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A parametric family of ternary purely exponential Diophantine equation $A^x + B^y = C^z$

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Abstract: Let a, b, c be fixed positive integers such that $a + b = c^2$, $2 \nmid c$ and $(b/p) \neq 1$ for every prime divisor p of c, where (b/p) is the Legendre symbol. Further let m be a positive integer with $m > 1$. In this paper, using the Baker method, we prove that if $m > \max\{10^8, c^2\}$, then the equation $(am^2+1)^x + (bm^2-1)^y = (cm)^z$ has only one positive integer solution $(x, y, z) = (1, 1, 2)$.

Key words: ternary purely exponential Diophantine equation, application of the Baker method

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let A, B, C be fixed coprime positive integers with $\min\{A, B, C\} > 1$. The solution of ternary exponential Diophantine equation

$$A^x + B^y = C^z, \quad x, y, z \in \mathbb{N} \tag{1.1}$$

is a research topic with a long history and rich contents in number theory (see [11]). Let a, b, c be fixed positive integers such that

$$a + b = c^2, \ 2 \nmid c, \ \left(\frac{b}{p}\right) \neq 1 \tag{1.2}$$

for every prime divisor p of c, where (b/p) is the Legendre symbol. Further let m be a positive integer with $m > 1$. In the last decade, several authors have come up with (1.1) for

$$A = am^2 + 1, \ B = bm^2 - 1, \ C = cm. \tag{1.3}$$

According to Lemma 2.3 to be proved later in this paper, the last condition $(b/p) \neq 1$ in (1.2) is a sufficient and necessary condition to ensure that the positive integers A, B, C of (1.3) are coprime for any m. Then, (1.1) can be rewritten as

$$(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z, \quad x, y, z \in \mathbb{N}. \tag{1.4}$$

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Obviously, we see from (1.2) that (1.4) always has a solution $(x, y, z) = (1, 1, 2)$. Meanwhile, according to a far-sighted conjecture on (1.1) proposed by R. Scott and R. Styer [14], we can put forward the following conjecture about (1.4).

Conjecture 1.1 (1.4) *has only one solution* $(x, y, z) = (1, 1, 2)$.

From the results known so far, the following cases of Conjecture 1.1 have been confirmed.

(i) (N. Terai [16]) $(a, b, c) = (4, 5, 3)$, $m \le 20$ or $m \not\equiv 3 \pmod 6$.

(ii) (J.-P. Wang, T.-T. Wang and W.-P. Zhang [20]) $(a, b, c) = (4, 5, 3)$, $m \not\equiv 0 \pmod 3$.

(iii) (J.-L. Su and X.-X. Li [15]) $(a, b, c) = (4, 5, 3)$, $m > 90$, $m \equiv 0 \pmod 3$.

(iv) (C. Bertók [2]) $(a, b, c) = (4, 5, 3)$, $20 < m \le 90$.

(v) (M. Alan [1]) $(a, b, c) = (18, 7, 5)$, $m \not\equiv 23, 47, 63$ or $87 \pmod{120}$.

(vi) (N. Terai [17]) $(a, b, c) = (4, 21, 5)$, m satisfies some conditions.

(vii) (N. Terai and T. Hibino [18]) $(a, b, c) = (12, 13, 5)$, $m \not\equiv 17$ or $33 \pmod{40}$.

(viii) (N. Terai and T. Hibino [19]) $(a, b, c) = (3p, (p-3)p, p)$, where p is an odd prime with $3 < p < 3784$ and $p \equiv 1 \pmod 4$, $m \not\equiv 0 \pmod 3$, $m \equiv 1 \pmod 4$.

(ix) (N.-J. Deng, D.-Y. Wu and P.-Z. Yuan [4]) $(a, b, c) = (3c, (c-3)c, c)$, $c > 3$, $cm \not\equiv 0 \pmod 3$.

(x) (E. Kizildere and G. Soydan [9]) $(a, b, c) = (5p, (p-5)p, p)$, where p is an odd prime with $p > 5$ and $p \equiv 3 \pmod 4$, $pm \equiv \pm 1 \pmod 5$.

(xi) (E. Kizildere, M.-H. Le and G. Soydan [7]) $(a, b, c) = (rc, (c-r)c, c)$, where r is a positive integer with $r < c$ and $r \equiv 0 \pmod 3$, $\min\{rcm^2 + 1, (c-r)cm^2 - 1\} > 30$.

(xii) (T. Miyazaki and N. Terai [12]) $(a, b, c) = (1, c^2 - 1, c)$, $c \equiv \pm 3 \pmod 8$, $m \equiv \pm 1 \pmod c$.

(xiii) (N.-J. Deng and P.-Z. Yuan [5]) $(a, b, c) = ((c^2+1)/2, (c^2-1)/2, c)$, $c \equiv \pm 3 \pmod 8$, $m > c^2$, $am \equiv 1 \pmod 4$ or $am \equiv 7 \pmod 8$ and $am \not\equiv 0 \pmod 3$, or $am \equiv 11 \pmod{24}$.

(xiv) (E. Kizildere, T. Miyazaki, and G. Soydan [8]) $(a, b, c) = ((c^2+1)/2, (c^2-1)/2, c)$, $c \equiv \pm 11 \pmod{24}$, $m \equiv \pm 1 \pmod c$, $m > c^2$.

(xv) (R.-Q. Fu and H. Yang [6]) $a \equiv 0 \pmod 2$, $m \equiv 0 \pmod c$, $m > 36c^3 \log c$.

(xvi) (X.-W. Pan [13]) $a \equiv 4$ or $5 \pmod 8$, $m \equiv \pm 1 \pmod c$, $m > 6c^2 \log c$.

However, in general, Conjecture 1.1 is still an unsolved problem. In this paper, using a lower bound for linear forms in two logarithms and an upper bound for 2-adic logarithms due to M. Laurent [10] and Y. Bugeaud [3] respectively, we prove the following result:

Theorem 1.2 *If* $m > \max\{10^8, c^2\}$, *then Conjecture* 1.1 *is true.*

It follows from the above result that, for any given parameters (a, b, c) with (1.2), the proof of Conjecture 1.1 only needs to discuss finitely many smaller values of m.

2. Preliminaries

Lemma 2.1 *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *be positive integers with* $\min\{\alpha_1, \alpha_2\} > 1$, *and let* $\Lambda = \beta_1 \log \alpha_1 - \beta_2 \log \alpha_2$. *Suppose that* α_1 *and* α_2 *are multiplicatively independent. If* $\Lambda \neq 0$, *then*

$$\log |\Lambda| \geq -25.2(\log \alpha_1)(\log \alpha_2) \left(\max \left\{ 10, 0.38 + \log \left(\frac{\beta_1}{\log \alpha_2} + \frac{\beta_2}{\log \alpha_1} \right) \right\} \right)^2.$$

Proof This is the special case of Corollary 2 of [10] for α_1, α_2 positive integers and $m = 10$. □

For any positive integer n, let $\operatorname{ord}_2 n$ denote the order of 2 in n.

Lemma 2.2 *Let* α_1, α_2 *be multiplicatively independent odd integers with* $\min\{|\alpha_1|, |\alpha_2|\} > 1$, *and let* β_1, β_2 *be positive integers. Further let* $\Lambda' = \alpha_1^{\beta_1} - \alpha_2^{\beta_2}$. *If* $\Lambda' \neq 0$ *and* $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod 4$, *then*

$$\operatorname{ord}_2 |\Lambda'| < 19.55 (\log |\alpha_1|)(\log |\alpha_2|) \times \left(\max \left\{ 2\log 2, 0.4 + \log(2\log 2) + \log \left(\frac{\beta_1}{\log |\alpha_2|} + \frac{\beta_2}{\log |\alpha_1|} \right) \right\} \right)^2.$$

Proof This is the special case of Theorem 2 of [3] for $p = 2$, $y_1 = y_2 = 1$, $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod 4$, $g = 1$ and $E = 2$. □

Here and below, we assume that (a, b, c) and (A, B, C) satisfy (1.2) and (1.3), respectively, and (x, y, z) is a solution of (1.4) with $(x, y, z) \neq (1, 1, 2)$.

Lemma 2.3 *For any positive integer* m *and any fixed parameters* a, b, c *with* (1.2), A, B, C *are always coprime if and only if* $(b/p) \neq 1$ *for every prime divisor* p *of* c.

Proof Let $d = \gcd(A, B)$. Since

$$A + B = (am^2 + 1) + (bm^2 - 1) = (cm)^2 = C^2, \tag{2.1}$$

A, B, C are coprime if and only if $d = 1$.

If $(b/p) = 1$, then there exists a positive integer m such that $bm^2 - 1 \equiv 0 \pmod p$. Since $p \mid c$, by (2.1), we have $am^2 + 1 \equiv 0 \pmod p$. It implies that $d \geq p > 1$.

Conversely, if there is a positive integer m such that $\gcd(am^2 + 1, bm^2 - 1) = d > 1$, then d has a prime divisor p. Since

$$am^2 \equiv -1 \pmod p, \quad bm^2 \equiv 1 \pmod p, \tag{2.2}$$

by (2.1), we have

$$p \mid cm. \tag{2.3}$$

Further, since $\gcd(am^2 + 1, m) = 1$, by (2.2), we get $p \nmid m$. Hence, by (2.3), we have

$$p \mid c. \tag{2.4}$$

Therefore, by (2.4) and the second congruence of (2.2), we get $(b/p) = 1$ for a prime divisor of c. Thus, the lemma is proved. □

Lemma 2.4 *If $m > c^2$, then we have*

(i) $2 \nmid y$.

(ii) $z \geq 4$.

(iii) $z > m$.

Proof Since $(am^2+1)^x + (bm^2-1)^y \geq (am^2+1) + (bm^2-1) = (cm)^2$, we have $z \geq 2$. Hence, by (1.4), we get $0 \equiv (cm)^z \equiv (am^2+1)^x + (bm^2-1)^y \equiv 1 + (-1)^y \pmod{m^2}$. Therefore, since $m^2 > 2$, we obtain $2 \nmid y$ and (i) is proved.

We now assume that $z = 3$. If $y > 1$, since $2 \nmid y$, then we have $y \geq 3$. Hence, since $m > c^2$, we get $m^{9/2} > (cm)^3 \geq (am^2+1) + (bm^2-1)^3 > (bm^2-1)^3 \geq (m^2-1)^3$, whence we obtain $m^{3/2} > m^2 - 1$. But, since $m > c^2 \geq 9$, it is impossible. So we have $y = 1$. Similarly, we can prove that if $y = 1$, then $x < 3$. Finally, if $y = 1$ and $x = 2$, then from (1.4) we have

$$a(am^2+1) = c^2(cm-1). \tag{2.5}$$

Since $\gcd(am^2+1, bm^2-1) = 1$, we see from (2.1) that $\gcd(am^2+1, c^2) = 1$. Hence, by (2.5), we get $c^2 \mid a$ and $a \geq c^2 = a + b > a$, a contradiction. Therefore, we obtain $z \neq 3$, $z \geq 4$, and (ii) is proved.

Since $2 \nmid y$ and $z \geq 4$, by (1.4), we have $0 \equiv (cm)^z \equiv (am^2+1)^x + (bm^2-1)^y \equiv (am^2x+1) + (bm^2y-1) \equiv (ax+by)m^2 \pmod{m^4}$, whence we get

$$ax + by \equiv 0 \pmod{m^2}. \tag{2.6}$$

Since $ax + by$ is a positive integer, by (2.6), we have

$$ax + by \geq m^2. \tag{2.7}$$

By (1.4), we have $(cm)^z > \max\{(am^2+1)^x, (bm^2-1)^y\}$, which together with $c^2 < m$ yields

$$x < z\frac{\log(cm)}{\log(am^2+1)} < z, \quad y < z\frac{\log(cm)}{\log(bm^2-1)} < z. \tag{2.8}$$

Hence, by (2.8), we get

$$ax + by < (a+b)z = c^2 z < mz. \tag{2.9}$$

Therefore, by (2.7) and (2.9), we get $z > m$ and (iii) is proved. Thus, the proof of this lemma is complete. □

3. Proof of Theorem 1.2

We now assume that $m > \max\{10^8, c^2\}$ and (x, y, z) is a solution of (1.4) with $(x, y, z) \neq (1, 1, 2)$. Obviously, the theorem holds if it can be proved that the solution does not exist.

We first discuss the case $\min\{(am^2+1)^{2x}, (bm^2-1)^{2y}\} < (cm)^z$. If $(am^2+1)^{2x} < (cm)^z$, then from (1.4) we get $(bm^2-1)^y = (cm)^z - (am^2+1)^x > (am^2+1)^x\left((am^2+1)^x - 1\right) > (am^2+1)^x$. So we have $2(bm^2-1)^y > (cm)^z$. Taking the logarithms of both sides of (1.4), we have

$$z\log(cm) = y\log(bm^2-1) + \log\left(1 + \frac{(am^2+1)^x}{(bm^2-1)^y}\right). \tag{3.1}$$

It is well known that $\log(1+\alpha) < \alpha$ for any $\alpha > 0$. By (3.1), we get

$$0 < z\log(cm) - y\log(bm^2-1) < \frac{(am^2+1)^x}{(bm^2-1)^y} < \frac{2(am^2+1)^x}{(cm)^z} < \frac{2}{(cm)^{z/2}}. \tag{3.2}$$

Taking $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (cm, bm^2-1, z, y)$ and $\Lambda = \beta_1 \log\alpha_1 - \beta_2 \log\alpha_2$, by (3.2), we have

$$0 < \Lambda < \frac{2}{(cm)^{z/2}},$$

whence we get

$$\log 2 > \log\Lambda + \frac{z}{2}\log(cm). \tag{3.3}$$

Since $\Lambda > 0$, applying Lemma 2.1, we have

$$\log\Lambda \geq -25.20\,(\log(cm))\,\big(\log(bm^2-1)\big)\,H^2, \tag{3.4}$$

where

$$H = \max\left\{10, 0.38 + \log\left(\frac{z}{\log(bm^2-1)} + \frac{y}{\log(cm)}\right)\right\}. \tag{3.5}$$

The combination of (3.3) and (3.4) yields

$$2\log 2 + 50.40\,(\log(cm))\,\big(\log(bm^2-1)\big)\,H^2 > z\log(cm). \tag{3.6}$$

Since $c \geq 3$ and $m > 10^8$, we have

$$cm > 3 \times 10^8. \tag{3.7}$$

Hence, by (3.6) and (3.7), we get

$$0.01 + 50.40H^2 > \frac{z}{\log(bm^2-1)}. \tag{3.8}$$

When $10 \geq 0.38 + \log\big(z/(\log(bm^2-1)) + y/(\log(cm))\big)$, by (3.5) and (3.8), we have $H = 10$ and

$$\frac{z}{\log(bm^2-1)} < 5040.01. \tag{3.9}$$

Further, since

$$\max\{am^2+1, bm^2-1\} < (cm)^2 < m^3, \tag{3.10}$$

by (3.9) and (3.10), we get

$$z < 5040.01\log(bm^2-1) < 15120.03\log m. \tag{3.11}$$

On the other hand, by (iii) of Lemma 2.4, we have

$$z > m. \tag{3.12}$$

Hence, the combination of (3.11) and (3.12) yields $m < 15120.03 \log m$, whence we calculate that $m < 2 \times 10^5$, a contradiction.

When $10 < 0.38 + \log\left(z/(\log(bm^2-1)) + y/(\log(cm))\right)$, since $z/(\log(bm^2-1)) > y/(\log(cm))$ by (3.2), we have

$$\frac{z}{\log(bm^2-1)} > \frac{1}{2}\left(\frac{z}{\log(bm^2-1)} + \frac{y}{\log(cm)}\right) > \frac{1}{2}e^{10-0.38} > 7531. \tag{3.13}$$

However, by (3.5) and (3.8), we get

$$0.01 + 50.40\left(0.38 + \log 2 + \log\left(\frac{z}{\log(bm^2-1)}\right)\right)^2 > \frac{z}{\log(bm^2-1)},$$

whence we calculate that $z/(\log(bm^2-1)) < 4600$, which contradicts (3.13). Therefore, if $m > \max\{10^8, c^2\}$, then (1.4) has no solutions (x, y, z) with $(x, y, z) \neq (1, 1, 2)$ and $(am^2+1)^{2x} < (cm)^z$.

Using the same method, we can obtain a similar result for $(bm^2-1)^{2y} < (cm)^z$. Thus, we may assume that

$$\min\left\{(am^2+1)^{2x}, (bm^2-1)^{2y}\right\} > (cm)^z. \tag{3.14}$$

Next, we discuss the case $2 \mid m$. Take $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (am^2+1, -(bm^2-1), x, y)$ and $\Lambda' = \alpha_1^{\beta_1} - \alpha_2^{\beta_2}$. By (i) of Lemma 2.4, we have $2 \nmid y$. Hence, by (1.4), we get $\Lambda' = (am^2+1)^x + (bm^2-1)^y = (cm)^z$ and

$$\mathrm{ord}_2\, \Lambda' \geq z. \tag{3.15}$$

Since $\alpha_1 \equiv am^2+1 \equiv 1 \pmod 4$ and $\alpha_2 \equiv -(bm^2-1) \equiv 1 \pmod 4$, by Lemma 2.2, we have

$$\mathrm{ord}_2\, \Lambda' < 19.55\left(\log(am^2+1)\right)\left(\log(bm^2-1)\right)(H')^2, \tag{3.16}$$

where

$$H' = \max\left\{12\log 2, 0.4 + \log(2\log 2) + \log\left(\frac{x}{\log(bm^2-1)} + \frac{y}{\log(am^2+1)}\right)\right\}. \tag{3.17}$$

The combination of (3.15) and (3.16) yields

$$z < 19.55\left(\log(am^2+1)\right)\left(\log(bm^2-1)\right)(H')^2. \tag{3.18}$$

When $12\log 2 \geq 0.4 + \log(2\log 2) + \log\left(x/(\log(bm^2-1)) + y/(\log(am^2+1))\right)$, by (3.17) and (3.18), we have

$$z < 1352.58\left(\log(am^2+1)\right)\left(\log(bm^2-1)\right). \tag{3.19}$$

Further, by (3.10) and (3.19), we get

$$z < 12173.22(\log m)^2. \tag{3.20}$$

Furthermore, by (3.12) and (3.20), we obtain $m < 12173.22(\log m)^2$, whence we calculate that $m < 3 \times 10^6$, a contradiction.

When $12\log 2 < 0.4 + \log(2\log 2) + \log\left(x/(\log(bm^2-1)) + y/(\log(am^2+1))\right)$, by (3.17) and (3.18), we have

$$z < 19.55\left(\log(am^2+1)\right)\left(\log(bm^2-1)\right) \times \left(0.4 + \log(2\log 2) + \log\left(\frac{x}{\log(bm^2-1)} + \frac{y}{\log(am^2+1)}\right)\right)^2. \tag{3.21}$$

From (2.8), we get

$$\max\left\{\frac{x}{\log(bm^2-1)}, \frac{y}{\log(am^2+1)}\right\} < \frac{z\log(cm)}{(\log(am^2+1))(\log(bm^2-1))}. \tag{3.22}$$

Hence, by (3.21) and (3.22), we have

$$\begin{aligned} z' &< 19.55\,(\log(cm))\,(0.4 + \log(2\log 2) + \log 2 + \log z')^2 \\ &< 19.55\,(\log(cm))\,(1.42 + \log z')^2, \end{aligned} \tag{3.23}$$

where

$$z' = \frac{z\log(cm)}{(\log(am^2+1))\,(\log(bm^2-1))}. \tag{3.24}$$

By (3.7), we have $\log\log(cm) = 2.97$ and $2\,(\log\log\log(cm))\,/\,(\log\log(cm)) < 0.74$. Therefore, we can deduce from (3.23) that

$$z' < 343\,(\log(cm))\,(\log\log(cm))^2. \tag{3.25}$$

For a detailed proof of (3.25), see Appendix at the end of this paper. Further, by (3.10), (3.24), and (3.25), we get

$$\begin{aligned} z &< 343\left(\log(am^2+1)\right)\left(\log(bm^2-1)\right)(\log\log(cm))^2, \\ &< 3087(\log m)^2\left(\log\left(\frac{3}{2}\log m\right)\right)^2. \end{aligned} \tag{3.26}$$

Furthermore, the combination of (3.12) and (3.26) yields

$$m < 3087(\log m)^2\left(\log\left(\frac{3}{2}\log m\right)\right)^2,$$

whence we calculate that $m < 7.9\times 10^6$, a contradiction. Thus, the theorem holds for $2 \mid m$.

Finally, we discuss the case $2 \nmid m$. Since $2 \nmid c$, we see from (1.2) that am^2+1 and bm^2-1 must have opposite parity. Since am^2+1 and bm^2-1 are symmetric in (1.4) and other conditions, we may therefore assume without loss of generality that $2 \mid am^2+1$. Take

$$\begin{aligned} (\alpha_1,\alpha_2,\beta_1,\beta_2) &= \begin{cases} \left(-cm, -(bm^2-1), z, y\right), \\ \left((-1)^{(cm-1)/2}cm, bm^2-1, z, y\right), \end{cases} \\ \Lambda' &= \begin{cases} \alpha_2^{\beta_2} - \alpha_1^{\beta_1}, & \text{if } cm \equiv 3 \pmod 4 \text{ and } 2 \nmid z, \\ \alpha_1^{\beta_1} - \alpha_2^{\beta_2}, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.27}$$

Since $z \geq 4$ by (ii) of Lemma 2.4, it is easy to see from (3.14) that $x > 1$. Hence, by (1.4) and (3.27), we have $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod 4$, $\Lambda' = (am^2+1)^x$ and

$$\operatorname{ord}_2 \Lambda' \geq x. \tag{3.28}$$

Applying Lemma 2.2 to (3.27), we get

$$\operatorname{ord}_2 \Lambda' < 19.55\,(\log(cm))\,\big(\log(bm^2-1)\big)\,(H')^2, \tag{3.29}$$

where

$$H' = \max\left\{12\log 2, 0.4 + \log(2\log 2) + \log\left(\frac{z}{\log(bm^2-1)} + \frac{y}{\log(cm)}\right)\right\}. \tag{3.30}$$

The combination of (3.28) and (3.29) yields

$$x < 19.55\,(\log(cm))\,\big(\log(bm^2-1)\big)\,(H')^2. \tag{3.31}$$

Further, by (3.14), we have

$$x > \frac{z\log(cm)}{2\log(am^2+1)}. \tag{3.32}$$

Hence, by (3.31) and (3.32), we get

$$z < 39.1\,\big(\log(am^2+1)\big)\,\big(\log(bm^2-1)\big)\,(H')^2. \tag{3.33}$$

When $12\log 2 \geq 0.4 + \log(2\log 2) + \log\big(z/(\log(bm^2-1)) + y/(\log(cm))\big)$, by (3.30) and (3.33), we have

$$z < 2705.15\,\big(\log(am^2+1)\big)\,\big(\log(bm^2-1)\big)\,. \tag{3.34}$$

Therefore, by (3.10), (3.12), and (3.34), we get $m < 24346.35(\log m)^2$, whence we calculate that $m < 6 \times 10^6$, a contradiction.

When $12\log 2 < 0.4 + \log(2\log 2) + \log\big(z/(\log(bm^2-1)) + y/(\log(cm))\big)$, by (2.8), (3.30), and (3.33), we have

$$\begin{aligned} z &< 39.1\,\big(\log(am^2+1)\big)\,\big(\log(bm^2-1)\big) \\ &\quad \times \left(0.4 + \log(2\log 2) + \log\left(\frac{z}{\log(bm^2-1)} + \frac{y}{\log(cm)}\right)\right)^2 \\ &< 39.1\,\big(\log(am^2+1)\big)\,\big(\log(bm^2-1)\big)\left(1.42 + \log\left(\frac{z}{\log(bm^2-1)}\right)\right)^2, \end{aligned}$$

whence we get

$$\frac{z}{\log(bm^2-1)} < 39.1\,\big(\log(am^2+1)\big)\left(1.42 + \log\left(\frac{z}{\log(bm^2-1)}\right)\right)^2. \tag{3.35}$$

Further, since $m > 10^8$, we have $am^2+1 > 10^{16}$, $\log\log(am^2+1) > 3.60$, and $2\,\big(\log\log\log(am^2+1)\big)/(\log\log(am^2+1)) < 0.72$. Hence, using the same method as in the proof of (3.25), we can deduce from (3.35) that

$$\frac{z}{\log(bm^2-1)} < 591\,\big(\log(am^2+1)\big)\,\big(\log\log(am^2+1)\big)^2. \tag{3.36}$$

Furthermore, by (3.10), (3.12), and (3.36), we have $m < 5319(\log m)^2\,\big(\log(3\log m)\big)^2$, whence we calculate that $m < 2.4 \times 10^7$, a contradiction. Thus, the theorem holds for $2 \nmid m$. To sum up, the proof is complete.

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S1. Appendix : Detailed proof of (3.25)

Let t be a real variable with $t > 1$, and let

$$f(t) = t - 19.55\,(\log(cm))\,(1.42 + \log t)^2\,. \tag{S1.1}$$

We see from (3.23) that

$$f(z') < 0. \tag{S1.2}$$

Further let

$$t_0 = 343\,(\log(cm))\,(\log\log(cm))^2\,. \tag{S1.3}$$

If $f'(t_0) \leq 0$, then from (S1.1) and (S1.3) we have

$$\begin{aligned}&323\,(\log(cm))\,(\log\log(cm))^2\\ &\quad < 19.55\,(\log(cm))\,(1.42 + \log 343 + \log\log(cm) + 2\,(\log\log\log(cm)))^2\,,\end{aligned}$$

whence we get

$$343 < 19.55\left(\frac{1.42 + \log 343}{\log\log(cm)} + 1 + \frac{2\,(\log\log\log(cm))}{\log\log(cm)}\right)^2. \tag{S1.4}$$

By (3.7), we have $\log\log(cm) > 2.97$ and $2\,(\log\log\log(cm))\,/\,(\log\log(cm)) < 0.74$. Hence, by (S1.4), we get $343 < 19.55(2.444 + 1 + 0.74)^2 < 343$, a contradiction. So we have

$$f(t_0) > 0. \tag{S1.5}$$

By (S1.1), we have

$$f'(t) = 1 - 39.1\,(\log(cm))\left(\frac{1.42 + \log t}{t}\right) \tag{S1.6}$$

and

$$f''(t) = 39.1\,(\log(cm))\left(\frac{(1.42 + \log t) - 1}{t^2}\right), \tag{S1.7}$$

where $f'(t)$ and $f''(t)$ are derivative and divalent derivative of $f(t)$, respectively. Obviously, we see from (S1.7) that $f''(t) > 0$ for $t > 1$. It implies that $f'(t)$ is an increasing function for $t > 1$. If $f'(t_0) \leq 0$, then from (S1.3) and (S1.6) we have $t_0 \leq 39.1\,(\log(cm))\,(1.42 + \log t_0)$ and

$$\begin{aligned}&343\,(\log(cm))\,(\log\log(cm))^2\\ &\quad \leq 39.1\,(\log(cm))\,(1.42 + \log 343 + \log\log(cm) + 2\log\log\log(cm))\,,\end{aligned}$$

whence we get

$$\begin{aligned}343\log\log(cm) &\leq 39.1\left(\frac{1.42 + \log 343}{\log\log(cm)} + 1 + \frac{2\log\log\log(cm)}{\log\log(cm)}\right)\\ &< 39.1(2.45 + 1 + 0.74) < 164,\end{aligned}$$

a contradiction. So we have

$$f'(t_0) > 0. \tag{S1.8}$$

Recall that $f'(t)$ is an increasing function for $t > 1$. We see from (S1.8) that $f'(t) > 0$ for $t \geq t_0$. It implies that $f(t)$ is also an increasing function for $t \geq t_0$. Therefore, by (S1.5), we have

$$f(t) > 0 \quad \text{for } t \geq t_0. \tag{S1.9}$$

Thus, we find from (S1.2) and (S1.9) that z' satisfies (3.25).