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Inverse coefficient identification problem for a hyperbolic equation with nonlocal integral condition

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Abstract: This paper is concerned with an inverse coefficient identification problem for a hyperbolic equation in a rectangular domain with a nonlocal integral condition. We introduce the definition of the classical solution, and then the considered problem is reduced to an auxiliary equivalent problem. Further, the existence and uniqueness of the solution of the equivalent problem are proved using a contraction mapping principle. Finally, using equivalency, the unique existence of a classical solution is proved.

Key words: Iinverse problems, hyperbolic equations, nonlocal integral condition, classical solution, existence, uniqueness

1. Introduction and problem formulation

As is well known, in many physical problems, determination of coefficients and right-hand side simultaneous in a differential equation from some known data is required. Such problems are called inverse problems of mathematical physics. They arise in various fields of human activity, such as acoustics, seismology, electromagnetics, fluid dynamics, remote sensing, nondestructive evaluation, and many other areas. It will be noted that these kinds of problems are ill-posed in the sense of Hadamard.

Nowadays, in the modern mathematical literature, the theory of inverse boundary-value problems for equations of hyperbolic type of the second-order is stated rather satisfactory. The inverse and ill-posed problems associated with the hyperbolic/wave equation have drawn the attention of many authors. A more detailed bibliography and a classification of inverse problems are found in monographs or books (see for example, [2], [4], [10], [14], [15], [18], [20], [21], [22], and the references therein). Note that in most of the publications devoted to problems with nonlocal integral conditions, spatially nonlocal conditions are considered. In this article, we consider a time nonlocal inverse problem for a hyperbolic equation with integral conditions.

Let us now survey the content of some works devoted to inverse coefficient problems for hyperbolic equations. A.M. Denisov [5] suggested an iterative method for solving the inverse coefficient problem for a hyperbolic equation based on a reduction to a nonlinear operator equation for the unknown coefficient and proved the uniform convergence of the iterations to a solution of the inverse problem. In the paper of G. Eskin [6] the inverse problems for the second-order hyperbolic equations of general form with time-dependent coefficients are investigated and the time-dependent Lorentzian metric by the boundary measurements is

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determined. The authors G. Hu, Y. Kian, and Y. Zhao [7] investigated the inverse acoustic source problems in an unbounded domain with dynamical boundary surface data of Dirichlet kind and they proved the uniqueness in recovering source terms. Moreover, simultaneous determination of embedded obstacles and source terms in an inhomogeneous background medium using the observation data of an infinite time period was verified. In the article published by M.I. Ismailov and I. Tekin [8], the direct and inverse initial boundary-value problems for a first-order system of two hyperbolic equations are considered. The suitability of the method of characteristics for the inverse problem of finding solely space-dependent coefficients and the finite difference method for solely time-dependent coefficients of the first order hyperbolic system are shown. The inverse problems to recover coefficients of the fifth-order Korteweg-de Vries equation, sixth-order generalized Boussinesq equation, and two sixth-order equations occurring in the dynamics of multiscale microstructure were considered in the work of J. Janno and A. Seletski [9]. The authors were proposed to use characteristics of solitary waves for solving the inverse problems and the uniqueness of the solution was proved. G. Nakamura, M. Watanabe, and B. Kaltenbacher [17] studied the inverse boundary-value problem for nonlinear wave equations with a field-dependent coefficient in one space dimension. This article shows that the linear part and the quadratic part of a field-dependent coefficient are approximately reconstructed from two measurements at the boundary. The inverse problem of recovering a solution-dependent coefficient multiplying the lowest derivative in a hyperbolic equation is investigated in the paper by A.Y. Shcheglov [23]. The theorems of global uniqueness and local existence for the solution to the inverse problem are also proved. By M. Slodicka and L. Seliga [24] was studied a non-linear wave equation with an unknown time-convolution kernel. The missing kernel from an additional integral measurement has been recovered.

Problems with time nonlocal conditions are closely related to inverse problems in which the overdetermination condition is integral form [3], [11], [19]. The conditions set in this way can be considered as a model of the action of a certain device that registers physical fields [4].

The study of the time nonlocal inverse problem with integral conditions, whose results are presented in this paper, showed that the dimension of the domain in which the solution is sought is important, and also that the solvability conditions can relate both the domain dimensions and the restrictions on other initial data.

Motivated by these works, in the present work we study an inverse boundary-value problem for a second-order hyperbolic equation with nonlocal conditions. A distinctive feature of the presented article is the investigation of an inverse hyperbolic problem with both spatial and time nonlocal conditions.

Let $T>0$ be some fixed number and let D_T be a rectangular region defined by $D_T: 0\le x\le 1,\ 0\le t\le T$. Consider the one-dimensional inverse problem of identifying an unknown pair of functions $\{u(x,t),a(t)\}$ for the following hyperbolic equation

$$u_{tt}(x,t)=u_{xx}(x,t)+a(t)u(x,t)+f(x,t),\quad (x,t)\in D_T, \tag{1.1}$$

with the nonlocal initial conditions

$$u(x,0)=\int_0^T P_1(t)u(x,t)dt+\varphi(x),\quad u_t(x,0)=\int_0^T P_2(t)u(x,t)dt+\psi(x),\quad x\in[0,1], \tag{1.2}$$

Neumann boundary condition

$$u_x(0,t)=0,\quad t\in[0,T], \tag{1.3}$$

nonlocal integral condition

$$\int_0^1 u(x,t)dx = 0, \quad t \in [0,T], \tag{1.4}$$

and overdetermination condition

$$u(0,t) = h(t), \quad t \in [0,T]. \tag{1.5}$$

where $f(x,t), \varphi(x), \psi(x), P_i(t)(i = 1,2)$, and $h(t)$ are given sufficiently smooth functions of $x \in [0,1]$ and $t \in [0,T]$.

It should be noted that the direct problem for hyperbolic equations with the conditions (1.2) has been investigated in the works [12],[13], and the references therein.

Definition 1.1 *The pair $\{u(x,t), a(t)\}$ is said to be a classical solution to the problem (1.1)–(1.5), if the functions $u(x,t) \in C^2(D_T)$ and $a(t) \in C[0,T]$ satisfyies an Equation (1.1) in the region D_T, the condition (1.2) on $[0,1]$, and the statements (1.3)–(1.5) on the interval $[0,T]$.*

In order to investigate the problem (1.1)–(1.5), first we consider the following auxiliary problem

$$y''(t) = a(t)y(t), \; t \in [0,T], \tag{1.6}$$

$$y(0) = \int_0^T P_1(t)y(t)dt, \quad y'(0) = \int_0^T P_2(t)y(t)dt, \tag{1.7}$$

where $P_1(t), P_2(t), a(t) \in C[0,T]$ are given functions, and $y = y(t)$ is desired function. Moreover, by the solution of the problem (1.6),(1.7), we mean a function $y(t)$ belonging to $C^2[0,T]$ and satisfying the conditions (1.6),(1.7) in the usual sense.

Lemma 1.2 ([16]) *Assume that $P_1(t), P_2(t), a(t) \in C[0,T]$, and the condition*

$$\left(T\,\|P_2(t)\|_{C[0,T]} + \|P_1(t)\|_{C[0,T]} + \frac{T}{2}\,\|a(t)\|_{C[0,T]}\right)T < 1$$

hold. Then the problem (1.6),(1.7) has a unique trivial solution.

Now along with the inverse boundary-value problem (1.1)–(1.5), we consider the following auxiliary inverse boundary-value problem: It is required to determine a pair $\{u(x,t), a(t)\}$ of functions $u(x,t) \in C^2(D_T)$ and $a(t) \in C[0,T]$ from relations (1.1)–(1.5), and

$$u_x(1,t) = 0, \; t \in [0,T], \tag{1.8}$$

$$h''(t) - u_{xx}(0,t) = a(t)h(t) + f(0,t), \; t \in [0,T]. \tag{1.9}$$

The following theorem is valid.

Theorem 1.3 *Suppose that* $\varphi(x), \psi(x), P_i(t) \in C[0,T]$ $(i=1,2)$, $h(t) \in C^2[0,T]$, $h(t) \neq 0$, $f(x,t) \in C(D_T)$, $\int_0^1 f(x,t)dx = 0,\ t \in [0,T]$, *and the compatibility conditions*

$$\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0, \tag{1.10}$$

$$h(0) = \int_0^T P_1(t)h(t)dt + \varphi(0), \quad h'(0) = \int_0^T P_2(t)h(t)dt + \psi(0) \tag{1.11}$$

holds. Then the following assertions are valid:

(i) *each classical solution* $\{u(x,t), a(t)\}$ *of the problem* (1.1)–(1.5) *is a solution of problem* (1.1)–(1.3), (1.8), (1.9), *as well;*

(ii) *each solution* $\{u(x,t), a(t)\}$ *of the problem* (1.1)–(1.3), (1.8), (1.9), *if*

$$\left(T\left\|P_2(t)\right\|_{C[0,T]} + \left\|P_1(t)\right\|_{C[0,T]} + \frac{T}{2}\left\|a(t)\right\|_{C[0,T]}\right)T < 1, \tag{1.12}$$

is a classical solution of problem (1.1)–(1.5).

Proof Let $\{u(x,t), a(t)\}$ be any classical solution to problem (1.1)–(1.5). By integrating both sides of Equation (1.1) with respect to x from 0 to 1, we find

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx - (u_x(1,t) - u_x(0,t)) = a(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx,\ t \in [0,T]. \tag{1.13}$$

Using the fact that $\int_0^1 f(x,t)dx = 0,\ t \in [0,T]$, and the boundary condition (1.3), we conclude that the statement (1.8) is true.

Setting $x = 0$ in Equation (1.1), we find

$$u_{tt}(0,t) - u_{xx}(0,t) = a(t)u(0,t) + f(0,t),\ t \in [0,T]. \tag{1.14}$$

Taking into consideration $h(t) \in C^2[0,T]$ and twice differentiating (1.5) yields

$$u_{tt}(0,t) = h''(t),\ t \in [0,T]. \tag{1.15}$$

From (1.14), taking into account (1.5) and (1.15), we conclude that the relation (1.9) is fulfilled.

Now suppose that $\{u(x,t), a(t)\}$ is the solution to problem (1.1)–(1.3), (1.8), (1.9). Then from (1.13), by the condition

$$\int_0^1 f(x,t)dx = 0,\ t \in [0,T],$$

and relations (1.3), (1.8) we have

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx = a(t)\int_0^1 u(x,t)dx,\ t\in[0,T]. \tag{1.16}$$

Furthermore, from (1.2) and (1.10) it is easy to see that

$$\int_0^1 u(x,0)dx - \int_0^T P_1(t)\left(\int_0^1 u(x,t)dx\right)dt = \int_0^1\left(u(x,0) - \int_0^T P_1(t)u(x,t)dt\right)dx = \int_0^1 \varphi(x)dx = 0,$$

$$\int_0^1 u_t(x,0)dx - \int_0^T P_2(t)\left(\int_0^1 u(x,t)dx\right)dt = \int_0^1\left(u_t(x,0) - \int_0^T P_2(t)u(x,t)dt\right)dx = \int_0^1 \psi(x)dx = 0. \tag{1.17}$$

Since, by Lemma 1.2, problem (1.16), (1.17) has only a trivial solution. It means that $\int_0^1 u(x,t)dx = 0,\ t\in[0,T]$, i.e. the condition (1.4) is satisfied.

Next, from (1.9) and (1.14), we obtain

$$\frac{d^2}{dt^2}(u(0,t)-h(t)) = a(t)(u(0,t)-h(t)),\quad 0\le t\le T. \tag{1.18}$$

By virtue of (1.2) and the compatibility conditions (1.11), we have

$$u(0,0)-h(0)-\int_0^T P_1(t)(u(0,t)-h(t))dt = u(0,0)-\int_0^T P_1(t)u(0,t)dt$$

$$-\left(h(0)-\int_0^T P_1(t)h(t)dt\right) = \varphi(0)-\left(h(0)-\int_0^T P_1(t)h(t)dt\right) = 0,$$

$$u_t(0,0)-h'(0)-\int_0^T P_2(t)(u(0,t)-h(t))dt$$

$$= u_t(0,0)-\int_0^T P_2(t)u(0,t)dt-\left(h'(0)-\int_0^T P_2(t)h(t)dt\right)$$

$$= \psi(0)-\left(h'(0)-\int_0^T P_2(t)h(t)dt\right) = 0. \tag{1.19}$$

Using Lemma 1.2, and relations (1.18), (1.19), we conclude that condition (1.5) is satisfied. The theorem is proved. □

2. Existence and uniqueness of the classical solution

We seek the first component of classical solution $\{u(x,t), a(t)\}$ of the problem (1.1)–(1.3), (1.8), (1.9) in the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x, \ \ \lambda_k = k\pi, \tag{2.1}$$

where

$$u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x dx, \ \ k = 0, 1, 2, \dots,$$

and

$$m_k = \begin{cases} 1, & k = 0, \\ 2, & k = 1, 2, \dots . \end{cases}$$

Then applying the formal scheme of the Fourier method, from (1.1) and (1.2) we have

$$u_k''(t) + \lambda_k^2 u_k(t) = F_k(t; u, a), \ \ k = 0, 1, 2 \dots; \ \ 0 \le t \le T, \tag{2.2}$$

$$u_k(0) = \varphi_k + \int_0^T P_1(t) u_k(t) dt, \quad u_k'(0) = \psi_k + \int_0^T P_2(t) u_k(t) dt, \ \ k = 0, 1, 2 \dots . \tag{2.3}$$

where

$$F_k(t; u, a) = f_k(t) + a(t) u_k(t), \quad f_k(t) = m_k \int_0^1 f(x,t) \cos \lambda_k x dx,$$

$$\varphi_k = m_k \int_0^1 \varphi(x) \cos \lambda_k x dx, \quad \psi_k(t) = m_k \int_0^1 \psi(x) \cos \lambda_k x dx, \ \ k = 0, 1, 2, \dots .$$

Solving the problem (2.2), (2.3) gives

$$u_0(t) = \varphi_0 + \int_0^T P_1(t) u_0(t) dt + t \left(\psi_0 + \int_0^T P_2(t) u_0(t) dt \right) + \int_0^t (t - \tau) F_0(\tau; u, a) d\tau, \tag{2.4}$$

$$u_k(t) = \left(\varphi_k + \int_0^T P_1(t) u_k(t) dt \right) \cos \lambda_k t + \frac{1}{\lambda_k} \left(\psi_k + \int_0^T P_2(t) u_k(t) dt \right) \sin \lambda_k t$$

$$+ \frac{1}{\lambda_k} \int_0^t F_k(\tau; u, a) \sin \lambda_k (t - \tau) d\tau, \ \ k = 1, 2, \dots, \ \ 0 \le t \le T. \tag{2.5}$$

Obviously,

$$u_k'(t) = -\lambda_k \left(\varphi_k + \int_0^T P_1(t) u_k(t) dt \right) \sin \lambda_k t + \left(\psi_k + \int_0^T P_2(t) u_k(t) dt \right) \cos \lambda_k t$$

$$+\int_0^t F_k(\tau;u,a)\cos\lambda_k(t-\tau)d\tau,\ \ k=1,2,...,\ \ 0\le t\le T. \tag{2.6}$$

To determine the first component of the classical solution to the problem (1.1)–(1.3), (1.8), (1.9) we substitute the expressions $u_k(t)$ $(k=0,1,...)$ into (2.1) and obtain

$$u(x,t)=\left(\varphi_0+\int_0^T P_1(t)u_0(t)dt\right)+t\left(\psi_0+\int_0^T P_2(t)u_0(t)dt\right)$$

$$+\int_0^t (t-\tau)F_0(\tau;u,a)d\tau+\sum_{k=1}^{\infty}\left\{\left(\varphi_k+\int_0^T P_1(t)u_k(t)dt\right)\cos\lambda_k t\right.$$

$$\left.+\frac{1}{\lambda_k}\left(\psi_k+\int_0^T P_2(t)u_k(t)dt\right)\sin\lambda_k t+\frac{1}{\lambda_k}\int_0^t F_k(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\right\}\cos\lambda_k x. \tag{2.7}$$

It follows from (1.9) and (2.1) that

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(0,t)+\sum_{k=1}^{\infty}\lambda_k^2 u_k(t)\right\}. \tag{2.8}$$

By substituting expression (2.5) into (2.8), we obtain the equation for the second component of the solution to problem (1.1)–(1.3), (1.8), (1.9):

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(0,t)+\sum_{k=1}^{\infty}\lambda_k^2\left[\left(\varphi_k+\int_0^T P_1(t)u_k(t)dt\right)\cos\lambda_k t\right.\right.$$

$$\left.\left.+\frac{1}{\lambda_k}\left(\psi_k+\int_0^T P_2(t)u_k(t)dt\right)\sin\lambda_k t+\frac{1}{\lambda_k}\int_0^t F_k(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\right]\right\}. \tag{2.9}$$

Thus, the solution of problem (1.1)–(1.3), (1.8), (1.9) was reduced to the solution of system (2.7), (2.9) with respect to unknown functions $u(x,t)$ and $a(t)$.

Lemma 2.1 *If $\{u(x,t),a(t)\}$ is any solution to problem (1.1)–(1.3), (1.8), (1.9), then the functions*

$$u_k(t)=m_k\int_0^1 u(x,t)\cos\lambda_k x dx,\ \ k=0,1,2,\dots,$$

satisfyies the system (2.4), (2.5) in $C[0,T]$.

It follows from Lemma 2.1 that

Corollary 2.2 *Let system (2.7), (2.9) have a unique solution. Then problem (1.1)-(1.3), (1.8), (1.9) cannot have more than one solution, i.e. if the problem (1.1)-(1.3), (1.8), (1.9) has a solution, then it is unique.*

With the purpose to study the problem (1.1)-(1.3), (1.8), (1.9), we consider the following functional spaces.

Denote by $B_{2,T}^3$ a set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x,\ \ \lambda_k=k\pi,$$

considered in the region D_T, where each of the function $u_k(t)$ $(k=0,1,2,...)$ is continuous over an interval $[0,T]$ and satisfies the following condition:

$$J(u)\equiv\|u_0(t)\|_{C[0,T]}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|u_k(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<+\infty.$$

The norm in this set is defined by

$$\|u(x,t)\|_{B_{2,T}^3}=J(u).$$

It is known that $B_{2,T}^3$ is Banach space [1].

Obviously, $E_T^3=B_{2,T}^3\times C[0,T]$ is also Banach space, where the norm of an element $z=\{u,a\}$ is determined by the formula

$$\|z(x,t)\|_{E_T^3}=\|u(x,t)\|_{B_{2,T}^3}+\|a(t)\|_{C[0,T]}\,.$$

Now consider the operator

$$\Phi(u,a)=\{\Phi_1(u,a),\ \Phi_2(u,a)\},$$

in the space E_T^3, where

$$\Phi_1(u,a)=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_k(t)\cos\lambda_k x,\quad \Phi_2(u,a)=\tilde{a}(t),$$

and the functions $\tilde{u}_0(t),\tilde{u}_k(t),k=1,2,...,$ and $\tilde{a}(t)$ are equal to the right-hand sides of (2.4), (2.5), and (2.9), respectively.

Hence we have

$$\|\tilde{u}_0(t)\|_{C[0,T]}\le|\varphi_0|+T(\|P_1(t)\|_{C[0,T]}+T\,\|P_2(t)\|_{C[0,T]})\,\|u_0(t)\|_{C[0,T]}$$

$$+T\,|\psi_0|+T\sqrt{T}\left(\int_0^T|f_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}}+T^2\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]}\,,\tag{2.10}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le\sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_k|)^2\right)^{\frac{1}{2}}+\sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k^2\,|\psi_k|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{6}(\|P_1(t)\|_{C[0,T]}+\|P_2(t)\|_{C[0,T]})T\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{6T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+\sqrt{6}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{2.11}$$

$$\|\tilde{a}(t)\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Bigg\{\|h''(t)-f(0,t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$

$$\times\Bigg[\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_k|)^2\right)^{\frac{1}{2}}+T\,\|P_1(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\left(\sum_{k=1}^{\infty}(\lambda_k^2\,|\psi_k|)^2\right)^{\frac{1}{2}}+T\,\|P_2(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}. \tag{2.12}$$

Suppose that the data for problem (1.1)–(1.3), (1.8), (1.9) satisfy the assumptions:

(A_1) $\varphi(x)\in C^2[0,1],\ \varphi'''(x)\in L_2(0,1),\ \varphi'(0)=\varphi'(1)=0;$

(A_2) $\psi(x)\in C^2[0,1],\ \psi''(x)\in L_2(0,1),\ \psi'(0)=\psi'(1)=0;$

(A_3) $f(x,t),f_x(x,t)\in C(D_T),\ f_{xx}(x,t)\in L_2(D_T),\ f_x(0,t)=f_x(1,t)=0,\ 0\le t\le T;$

(A_4) $h(t)\in C^2[0,T],\ h(t)\ne 0,\ 0\le t\le T.$

Then from (2.10)–(2.12) we correspondingly find

$$\|\tilde{u}_0(t)\|_{C[0,T]}\le\|\varphi(x)\|_{L_2(0,1)}+T\,\|\psi(x)\|_{L_2(0,1)}+T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)}$$

$$+T(\|P_1(t)\|_{C[0,T]}+T\,\|P_2(t)\|_{C[0,T]}+T\,\|a(t)\|_{C[0,T]})\,\|u_0(t)\|_{C[0,T]}\,, \tag{2.13}$$

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|\tilde{u}_k(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}\le\sqrt{6}\,\|\varphi'''(x)\|_{L_2(0,1)}+\sqrt{6}\,\|\psi''(x)\|_{L_2(0,1)}$$

$$+\sqrt{6T}\,\|f_{xx}(x,t)\|_{L_2(D_T)}+\sqrt{6}T(\|P_1(t)\|_{C[0,T]}+\|P_2(t)\|_{C[0,T]}$$

$$+\|a(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{2.14}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left\{ \|h''(t) - f(0,t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \right.$$

$$\times[\|\varphi'''(x)\|_{L_2(0,1)} + \|\psi''(x)\|_{L_2(0,1)} + \sqrt{T}\,\|f_{xx}(x,t)\|_{L_2(D_T)} + T(\|P_2(t)\|_{C[0,T]}$$

$$\left.\left. + \|P_1(t)\|_{C[0,T]} + \|a(t)\|_{C[0,T]}) \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \right] \right\}. \tag{2.15}$$

It follows from (2.13) and (2.14) that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \leq A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C_1(T)\,\|u(x,t)\|_{B^3_{2,T}}, \tag{2.16}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)}$$

$$+\sqrt{6}\,\|\varphi'''(x)\|_{L_2(0,1)} + \sqrt{6}\,\|\psi''(x)\|_{L_2(0,1)} + \sqrt{6T}\,\|f_{xx}(x,t)\|_{L_2(D_T)},$$

$$B_1(T) = T^2 + \sqrt{6}T,$$

$$C_1(T) = T(1+\sqrt{6})\,\|P_1(t)\|_{C[0,T]} + T(T+\sqrt{6})\,\|P_2(t)\|_{C[0,T]}.$$

Further from (2.15), we may also write

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C_2(T)\,\|u(x,t)\|_{B^3_{2,T}}, \tag{2.17}$$

where

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \{\|h''(t) - f(0,t))\|_{C[0,T]}$$

$$\left. + \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left[\|\varphi'''(x)\|_{L_2(0,1)} + \|\psi''(x)\|_{L_2(0,1)} + \sqrt{T}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right] \right\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^{-2})\right)^{\frac{1}{2}} T,$$

$$C_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^{-2})\right)^{\frac{1}{2}} T(\|P_1(t)\|_{C[0,T]} + \|P_2(t)\|_{C[0,T]}).$$

From the inequalities (2.16) and (2.17), we conclude that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C(T)\,\|u(x,t)\|_{B^3_{2,T}}, \tag{2.18}$$

where

$$A(T) = A_1(T) + A_2(T),\;\; B(T) = B_1(T) + B_2(T),\;\; C(T) = C_1(T) + C_2(T).$$

Thus, we can prove the following theorem

Theorem 2.3 *Let* $R = A(T) + 2$. *If the statements* $(A_1) - (A_4)$ *and the condition*

$$R(B(T)R + C(T)) < 1, \tag{2.19}$$

holds, then problem (1.1)–(1.3), (1.8),(1.9) *has a unique solution in the ball* $K = K_R \subset E_T^3$.

Proof In the space E_T^3, consider the operator equation

$$z = \Phi z, \tag{2.20}$$

where $z = \{u, a\}$, and the components $\Phi_i(u, a)$ $(i = 1, 2)$, of operator $\Phi(u, a)$ defined by the right sides of (2.7) and (2.9), respectively and the following inequalities hold:

$$\|\Phi z\|_{E_T^3} \leq A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3} + C(T) \|u(x,t)\|_{B_{2,T}^3}$$

$$\leq A(T) + B(T)R^2 + C(T)R = A(T) + (B(T)(A(T) + 2) + C(T))(A(T) + 2), \tag{2.21}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \leq B(T)R(\|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^3}$$

$$+ \|a_1(t) - a_2(t)\|_{C[0,T]}) + C(T) \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^3}. \tag{2.22}$$

Then it follows from (2.19), (2.21), and (2.22) that the operator Φ acts in the ball $K = K_R$, and satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{z\} = \{u, a\}$ in the ball $K = K_R$, which is a solution of equation (2.20); i.e. the pair $\{u, a\}$ is the unique solution of the systems (2.7) and (2.9) in $K = K_R$.

Then the function $u(x,t)$ as an element of space $B_{2,T}^3$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Next, from (2.2) it follows that $u_k''(t)$ $(k = 1, 2, \ldots)$ are continuous in $[0, T]$ and consequently we have:

$$\left(\sum_{k=1}^{\infty} (\lambda_k \|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{2}\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \sqrt{2}\left(\sum_{k=1}^{\infty} (\lambda_k \|f_k(t) + a(t)u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

or

$$\left(\sum_{k=1}^{\infty} (\lambda_k \|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{2}\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \sqrt{2}\left\|\|f_x(x,t) + a(t)u_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,T)}.$$

From the last relations it is obvious that the function $u_{tt}(x,t)$ is continuous in the region D_T.

It is easy to verify that Eq. (1.1) and conditions (1.2), (1.3), (1.8), (1.9) satisfy in the usual sense. So, $\{u(x,t), a(t)\}$ is a solution of (1.1)–(1.3), (1.8),(1.9), and by Lemma 2.1 it is unique in the ball $K = K_R$. The proof is complete. □

In summary, from Theorem 1.3 and Theorem 2.3, straightforward implies the unique solvability of the original problem (1.1)–(1.5).

Theorem 2.4 *Suppose that all assumptions of Theorem 2.3, and the conditions*

$$\int_0^1 \varphi(x)dx = 0,\ \int_0^1 \psi(x)dx = 0,\ \int_0^1 f(x,t)dx = 0,\ t \in [0,T],$$

$$h(0) = \int_0^T P_1(t)h(t)dt + \varphi(0),\ h'(0) = \int_0^T P_2(t)h(t)dt + \psi(0),$$

$$\left(T \left\| P_2(t) \right\|_{C[0,T]} + \left\| P_1(t) \right\|_{C[0,T]} + \frac{T}{2}(A(T)+2) \right) T < 1,$$

holds. Then problem (1.1)–(1.5) has a unique classical solution in the ball $K \subset E_T^3$.

3. Conclusions

In the work, the classical solvability of a nonlinear coefficient identification problem for a second-order hyperbolic equation with nonlocal conditions was investigated. The considered problem was reduced to an auxiliary inverse boundary value problem in a certain sense and its equivalence to the original problem is shown. Then using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for the auxiliary problem is proved. Further, on the basis of the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse coefficient problem is established for the smaller value of time.

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