proof Suppose that $M_\psi D C_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is bounded. Let $f \in A^p_{\alpha+p}(\mathbb{D})$ and take $g(z) = \int_0^z f(w)dw$. Then $g'(z) \in A^p_{\alpha+p}(\mathbb{D})$ and $g(0) = 0$. However, according to Notation 2.2, $g \in A^p_\alpha(\mathbb{D})$. So

$$\begin{aligned}
\int_{\mathbb{D}} |g'(w)|^p d\mu_{p,\alpha,\psi} &= \int_{\mathbb{D}} |\psi\varphi' g' o\varphi|^p(1-|z|^2)^\alpha dA(z) = ||M_\psi D C_\varphi g||^p_{A^p_\alpha(\mathbb{D})} \\
&\leq C||g||^p_{A^p_\alpha(\mathbb{D})} \leq C[|g(0)|^p + ||g'||^p_{A^p_{\alpha+p}(\mathbb{D})}] = C||g'||^p_{A^p_{\alpha+p}(\mathbb{D})}.
\end{aligned}$$

The argument above shows that for any $f \in A^P_{\alpha+p}(\mathbb{D})$,

$$\int_{\mathbb{D}} |f|^p d\mu_{p,\alpha,\psi} \leq C||f||_{A^p_{\alpha+p}(\mathbb{D})},$$

for a constant C and it follows that $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$.
For the converse, suppose that $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$. For any $f \in A^p_\alpha(\mathbb{D})$ according to Notation 2.2, $f' \in A^p_{\alpha+p}(\mathbb{D})$. So with an application of second part of Lemma 2.1, we have

$$\begin{aligned}||M_\psi DC_\varphi f||^p_{A^p_\alpha(\mathbb{D})} &= \int_{\mathbb{D}} |\psi\varphi' f' o\varphi|^p (1-|z|^2)^\alpha dA(z) = \int |f'(w)|^p N_{p,\alpha,\psi}(\varphi, w) dA(w)\\ &= \int_{\mathbb{D}} |f'(w)|^p d\mu_{p,\alpha,\psi}(w) \leq C \int_{\mathbb{D}} |f'(w)|^p (1-|z|^2)^{\alpha+p} dA(w)\\ &\leq C_1 \int_{\mathbb{D}} |f(w)|^p (1-|z|^2)^\alpha dA(w) = C_1 ||f||^p_{A^p_\alpha(\mathbb{D})},\end{aligned}$$

for constants C and C_1. This completes the proof. □

The next lemma gives us a characterization for boundedness of operator $M_\psi C_\varphi$ on $A^p_\alpha(\mathbb{D})$.

Lemma 2.8 *[6] Let φ be an analytic self map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. If $0 < p < \infty$, then the weighted composition operator $M_\psi C_\varphi$ on $A^p_\alpha(\mathbb{D})$ is bounded if and only if:*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \Big(\frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2}\Big)^{\alpha+2} |\psi(w)|^p (1-|w|^2)^\alpha dA(w) < \infty.$$

In the next theorem, we have some equivalence conditions for boundedness of operators on $M_\psi C_\varphi D$ on $w\mathcal{B}_{p,\alpha}(X)$.

Theorem 2.9 *Let $1 \leq p < \infty$, φ be an analytic self map on $\mathbb{D}$ and $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$. Then the following statements are equivalent:*
a) Operator $M_\psi C_\varphi D : w\mathcal{B}_{p,\alpha}(X) \to w\mathcal{B}_{p,\alpha}(X)$ is bounded.
b) Operator $M_\psi C_\varphi D : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded.
c) Operator $M_\psi C_\varphi : A^p_\alpha(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded.
d) Operator $M_{\psi'} C_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is bounded and $\psi \in \mathcal{B}_{p,\alpha}(\mathbb{D})$.
e) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \big(\frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2}\big)^{\alpha+2} |\psi'(w)|^p (1-|w|^2)^\alpha dA(w) < \infty$ and $\psi \in \mathcal{B}_{p,\alpha}(\mathbb{D})$.

Proof (a)⇒ (b). It is similar to the proof of part (a) to (b) of Theorem 2.4.
(b)⇒ (c). Let $g \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ with $g(0) = 0$ with the boundedness of $M_\psi C_\varphi D$ on $\mathcal{B}_{p,\alpha}(\mathbb{D})$, we have that

$$||\psi g' o\varphi||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq C||g||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = C||g'||_{A^p_\alpha(\mathbb{D})}.$$

Let $f \in A^p_\alpha(\mathbb{D})$ define $g(z) = \int_0^z f(w)dw$. Then the argument above gives us

$$||\psi C_\varphi f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq C||f||_{A^p_\alpha(\mathbb{D})}.$$

(c) ⇒ (d). If $M_\psi C_\varphi : A^p_\alpha(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded, then by choosing $f = 1$, we get that $||\psi||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} < \infty$ and so $\psi \in \mathcal{B}_{p,\alpha}(\mathbb{D})$. On the other hand, by the hypothesis, $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$, so by

Lemma 2.7, we have that $M_\psi DC_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is bounded. However, for any $f \in A^p_\alpha(\mathbb{D})$, boundedness of $M_\psi C_\varphi$ from $A^p_\alpha(\mathbb{D})$ to $\mathcal{B}_{p,\alpha}(\mathbb{D})$, gives us

$$||M_{\psi'}C_\varphi f||_{A^p_\alpha(\mathbb{D})} \leq ||M_\psi C_\varphi f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} + ||M_\psi DC_\varphi f||_{A^p_\alpha(\mathbb{D})} \leq C||f||_{A^p_\alpha(\mathbb{D})},$$

for a constant C. So $M_{\psi'}C_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is bounded.

(d)$\Rightarrow$ (e). According to Lemma 2.8, it is clear.

(e)$\Rightarrow$ (a). Suppose that (e) holds, then with Lemma 2.8, we have (d). Since we also assume that $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$, so Lemma 2.7 gives us the boundedness of $M_\psi DC_\varphi$ on $A^p_\alpha(\mathbb{D})$. However, the evaluation at $\varphi(0)$ is bounded on $A^p_\alpha(\mathbb{D})$, so for any $f \in A^p_\alpha(\mathbb{D})$ we have

$$\begin{aligned} ||M_\psi C_\varphi(f)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} &= ||(M_\psi C_\varphi(f))'||_{A^p_\alpha(\mathbb{D})} + |\psi(0)f(\varphi(0))| \\ &\leq ||M_{\psi'}C_\varphi(f)||_{A^p_\alpha} + ||M_\psi DC_\varphi(f)||_{A^p_\alpha} + |\psi(0)||f(\varphi(0))| \leq C||f||_{A^p_\alpha(\mathbb{D})}, \end{aligned}$$

for a constant C. Therefore,

$$M_\psi C_\varphi : A^p_\alpha(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D}) \tag{2.10}$$

is bounded. Now suppose that $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$, then $Df = f' \in A^p_\alpha(\mathbb{D})$. Hence, for any $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$, (2.10) gives us the boundedness of $M_\psi C_\varphi D : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$. However, for $f \in w\mathcal{B}_{p,\alpha}(X)$ and $x^* \in X^*$, we have $x^* o f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$. So

$$\begin{aligned} ||\psi C_\varphi Df||_{w\mathcal{B}_{p,\alpha}(X)} &= \sup_{||x^*||\leq 1} ||x^*\psi C_\varphi f'||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = \sup_{||x^*||\leq 1} ||\psi C_\varphi x^* o f'||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \\ &\leq C \sup_{||x^*||\leq 1} ||\psi C_\varphi D(x^* o f)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq \sup_{||x^*||\leq 1} ||x^* o f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = C||f||_{w\mathcal{B}_{p,\alpha}(X)}, \end{aligned}$$

for a constant C and this completes the proof. □

Now we can characterize boundedness of the Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi} = (M_{\psi_1}C_\varphi + M_{\psi_2}C_\varphi D)$, on $w\mathcal{B}_{p,\alpha}(X)$.

Theorem 2.10 *Let $\alpha + 2 < p < \infty$, φ be an analytic self map on $\mathbb{D}$ and μ_{p,α,ψ_1} is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$. Then the following statements are equivalent:*

a) Operator $T_{\psi_1,\psi_2,\varphi} : w\mathcal{B}_{p,\alpha}(X) \to w\mathcal{B}_{p,\alpha}(X)$ is bounded.

b) Operator $T_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded.

c) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(w)|^2}\right)^{\alpha+2} |\psi_2'(w)|^p (1-|w|^2)^\alpha dA(w) < \infty$ and $\psi_1 \in \mathcal{B}_{p,\alpha}(\mathbb{D})$, $\psi_2 \in \mathcal{B}_{p,\alpha}(\mathbb{D})$.

Proof (a) $\Rightarrow$ (b). It is similar to the proof of Theorem 2.4.

(b) $\Rightarrow$ (c). Suppose that $T_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded. Then for any $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$,

$$||T_{\psi_1,\psi_2,\varphi}(f)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = ||M_{\psi_1}C_\varphi + M_{\psi_2}C_\varphi D)(f)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq ||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}. \tag{2.11}$$

By choosing $f = 1$, we get that

$$||\psi_1||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} < \infty. \tag{2.12}$$

However, we have assumed that $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$, so $\mu_{p,\alpha,\psi}$ will be a Carleson measure on $A^p_\alpha(\mathbb{D})$ and Theorem 2.6 gives us the boundedness of $M_{\psi_1}C_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$. Now with the triangle inequality and (2.11), we obtain

$$||(M_{\psi_2}C_\varphi D)(f)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq ||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} + ||M_{\psi_1}C_\varphi(f)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq ||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} + C||f||_{\mathcal{B}_{p,\alpha}(\mathbb{D})}.$$

Therefore, operator $M_{\psi_2}C_\varphi D$ is bounded on $\mathcal{B}_{p,\alpha}(\mathbb{D})$ and part (e) of Theorem 2.9 completes the proof.

(c) $\Rightarrow$(a). Suppose that $\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\varphi(w)|^2}\right)^{\alpha+2}|\psi_2'(w)|^p(1-|w|^2)^\alpha dA(w) < \infty$ and $\psi_1 \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ and $\psi_2 \in \mathcal{B}_{p,\alpha}(\mathbb{D})$. We also assumed that $\mu_{p,\alpha,\psi}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$ so it is a Carleson measure on $A^p_\alpha(\mathbb{D})$. Therefore, with application of Theorems 2.6 and 2.9, we have the boundedness of operators $M_{\psi_1}C_\varphi$ and $M_{\psi_2}C_\varphi D$ on $w\mathcal{B}_{p,\alpha}(X)$ and this completes the proof. □

In [19], boundedness of the Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ on weighted Bergman space $A^p_\alpha(\mathbb{D})$ has been characterized as follows:

Theorem 2.11 *Let $1 \leq p < \infty$, $\psi_1,\psi_2 \in H(\mathbb{D})$, φ be an analytic self-map on $\mathbb{D}$ and $\sup_{a\in\mathbb{D}}\frac{|\psi_2(a)|}{1-|\varphi(a)|^2} < \infty$. Then $T_{\psi_1,\psi_2,\varphi}$ is bounded on $A^p_\alpha(\mathbb{D})$ if and only if $\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\varphi(w)|^2}\right)^{\alpha+2}|\psi_1(w)|^p(1-|w|^2)^\alpha dA(w) < \infty$*

By using the above theorem and Theorem 2.10, we get the following corollary.

Corollary 2.12 *Let $1 \leq p < \infty$, $\psi_1,\psi_2 \in H(\mathbb{D})$, φ be an analytic self-map on $\mathbb{D}$ and $\sup_{a\in\mathbb{D}}\frac{|\psi_2(a)|}{1-|\varphi(a)|^2} < \infty$. Then the following statements are equivalent:*
(a) $T_{\psi_1,\psi_2,\varphi}$ is bounded on $\mathcal{B}_{p,\alpha}(\mathbb{D})$.
(b) $T_{\psi_2',\psi_2,\varphi}$ is bounded on $A^p_\alpha(\mathbb{D})$ and $\psi_1',\psi_2' \in A^p_\alpha(\mathbb{D})$.

3. Boundedness and compactness of C_φ on weak vector valued α-Besov spaces

In this section, we investigate the boundedness and compactness of composition operator C_φ on weak vector valued α-Besov space $w\mathcal{B}_{p,\alpha}(X)$.

Lemma 3.1 *Let $1 < p < \infty$, φ be an analytic self map on $\mathbb{D}$. Then the bounded composition operator $C_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is compact if and only if $||f_n o\varphi||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \to 0$ for any bounded sequence f_n in $\mathcal{B}_{p,\alpha}(\mathbb{D})$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. (In a similar way, we can define the weakly-compactness of the bounded operator $C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to w\mathcal{B}_{p,\alpha}(X)$.)*

From Theorem 2.6, we deduce the necessary and sufficient conditions for the boundedness of composition operator $T_{1,0,\varphi} = C_\varphi$ on $w\mathcal{B}_{p,\alpha}(X)$.

Corollary 3.2 *If $\alpha+2 < p$ and φ is an analytic self-map on $\mathbb{D}$. Then the following statements are equivalent:*
a) Operator $C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to w\mathcal{B}^p_\alpha(X)$ is bounded.
b) Operator $C_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}^p_\alpha(\mathbb{D})$ is bounded.
c) $\mu_{p,\alpha,1}$ is a Carleson measure on $A^p_\alpha(\mathbb{D})$.
d) Operator $C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to \mathcal{B}^p_\alpha(X)$ is bounded.

Theorem 2.9 gives us conditions for the boundedness of operator $T_{0,1,\varphi} = C_\varphi D$ on $w\mathcal{B}_{p,\alpha}(X)$.

Corollary 3.3 *Let $1 \le p < \infty$, φ be an analytic self map on $\mathbb{D}$ and $\mu_{p,\alpha,1}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$. Then*
a) Operator $C_\varphi D : w\mathcal{B}_{p,\alpha}(X) \to w\mathcal{B}_{p,\alpha}(X)$ is bounded.
b) Operator $C_\varphi D : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded.
c) Operator $C_\varphi : A^p_\alpha(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded.

Theorem 2.10 gives us conditions for the boundedness of operator $T_{0,\varphi',\varphi} = DC_\varphi$ on $w\mathcal{B}_{p,\alpha}(X)$.

Corollary 3.4 *Let $\alpha + 2 < p < \infty$, φ be an analytic self map on $\mathbb{D}$ and $\mu_{p,\alpha,1}$ is a Carleson measure on $A^p_{\alpha+p}(\mathbb{D})$. Then the following statements are equivalent:*
a) Operator $DC_\varphi : w\mathcal{B}_{p,\alpha}(X) \to w\mathcal{B}_{p,\alpha}(X)$ is bounded.
b) Operator $DC_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is bounded.
d) $\varphi' \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ and $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(w)|^2}\right)^2 |\varphi''(w)|^p (1-|w|^2)^\alpha dA(w) < \infty$.

Now for the compactness of the composition operator C_φ on $\mathcal{B}_{p,\alpha}(\mathbb{D})$, we have the following theorem.

Theorem 3.5 *Suppose that $\alpha + 2 < p < \infty$, φ be an analytic self map on $\mathbb{D}$. Then the following statements are equivalent:*
(a) $C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to w\mathcal{B}_{p,\alpha}(X)$ is weakly compact
(b) $C_\varphi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \to \mathcal{B}_{p,\alpha}(\mathbb{D})$ is compact.
(c) $\mu_{p,\alpha,1}$ is a compact Carleson measure on $A^p_\alpha(\mathbb{D})$.
(d) $C_\varphi : w\mathcal{B}_{p,\alpha}(X) \to \mathcal{B}_{p,\alpha}(X)$ is compact.

Proof (a) $\Rightarrow$ (b). Suppose that C_φ is compact on $w\mathcal{B}^p_\alpha(X)$. Let f_n be a bounded sequence in $\mathcal{B}^p_\alpha(\mathbb{D})$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. If $x \in X$ with $||x|| = 1$, define $g_n(z) = xf_n(z)$. Then as we showed in the proof of previous theorems, $g_n \in w\mathcal{B}^p_\alpha(X)$ and $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. So we have

$$||C_\varphi g_n||_{w\mathcal{B}^p_\alpha(X)} = ||g_n \circ \varphi||_{w\mathcal{B}^p_\alpha(X)} \to 0.$$

However, as we show in the proof of previous theorems,

$$||g_n o\varphi||_{w\mathcal{B}^p_v(X)} = ||f_n \circ \varphi||_{\mathcal{B}^p_\alpha(\mathbb{D})}.$$

So $||C_\varphi f_n||_{\mathcal{B}^p_\alpha(\mathbb{D})} \to 0$ and C_φ is compact on $\mathcal{B}^p_\alpha(\mathbb{D})$.

(b) $\Rightarrow$ (c). Suppose that C_φ is compact on $\mathcal{B}_{p,\alpha}(\mathbb{D})$. Let g_n be a bounded sequence in $A^p_\alpha(\mathbb{D})$ with $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ and $g_n(0) = 0$. Then there exists f_n such that $f'_n(z) = g_n(z)$, for $z \in \mathbb{D}$, $f_n \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. So we have

$$||g_n||^p_{L^p(\mu_{p,\alpha,1},\mathbb{D})} = \int_{\mathbb{D}} ||g_n(w)||^p d\mu_{p,\alpha,1}(w) = \int_{\mathbb{D}} ||g_n(w)||^p N_{p,\alpha,1}(\varphi, w) dA(w).$$

By the usual change of variable formula, if $w = \varphi(z)$ then $dA(w) = |\varphi'(z)|^2 dA(z)$. So

$$\begin{aligned}||g_n||^p_{L^p(\mu_{p,\alpha,1},\mathbb{D})} &= \int_{\mathbb{D}} ||g_n(\varphi(z)||^p |\varphi'(z)|^p (1-|z|^2)^\alpha dA(z) \\ &= \int_{\mathbb{D}} ||f_n'(\varphi(z))||^p |\varphi'(z)||^p (1-|z|^2)^\alpha dA(z) \\ &= \int_{\mathbb{D}} ||f_n(\varphi(z))'||^p (1-|z|^2)^\alpha dA(z) \\ &= ||(f_n o \varphi)'||^p_{A^p_\alpha(\mathbb{D})} \le C||f_n o\varphi||^p_{\mathcal{B}_{p,\alpha}(\mathbb{D})},\end{aligned}$$

for a constant C. However, φ is continuous on $\mathbb{D}$, so $||f_n o\varphi||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \to 0$ uniformly on compact subsets of $\mathbb{D}$. Hence, $||g_n||_{L^p(\mu_{p,\alpha,1},\mathbb{D})} \to 0$ and so $I_\alpha : A^p_\alpha(\mathbb{D}) \to L^p(\mu_{p,\alpha,1},\mathbb{D})$ is compact. It follows that $\mu_{p,\alpha,1}$ is a compact Carleson measure.

(c)$\Rightarrow$ (a). Assume that $\mu_{p,\alpha,1}$ is a compact Carleson measure. Suppose that (f_n) is a bounded sequence in $\mathcal{B}_{p,\alpha}(\mathbb{D})$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ and $f_n(0) = 0$. So $f_n' \in A^p_\alpha(\mathbb{D})$ and $f_n' \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since $I_\alpha : A^p_\alpha(\mathbb{D}) \to L^p(\mu_{p,\alpha,1}, X)$ is compact, it follows that $||f_n'||_{L^p(\mu_{p,\alpha,1},\mathbb{D})} \to 0$. However,

$$\begin{aligned}||C_\varphi f_n||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} &= ||(f_n o\varphi)(0)|| + ||(f_n o\varphi)'||_{A^p_\alpha(\mathbb{D})} = ||f_n(\varphi(0))|| + \left(\int_{\mathbb{D}} ||f_n'(\varphi(z))||^p |\varphi'(z)|^p (1-|z|^2)^\alpha dA(z)\right)^{1/p} \\ &= ||f_n(\varphi(0))|| + \left(\int_{\mathbb{D}} ||f_n'(w)||^p N_{p,\alpha,1}(\varphi, w) dA(w)\right)^{1/p} \\ &= ||f_n(\varphi(0))|| + \left(\int_{\mathbb{D}} ||f_n'(w)||^p d\mu_{p,\alpha,1}(w)\right)^{1/p}.\end{aligned}$$

Hence, $||f_n o\varphi||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \to 0$. Thus, C_φ is compact on $\mathcal{B}_{p,\alpha}(\mathbb{D}.)$ Let f_n be a bounded sequence in $w\mathcal{B}_{p,\alpha}(\mathbb{D})$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then for each $x^* \in X^*$, $(x^* o f_n)$ is a bounded sequence in $\mathcal{B}_{p,\alpha}(\mathbb{D})$ and $(x^* o f_n) \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since C_φ is compact on $\mathcal{B}_{p,\alpha}(\mathbb{D})$, so $||C_\varphi(x^* o f_n)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \to 0$. However, we have

$$||x^* o(f_n o\varphi)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = ||(x^* o f_n) o\varphi)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \to 0.$$

Hence,

$$||C_\varphi f_n||_{w\mathcal{B}_{p,\alpha}(X)} = \sup_{||x^*||_{X^*} \le 1} ||x^* o(f_n o\varphi)||_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \to 0,$$

and this completes the proof.
(c) $\Leftrightarrow$ (d). It is similar to the previous proof. □

Acknowledgments

The authors are thankful to the referees and Dr. Mostafa Hassanlou for helpful suggestions for the improvement of this article.

References

[1] Arregui J, Blasco O. Bergman and Bloch spaces of vector-valued functions. Mathematische Nachrichten 2003; 261: 3-22. doi:10.1002/mana.200310109

[2] Blasco O. Operators on weighted Bergman spaces and applications. Duke Mathematical Journal 1992; 66: 443-467. doi: 10.1215/S0012-7094-92-06614-2

[3] Blasco O. Vector valued analytic functions of bounded mean oscillation and geometry of Banach spaces. Illinois Journal of Mathematics 1997; 41: 532-558. doi:10.1215/ijm/1256068979

[4] Branges L. A proof of the Bieberbach conjecture. Acta Mathematica 1985; 154: 137-152. doi: 10.1007/BF02392821

[5] Cowen C, MacCluer BD. Composition Operators on Spaces of Analytic Functions. Studies in Advanced Mathematics, Boca Raton, CRC Pres, 1995.

[6] Cuckovic Z, Zhao R. Weighted composition operators between different weighed Bergman spaces and different Hardy spaces. Illinois Journal of Mathematics 2007; 51: 1174-1181. doi: 10.1215/ijm/1258138425

[7] Forelli F. The isometries of H^p. Canadian Journal of Mathematics 1964; 16: 721-728. doi.org/10.4153/CJM-1964-068-3

[8] Geng LG, Zhou ZH, Dong ZT. Isometric composition operators on weighted Dirichlet type spaces. Journal of Inequalities and Applications 2012; 23: 1029-1036. doi: 10.1186/1029-242X-2012-23

[9] Gul U. Essential spectra of composition operators on the space of bounded analytic functions. Turkish Journal of Mathematics 2008; 32: 475-480.

[10] Hassanlou M, Vaezi H, Wang M. Weighted composition operators on weak vector-valued Bergman spaces and Hardy spaces. Banach Journal of Mathematical Analysis 2015; 9(2): 35-43. doi: 10.15352/bjma/09-2-4

[11] Hedenmalm H, Korenblum B, Zhu K. Theory of Bergman Spaces. New York, Springer, 2000.

[12] Jovovic M, MacCluer BD. Composition operator on Dirichlet spaces. Acta Scientiarum Mathematicarum (Szeged) 1997; 63: 229-247.

[13] Latilla J. Weakly compact composition operators on vector-valued BMOA. Journal of Mathematical Analysis and Applications 2005; 308: 730–745. doi: 10.1016/j.jmaa.2004.12.002

[14] MacCluer BD. Compact composition operator on $H^p(B_N)$. Michigan Mathematical Journal 1985; 32: 237-248. doi: 10.1307/mmj/1029003191

[15] MacCluer BD. Composition operators on S^p. Houston Journal of Mathematics 1987; 13: 245-254.

[16] Maccluer BD, Shapiro JH. Angular derivatives and compact composition operators on the Hardy and Bergman spaces. Canadian Journal of Mathematics 1986; 38: 878–906. doi: 10.4153/CJM-1986-043-4

[17] Nasresfahani S, Vaezi H. Weighted composition operators acting on Dirichlet type spaces. Journal of Mathematical Extention 2020; 17(2): 121-133.

[18] Shapiro, JH. Composition Operators and Classical Function Theory. Springer, New York, 1993.

[19] Stević S, Sharma AK, Bhat A. Product of multiplication composition and differentiation operators on weighted Bergman spaces. Applied Mathematics and Computation 2011; 217: 8115-8125. doi: 10.1016/j.amc.2011.03.014

[20] Wang M. Weighted composition operators between Dirichlet spaces. Acta Mathematica Scientia 2011; 2: 641–651. doi: 10.1016/S0252-9602(11)60264-6

[21] Wolf E. Weighted composition operators between weighted Bergman spaces. RACSAM - Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas 2009; 103: 11-15. doi: 10.1007/BF03191830

[22] Zorboska N. Composition operators on weighted Dirichlet spaces. Proceedings of the American Mathematical Society 1998; 126(7): 2013-2023. doi: S0002-9939(98)04266-X