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On elastic graph spines associated to quadratic Thurston maps

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Abstract: For a quadratic Thurston map having two distinct critical points and n postcritical points, we count the number of possible dynamical portraits. We associate elastic graph spines to several hyperbolic quadratic Thurston rational functions. These functions have four postcritical points, real coefficients, and invariant real intervals. The elastic graph spines are constructed such that each has embedding energy less than one. These are supporting examples to Dylan Thurston's recent positive characterization of rational maps. Using the same characterization, we prove that with a combinatorial restriction on the branched covering and a cycle condition on the dynamical portrait, a quadratic Thurston map with finite postcritical set of order n is combinatorially equivalent to a rational map. This is a special case of the Bernstein–Levy theorem.

Key words: Quadratic Thurston map, branched self-cover, elastic graph spine, embedding energy

1. Introduction

In order to provide a setting, we paraphrase definitions and state facts from Milnor [5], Beardon [1] and folklore. The Riemann sphere S^2 is the complex plane compactified by a point at infinity. A rational function is a holomorphic function defined on the Riemann sphere which is a ratio of two polynomials. A rational function $f = p/q$ is said to be a quadratic rational function provided that $\deg(f) := \max(\deg(p), \deg(q)) = 2$ where the polynomials do not have common roots. In general, the degree of a continuous map $f : S^2 \to S^2$ is some fixed integer α such that the induced homomorphism on the homology groups $f_* : H_2(S^2) \to H_2(S^2)$ maps x to αx. Degree is a homotopy invariant.

Let X and Y be compact oriented surfaces. A continuous and surjective map $f : X \to Y$ is called a branched covering map if for each point $x \in X$ there exists $d \in \mathbb{N}$, open sets $U \subset X$ and $V \subset Y$ with $x \in U$ and $y := f(x) \in V$, and orientation-preserving homeomorphisms $\varphi : U \to \mathbb{D}$ and $\psi : V \to \mathbb{D}$ with $\varphi(x) = 0$ and $\psi(y) = 0$ such that $\left(\psi \circ f \circ \varphi^{-1}\right)(z) = z^d$ for all $z \in \mathbb{D}$. For each $x \in X$, $\deg_f(x) := d \geq 1$ is uniquely determined by f. A point $c \in X$ is a critical point if $\deg_f(c) \geq 2$. Consequently, f fails to be injective in any neighborhood of c. Let $C(f)$ denote the set of critical points of f. A critical value is the image of a critical point. Forward orbit of a critical point c is $\{f^n(c) : n \geq 1\}$. The postcritical set $P(f)$ is defined as the union of forward orbits of all critical points, that is, $P(f) := \{f^n(c) : n \geq 1, \text{ c is any critical point}\}$. A function is said to be postcritically finite provided that every forward orbit is finite. That is, every forward orbit is either

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periodic or eventually periodic.

A branched self-cover $f:(\Sigma,P)\to(\Sigma,P)$ is a (possibly disconnected) compact closed surface Σ, a finite subset $P\subset\Sigma$, and a map $f:\Sigma\to\Sigma$ that is a branched covering map with branch values contained in P, has constant degree greater than one outside of P, maps P to P, and is a bijection on components of Σ [7]. A postcritically finite branched self-cover of the Riemann sphere is called a Thurston map. For our purposes, a map of hyperbolic type is a map where each forward orbit eventually reaches a cycle which includes at least one critical point. For a quadratic Thurston map having two distinct critical points and n postcritical points, the number of possible dynamical portraits (also called ramification portraits) is found by combinatorial enumeration in Section 2.

Theorem 1.1 *The number of possible dynamical portraits of a quadratic Thurston map having two distinct critical points and n postcritical points (up to permutation of critical values) is given by*

$$\frac{1}{4}(n^3-3n+2)+((n+1) \bmod 2)\frac{n}{2}+\frac{1}{2}\sum_{i=3}^{n-1}\left\lfloor\frac{i-1}{2}\right\rfloor.$$

A Thurston map is called Euclidean provided that its degree is at least two, its local degree at each of its critical points is two, it has at most four postcritical points and none of its postcritical points is critical [2]. Kelsey and Lodge [3] list all combinatorial classes of non-Euclidean quadratic Thurston maps with fewer than five postcritical points, give possible dynamical portraits and their characterization, and find associated functions. The present paper studies combinatorial objects, in particular, elastic graph spines, associated to postcritically finite quadratic Thurston maps.

Hereafter, for rigorous definitions and their discussion, refer to the article by Dylan Thurston [7]. An elastic graph is a graph in which each edge e is assigned a positive elastic constant $\alpha(e)$, where α is interpreted as both a metric and a measure on the graph [7, 9]. We can regard $\alpha(e)$ as elastic length of the edge e [8]. A graph spine Γ for $S^2\setminus P$ is a deformation retract of $S^2\setminus P$ where P is a finite subset of S^2. So, Γ is a connected, embedded graph such that each complementary region has a single puncture. Embedding energy of a piecewise linear map $\psi:G_1\to G_2$ between elastic graphs $G_1,\ G_2$ is given by the formula

$$\operatorname{Emb}(\psi):=\operatorname*{ess\,sup}_{y\in G_2}\sum_{x\in\psi^{-1}(y)}|\psi'(x)|. \tag{1.1}$$

On a homotopy class $[\psi]$, $\operatorname{Emb}[\psi]:=\inf_{\varphi\in[\psi]}\operatorname{Emb}(\varphi)$. For a branched self-cover f on S^2 with a finite set $P\subset S^2$, take G_2 as the spine $\Gamma_0\subset S^2\setminus P$ and G_1 as $\Gamma_1=f^{-1}(\Gamma_0)$. We say that Γ_1 is the pullback of the elastic graph spine Γ_0. Due to the covering map, Γ_1 inherits an elastic structure. We call $\psi:\Gamma_1\to\Gamma_0$ a projection. The projection is well-defined and unique up to homotopy because Γ_0 is a spine for $S^2\setminus P$ [7].

Embedding energy is assigned to a projection by considering corresponding edges in the complementary region of each postcritical point of the elastic graph spine Γ_0 and the pullback Γ_1. The derivative of the projection can be interpreted as the ratio of elastic length of an edge $\alpha\in\Gamma_0$ to an edge $\beta\in\Gamma_1=f^{-1}(\Gamma_0)$ where $\psi(\beta)=\alpha$. Then, the embedding energy can be realized simply as the maximum of the ratios whenever a single edge is projected to a single edge. However, when more than one edge is projected to a single edge, we will see a sum in the place of a single term in the expression of the embedding energy. We will explicitly construct elastic graph spines with embedding energy less than one for hyperbolic quadratic Thurston maps which are rational functions with real coefficients and invariant real interval in Section 3. Existence of these spines was guaranteed by a theorem of Dylan Thurston [7] which we cite below for accessibility. Given an elastic graph spine G for $\Sigma \setminus P$, φ_G^n denotes the nth iterate of the projection map φ_G, see ([7], Section 5).

Theorem 1.2 *[7] Let $f : (\Sigma, P) \to (\Sigma, P)$ be a branched self-cover of hyperbolic type. Then the following are equivalent.*

(1) The branched self-cover f is equivalent to a rational map.

(2) There is an elastic graph spine G for $\Sigma \setminus P$ and an integer $n > 0$ so that $Emb[\varphi_G^n] < 1$.

(3) For every elastic graph spine G for $\Sigma \setminus P$ and for every sufficiently large n (depending on f and G), we have $Emb[\varphi_G^n] < 1$.

To use Thurston's theorem to characterize a branched self-cover as a rational map, and to associate an elastic graph spine with embedding energy less than one to a known rational map, we use the following procedure: determine the dynamical portrait, choose an elastic graph spine around postcritical points, calculate the pullback of the spine, write the projection and the embedding energy, change variables (edge lengths) until the embedding energy is less than one, otherwise prove using inequalities that it is never less than one, and try a higher iterate of the projection or another spine.

Using the same theorem by Dylan Thurston, we will construct an elastic graph spine with embedding energy less than one for branched self-coverings with specific conditions. This will imply by Theorem 1.2 that maps having such combinatorics are combinatorially equivalent to rational maps. This constitutes a proof of a special case of the Bernstein–Levy theorem for topological polynomials ([4], Theorem 2.1) which states that a Thurston map with a fixed critical point at ∞ of local degree d and all other critical points being periodic is equivalent to a polynomial.

Theorem 1.3 *Let f be a quadratic Thurston map having an invariant real interval with $|P(f)| = n$. Assume that the dynamical portrait of f has ∞ as a fixed critical point and the remaining critical point is periodic of period $n-1$. Assume that $P(f) \setminus \{\infty\} \subset \mathbb{R}$ and there is a proper ordering on $P(f)$. Then f is combinatorially equivalent to a rational map.*

2. Counting the number of dynamical portraits

In this section, the number of possible dynamical portraits of a quadratic Thurston map having two distinct critical points and n postcritical points will be counted. There are no other restrictions on the map and on its combinatorics. For a quadratic Thurston map, Kelsey-Lodge [3] provide dynamical portraits with 3 and 4 postcritical points. For this section only, the term edge refers to a directed edge of a dynamical portrait, which is a directed graph. A point in $P(f)$ will be called a vertex. We need to determine the general rules vertices and edges of the dynamical portrait must satisfy. The number of edges that come out of each vertex is greater than zero as f is a mapping and it is less than two as f is a function, hence, a single edge must come out of every vertex. Any vertex has at most two points in its preimage because f is of degree two, therefore there cannot be more than two edges towards a vertex. Let c be a critical point, preimage of a critical value $p = f(c)$ contains at most one point because by definition $\deg_f(c) \geq 2$ and we have the formula $\sum_{q \in f^{-1}(p)} \deg_f(q) = \deg(f) = 2$. If $c \in P(f)$, there will be an edge from c to the corresponding critical value $f(c)$. If $c \notin P(f)$, there will be no edge towards $f(c)$. Therefore, there is at most one edge towards a critical value. There are two distinct critical points, hence, two distinct critical values.

We want to count the number of dynamical portrait patterns, in order to do so we will mark critical values as v_1 and v_2, then reduce the amount we have double counted while marking. We have two possibilities for a dynamical portrait: either there are two disjoint cycles or a single cycle. These possibilities are indexed by i. A dynamical portrait is called symmetric when one cannot distinguish the dynamical portrait if marked critical values v_1 and v_2 are permuted. Note that in any given dynamical portrait there are n postcritical points. For $i = 1, 2$; N_i^* denotes the number of dynamical portraits with marked critical values and S_i denotes the number of symmetric dynamical portraits. The number N_i^* counts nonsymmetric dynamical portraits twice and symmetric dynamical portraits once. Hence by marking critical values we get duplicate portraits which we reduce with $N_i = (N_i^* + S_i)/2$. We obtain our desired result with $N = N_1 + N_2$, which is the number of possible dynamical portraits for a quadratic Thurston map having two distinct critical points and n postcritical points. In calculations, if the upper limit of a sum is below the lower limit, assume that the sum is zero.

In the case of dynamical portraits with one cycle, $i = 1$, we may count the subcases based on the number of critical points that are contained in $P(f)$. When $|P(f) \cap C(f)| = 0$, we have a cycle and two branches in the portrait. Note that branches of the portrait consist of pre-periodic points that are not periodic. Either branches merge before entering a cycle or enter the cycle from different points, and this is only possible when n is at least 4. Each case has $\frac{1}{6}(n^3 - 6n^2 + 11n - 6) = \sum_{j=2}^{n-2}(j-1)(n-j-1)$ possible portraits where j is the number of points in a single branch and $(j-1)(n-j-1)$ is the number of possible configurations. When $|P(f) \cap C(f)| = 1$, we have a cycle and a branch. The critical point that is in $P(f)$ is either in the cycle or in the branch (map is of hyperbolic type or not) each case having $(n-1)(n-2) = 2\sum_{j=2}^{n-1}(j-1)$ possible portraits where j is the number of points in the branch and $(j-1)$ is the number of possible configurations, and the multiple two accounts for the permutation of critical values. When $|P(f) \cap C(f)| = 2$, an n-cycle is formed where the only distinguishing property is the distance between critical points in the cycle, hence, there are $(n-1)$ possible portraits. Thus,

$$N_1^* = \frac{1}{3}(n^3 - 6n^2 + 11n - 6) + 2(n-1)(n-2) + (n-1) = \frac{1}{3}(n-1)(n^2 + n - 3). \tag{2.1}$$

There are symmetric dynamical portraits when the cycle is even and branches of equal length join from antipodal points of the cycle. This includes the case where the branches have zero length, that is, critical points are inside the cycle and at antipodal positions. In this case we have $((n+1) \bmod 2)\frac{n}{2}$. We also have symmetric dynamical portraits when two orbits merge before entering a cycle. In this case we have $\sum_{i=3}^{n-1}\left\lfloor \frac{i-1}{2} \right\rfloor$ many symmetric dynamical portraits since for each increment of $n \mapsto n+1 = n'$ there are $\left\lfloor \frac{n'-2}{2} \right\rfloor$ new configurations and for $n = 4$ there is only 1. So,

$$S_1 = ((n+1) \bmod 2)\frac{n}{2} + \sum_{i=3}^{n-1}\left\lfloor \frac{i-1}{2} \right\rfloor. \tag{2.2}$$

Thus, the number of dynamical portraits with a single cycle and unmarked critical values is given by the formula

$$N_1 = \frac{1}{2}\left(\frac{1}{3}\left((n-1)(n^2+n-3)\right) + ((n+1) \bmod 2)\frac{n}{2} + \sum_{i=3}^{n-1}\left\lfloor \frac{i-1}{2} \right\rfloor\right). \tag{2.3}$$

In the case of dynamical portraits with two disjoint cycles, $i = 2$, by denoting the number of points in the forward orbit of, say, $c_1 = f^{-1}(v_1)$ by j and the number of possible configurations with $j(n-j)$, we have,

$$N_2^* = \sum_{j=1}^{n-1} j(n-j) = \binom{n+1}{3}. \tag{2.4}$$

There are $n/2$ symmetric dynamical portraits if n is even and 0 if n is odd, hence, $S_2 = ((n+1) \bmod 2)\frac{n}{2}$. Thus,

$$N_2 = \frac{1}{2}\left(\binom{n+1}{3} + ((n+1) \bmod 2)\frac{n}{2}\right). \tag{2.5}$$

The total number of possible dynamical portraits is

$$N = N_1 + N_2 = \frac{1}{4}(n^3 - 3n + 2) + ((n+1) \bmod 2)\frac{n}{2} + \frac{1}{2}\sum_{i=3}^{n-1}\left\lfloor \frac{i-1}{2} \right\rfloor.$$

3. Invariant real interval method and analysis of several hyperbolic cases with $n = 4$

We will associate elastic graph spines to several hyperbolic quadratic Thurston rational functions with 4 postcritical points in order to construct examples to Theorem 1.2 in (1) $\Rightarrow$ (2) direction. Reader can refer to [3] for the list of hyperbolic, nonhyperbolic and Thurston obstructed maps with up to 4 postcritical points. Maps we have selected are of hyperbolic type in order to satisfy the assumption of Theorem 1.2. They have real coefficients, real critical and postcritical points (except the point at infinity), and have invariant real interval in order to make use of a method, we call, invariant real interval method. The purpose of the method is to calculate the pullback of an elastic graph spine with minimal computation.

A tree is a simply connected graph, for our discussion considered as embedded in S^2. A vertex x of a tree T is called a branch point (in graph theoretical sense) if $T \setminus \{x\}$ has more than two components. Suppose that $P \subset S^2$ is some finite subset. A tree T in S^2 such that $P \subset V(T)$ is called a spanning tree for P if $V(T) \setminus P$ consists only of branch points. Let $f : S^2 \to S^2$ be a Thurston map. A spanning tree T for $P(f)$ is called an invariant spanning tree for f if we have $f(T) \subset T$, and vertices of T map to vertices of T [6]. An invariant spanning tree for f is called an invariant real interval for f provided that each point in $P(f)$ is either real or infinity.

The invariant real interval method is as follows. Let f be a quadratic Thurston map having an invariant real interval with $|P(f)| = n$. Suppose that the endpoints of the invariant real interval are critical values. The invariant real interval is of the form

$$v_1,\ p_1,\ \ldots,\ p_{n-2},\ v_2 \tag{3.1}$$

where v_i denote critical values and p_i are remaining postcritical points. Suppose that the invariant real interval (3.1) is ordered on the real circle of the Riemann sphere. For the sake of the method, since each p_i has two preimages, we indicate the interior of the invariant real interval with two dashed lines as in the middle picture of Figures 1 and 2. Since critical values have unique preimages, they are placed as the endpoints of both of the lines such as to form two copies the invariant real interval, endpoints attached. Then, the preimages of points are written above them so as to obtain the same ordering in (3.1) clockwise or counter-clockwise.

Now, take an elastic graph spine Γ_0, that is horizontally symmetric with respect to the invariant real interval. We can calculate the pullback $\Gamma_1 = f^{-1}(\Gamma_0)$ as follows. An edge $\alpha \in \Gamma_0$ that intersects the invariant real interval is placed once above and once below in the two copies of the invariant real interval, between the same postcritical points that α passes through in Γ_0. Edges that do not intersect the invariant real interval are placed between respective preimages of their endpoints. The invariant real interval method works simply because f is of degree two and the postcritical set is an invariant real interval such that the endpoints are critical values. In Figure 3 direct computation is used to calculate the pullback of the spine instead of the invariant real interval method which shows that direct computation can be less complicated in some cases. To visualize the invariant real interval method consider examples in Figures 1, 2, and 4. Greek letters in the definition of a projection indicate arcs and in the embedding energy indicate lengths of arcs. If α is an edge of the spine Γ_0 and more than one edge of the pullback is projected to α, that is $\beta_i \in \psi^{-1}(\alpha)$ where $\beta_i \in \Gamma_1$, the corresponding term in the embedding energy is $\sum_i \alpha/\beta_i$ due to the definition of the embedding energy.

Example 3.1 *For $f_1(z) = 1 + \frac{1-p}{4z^2-4z}$, $p = \frac{1}{4}$, we choose an elastic graph spine Γ_0 which is the graph on the left in Figure 1. The graph in the middle, call it Γ_1', is obtained by applying the invariant real interval method to Γ_0. For $x \in P(f)$, $f^{-1}(x) \cap P(f)$ is written above x with red color in the middle picture such that the ordering of the invariant real interval is preserved clockwise. The edge α of Γ_0 intersects the edge between p and 0 once, hence, two copies of α are drawn once above and once below, between p and 0. Vertices of the spine and other edges are pulled back similarly. We may continuously deform Γ_1' without passing over marked postcritical points (red), in order to be able to write the projection map without difficulty. We will refer to the end result with $\Gamma_1 = f^{-1}(\Gamma_0)$, which is the graph on the right.*

We introduce small edge segments δ in the spine Γ_0 and ϵ in the pullback Γ_1 in order to ensure that we get an embedding energy less than one. In the projection, segment δ in the spine is mapped by β, ϵ and ϵ' in Γ_1, hence the sum in the fourth term of the embedding energy below. We write the piecewise linear projection map $\psi_1 : \Gamma_1 \to \Gamma_0$ explicitly to describe which edges of Γ_1 map to which edges of Γ_0. Subscripts R and L that stand for right and left refer to the position of an edge in Γ_1.

$$\psi_1 : \gamma_L \mapsto \alpha,\ \tilde{\gamma}_L \mapsto \gamma,\ \tilde{\gamma}_R \mapsto \tilde{\gamma} - 2\delta,\ \{\epsilon, \epsilon', \beta\} \mapsto \delta,\ \{\gamma_R - 2\epsilon, \alpha - 2\epsilon', \alpha\} \mapsto \beta.$$

Notice that the objective of introducing δ in Γ_1 is to compensate the $\tilde{\gamma}/\tilde{\gamma}$ term that would otherwise give us embedding energy at least 1. Thus, the embedding energy of the projection ψ_1 is

$$Emb\,[\psi_1] = max\left(\frac{\alpha}{\gamma}, \frac{\gamma}{\tilde{\gamma}}, \frac{\tilde{\gamma} - 2\delta}{\tilde{\gamma}}, \frac{\delta}{\beta} + \frac{\delta}{\epsilon} + \frac{\delta}{\epsilon'}, \frac{\beta}{\gamma - 2\epsilon} + \frac{\beta}{\alpha - 2\epsilon'} + \frac{\beta}{\alpha}\right). \tag{3.2}$$

When, for example, $\alpha = 90$, $\beta = 5$, $\gamma = 91$, $\tilde{\gamma} = 100$, $\delta = 1$, $\epsilon = \epsilon' = 5$, we have $Emb\,[\psi_1] < 1$.

Example 3.2 *For $f_2(z) = \frac{p}{1-z^2}$, $p = \frac{-1+\sqrt{5}}{2}$, we choose an elastic graph spine Γ_0 similar to the one in Example 3.1 which is on the left in Figure 2. The result of the invariant real interval method is again labeled with Γ_1'; the graph obtained by deforming Γ_1' is labeled with Γ_1. The projection map $\psi_2 : \Gamma_1 \to \Gamma_0$ can be written as,*

$$\psi_2 : \gamma_L \mapsto \alpha,\ \tilde{\gamma}_L \mapsto \gamma,\ \beta \mapsto \delta,\ \{\alpha_L, \gamma_R, \tilde{\gamma}_R\} \mapsto \tilde{\gamma} - 2\delta,\ \alpha_R \mapsto \beta.$$

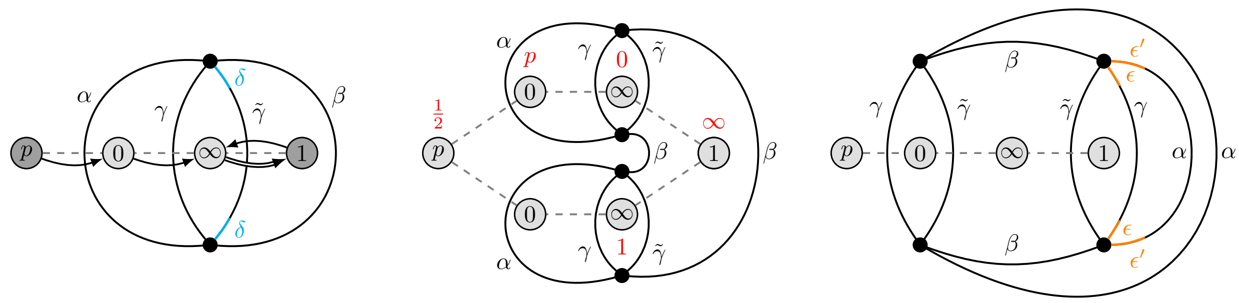

Figure 1. $f_1(z) = 1 + \frac{1-p}{4z^2-4z}$, $p = \frac{1}{4}$. We choose a spine Γ_0 and indicate its dynamics; obtain Γ_1' with invariant real interval method; deform Γ_1' to get Γ_1 and write the projection map.

Note that, similar to Example 3.1 we introduce a segment whenever we need to map an edge of Γ_1 to the same edge in Γ_0 in order to decrease the embedding energy below one. Embedding energy of ψ_2 is

$$Emb\,[\psi_2] = max\left(\frac{\alpha}{\gamma}, \frac{\gamma}{\tilde{\gamma}}, \frac{\delta}{\beta}, \frac{\tilde{\gamma}-2\delta}{\tilde{\gamma}} + \frac{\tilde{\gamma}-2\delta}{\gamma} + \frac{\tilde{\gamma}-2\delta}{\alpha}, \frac{\beta}{\alpha}\right). \tag{3.3}$$

When $\alpha = 24$, $\beta = 23$, $\gamma = 25$, $\tilde{\gamma} = 26$, $\delta = 12$, we have $Emb\,[\psi_2] < 1$.

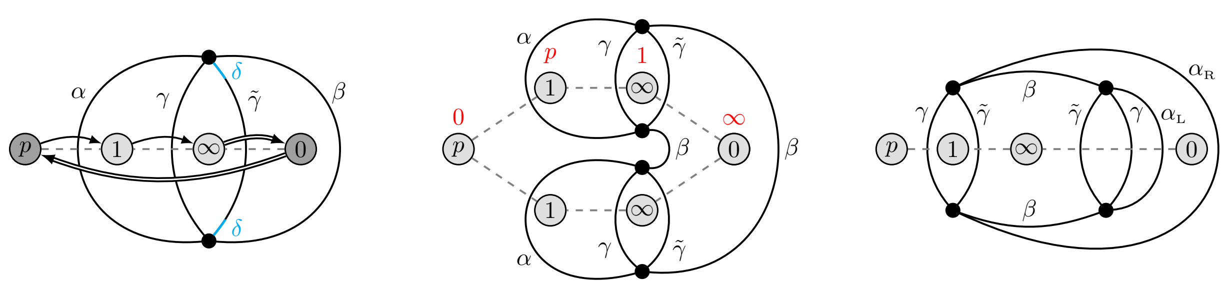

Figure 2. $f_2(z) = \frac{p}{1-z^2}$, $p = \frac{-1+\sqrt{5}}{2}$. The figure displays Γ_0, which is the elastic graph spine for f_2; Γ_1', the pullback of Γ_0; Γ_1, the deformed version of Γ_1', respectively.

Example 3.3 $f_3(z) = \frac{p}{1-z^2}$, $p = \frac{-1-\sqrt{5}}{2}$. *Observing the dynamical portrait for f_3, we can see that it is possible to apply the invariant real interval method on f_3 by considering the invariant real interval $(p,\ 0,\ 1,\ \infty)$. However, it is easier to write a projection map using direct calculation to compute Γ_1 corresponding to our choice of Γ_0. Let θ and $\tilde{\theta}$ have the same length. In the definition of the projection ψ_3, we project θ_L to the upper vertex of α in Γ_0 and project $\tilde{\theta}_L$ to the lower vertex of α. The rest of ψ_3 is given as follows.*

$$\psi_3 : \tilde{\gamma}_L \mapsto \alpha,\ \{\alpha_L, \beta_L\} \mapsto \gamma,\ \gamma - \epsilon \mapsto \theta,\ \{\epsilon, \epsilon'\} \mapsto \kappa,\ \alpha_R \mapsto \tilde{\gamma} - 2\kappa,\ \theta - \epsilon' \mapsto \delta,\ \{\beta_R, \tilde{\gamma}_R\} \mapsto \beta - 2\delta.$$

Embedding energy of ψ_3 is

$$Emb\,[\psi_3] = max\left(\frac{\alpha}{\tilde{\gamma}}, \frac{\gamma}{\alpha} + \frac{\gamma}{\beta}, \frac{\theta}{\gamma-\epsilon}, \frac{\kappa}{\epsilon} + \frac{\kappa}{\epsilon'}, \frac{\tilde{\gamma}-2\kappa}{\alpha}, \frac{\delta}{\theta-\epsilon'}, \frac{\beta-2\delta}{\beta} + \frac{\beta-2\delta}{\tilde{\gamma}}\right). \tag{3.4}$$

When $\alpha = 165,\ \beta = 13,\ \gamma = 12,\ \tilde{\gamma} = 167,\ \delta = \epsilon = \epsilon' = 3,\ \theta = 7,\ \kappa = 1.4,$ *we have* $Emb\,[\psi_3] < 1.$

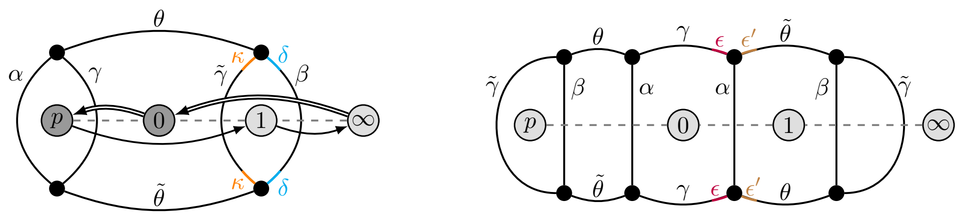

Figure 3. $f_3(z) = \frac{p}{1-z^2},\ p = \frac{-1-\sqrt{5}}{2}$. We determine an elastic graph spine Γ_0 without employing an invariant real interval and calculate its pullback graph Γ_1 with direct computation instead of using the invariant interval method.

Example 3.4 $f_4(z) = p(1-z^2),\ p \approx -1.3247.$ *In* $\Gamma_1,$ *collapse two* θ_L *and* β_R*. Then use the projection map* $\psi_4 : \Gamma_1 \to \Gamma_0,$ *defined as,*

$$\psi_4 : \alpha \mapsto \theta,\ \beta \mapsto \alpha,\ \theta \mapsto \beta/2.$$

Embedding energy for ψ_4 *is*

$$Emb\,[\psi_4] = max\left(\frac{\theta}{\alpha}, \frac{\alpha}{\beta}, \frac{\beta}{2\theta}\right). \tag{3.5}$$

When $\alpha = 5,\ \beta = 6,\ \theta = 4,$ *we have* $Emb\,[\psi_4] < 1.$ *This last example hints us to generalize the method to quadratic Thurston maps with similar dynamics, that is, the map has an invariant real interval, and the postcritical set of the map satisfies a proper ordering defined in the following section.*

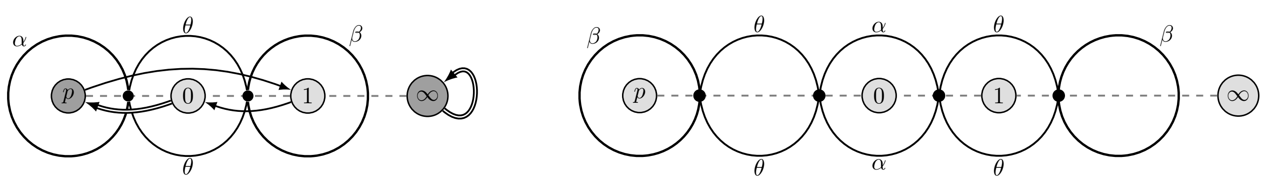

Figure 4. $f_4(z) = p(1-z^2),\ p \approx -1.3247$. We choose a spine Γ_0; calculate its pullback Γ_1 with invariant real interval method and write the projection ψ_4 by collapsing loops in Γ_1 not encircling any postcritical points.

4. Proper combinatorics on quadratic Thurston maps

Let f be a quadratic Thurston map. Let $P(f)$ denote the postcritical set of f with cardinality n. Assume that the dynamical portrait of f has ∞ as a fixed critical point and the remaining critical point is periodic of period $n-1$. Assume that $P(f) \setminus \{\infty\} \subset \mathbb{R}$. The dynamical portrait can be depicted as $a_0 \to a_1 \to \ldots \to a_{n-2} \Rightarrow a_0,\ \infty \Rightarrow \infty$. As before, the double arrow indicates that a_0 is a critical value and a_{n-2} is a critical point (the second critical value is ∞). Hence the invariant spanning tree for f is an invariant real interval with a_0 and ∞ as endpoints because a_0 is either the smallest or the largest real postcritical point. Invariant spanning trees for quadratic rational maps are further investigated in [5].

Under these assumptions, we investigate orderings of postcritical points, excluding ∞, that are realized by a branched self-cover having an invariant real interval. Define a proper ordering on $P(f)$ as follows. Consider a_i ordered on the real axis. Since a_0 is either the largest or the smallest postcritical point we have an ordering that ends or begins with a_0 (e.g., $a_1a_2a_0$ or $a_0a_2a_1$). For the sake of the argument, mirror points on one side of a_0 to the other side (e.g., obtain $a_1a_2a_0a_2a_1$). To a row underneath, write the preimage of each point; given a_i, a_{i-1} for $i \neq 0$, a_{n-2} for $i=0$ (e.g., $\left(\begin{smallmatrix} a_1 & a_2 & a_0 & a_2 & a_1 \\ a_0 & a_1 & a_2 & a_1 & a_0 \end{smallmatrix}\right)$). An ordering on $P(f)$ is called a proper ordering, if we can obtain the initial ordering in the second row (e.g., $\left(\begin{smallmatrix} a_1 & a_2 & \underline{a_0} & \underline{a_2} & \underline{a_1} \\ \underline{a_0} & a_1 & \underline{a_2} & \underline{a_1} & a_0 \end{smallmatrix}\right)$). In the invariant real interval, if $v_1 = a_0$ and $v_2 = \infty$ we have that $p_{n-2} = a_1$, and if $v_1 = \infty$ and $v_2 = a_0$ we have $p_1 = a_1$. We explicitly write proper orderings for $|P(f)| = 4, 5,$ and 6.

For a dynamical portrait consisting of 4 postcritical points, only 1 ordering out of 2 possible arrangements of postcritical points on the invariant real interval, $(a_0a_2a_1)$, is proper. For a dynamical portrait consisting of 5 postcritical points, 2 orderings out of 6 possible cases are proper. These are $(a_0a_2a_3a_1)$ and $(a_0a_3a_2a_1)$

$$\begin{pmatrix} a_1 & a_3 & a_2 & \underline{a_0} & \underline{a_2} & \underline{a_3} & \underline{a_1} \\ \underline{a_0} & \underline{a_2} & a_1 & \underline{a_3} & \underline{a_1} & a_2 & a_0 \end{pmatrix} \quad \begin{pmatrix} a_1 & a_2 & a_3 & \underline{a_0} & \underline{a_3} & \underline{a_2} & \underline{a_1} \\ \underline{a_0} & a_1 & a_2 & \underline{a_3} & \underline{a_2} & \underline{a_1} & a_0 \end{pmatrix}. \tag{4.1}$$

For $P(f) = 6$, 3 orderings out of 24 cases are proper. These are $(a_0a_3a_2a_4a_1)$, $(a_0a_3a_4a_2a_1)$, and $(a_0a_4a_3a_2a_1)$.

Theorem 1.3 *Let f be a quadratic Thurston map having an invariant real interval with $|P(f)| = n$. Assume that the dynamical portrait of f has ∞ as a fixed critical point and the remaining critical point is periodic of period $n-1$. Assume that $P(f) \setminus \{\infty\} \subset \mathbb{R}$ and there is a proper ordering on $P(f)$. Then f is combinatorially equivalent to a rational map.*

Proof Let Γ_0 be an elastic graph spine as in Figure 5 such that each postcritical point lies inside a loop and the point at infinity lies outside the spine. Label edges of the loops by α_i where a_i is the postcritical point inside the loop. We calculate the pullback Γ_1 with the invariant real interval method. Let a_0 denote the real critical value. By restricting f to the real line we can see that a_0 is either the smallest or the largest real postcritical point. Hence, we can arrange postcritical points so that the endpoints of the invariant real interval are the critical values a_0 and ∞. Now we can apply the invariant real interval method on Γ_0. Because a_0 is a critical value, the spine unfolds around a_0, and so we compute the pullback Γ_1 which is a deformation retract of $S^2 \setminus f^{-1}(P(f))$. Collapse loops in the pullback that do not encircle a postcritical point. Further deformations are not needed due to the proper ordering. A loop in Γ_1 encircling a postcritical point a_i is to be mapped by ψ to a loop of the spine Γ_0 that contains the same postcritical point. Thus, we can write the projection $\psi : \Gamma_1 \to \Gamma_0$ as $\alpha_{i+1} \mapsto \alpha_i$ for $i \in \{0, 2, 3, \cdots, n-3\}$, $\alpha_0 \mapsto \alpha_{n-2}$, $\alpha_2 \mapsto \alpha_1/2$. Hence the embedding energy is written as

$$\operatorname{Emb}[\psi] = \max\left(\frac{\alpha_2}{\alpha_3}, \cdots, \frac{\alpha_{n-3}}{\alpha_{n-2}}, \frac{\alpha_{n-2}}{\alpha_0}, \frac{\alpha_0}{\alpha_1}, \frac{\alpha_1}{2\alpha_2}\right). \tag{4.2}$$

We have a simple expression to make the embedding energy less than one:

$$\alpha_2 < \alpha_3 < \cdots < \alpha_{n-3} < \alpha_{n-2} < \alpha_0 < \alpha_1 < 2\alpha_2. \tag{4.3}$$

Positive real numbers satisfying these inequalities can easily be determined for arbitrary n. Due to Theorem 1.2 in $(2) \Rightarrow (1)$ direction, f is combinatorially equivalent to a rational map. $\square$

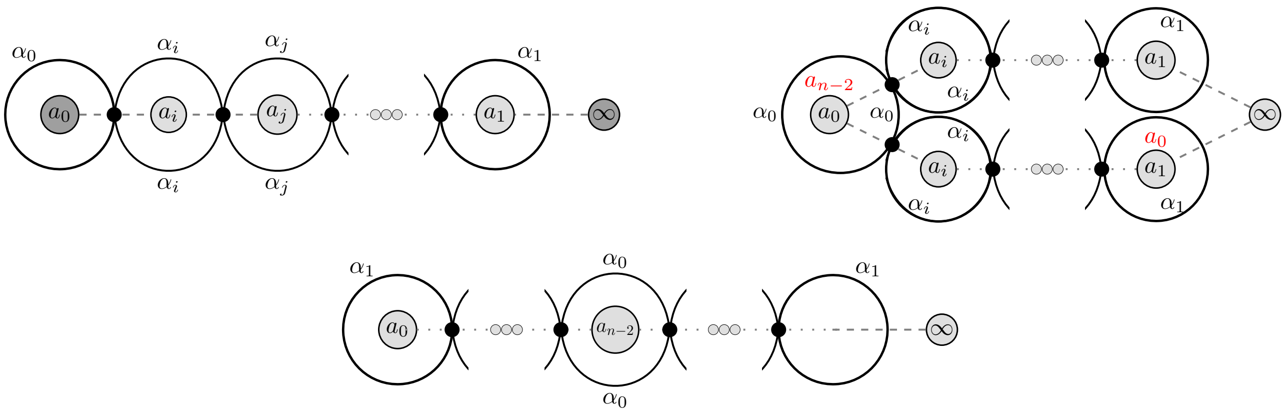

Figure 5. The spine Γ_0, its pullback and its deformed pullback Γ_1. i and j are placeholders that are determined by the proper ordering.

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