Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1256 – 1267
© TÜBİTAK
doi:10.3906/mat-2107-100

Hyperelastic curves along Riemannian maps

Tunahan TURHAN[1], **Gözde ÖZKAN TÜKEL**[2], **Bayram ŞAHİN**[3,*]

[1]Department of Mathematics and Science Education, Faculty of Education,
Süleyman Demirel University, Isparta, Turkey

[2]Department of Basic Sciences, Faculty of Technology, Isparta University of Applied Sciences, Isparta, Turkey

[3]Department of Mathematics, Faculty of Science, Ege University, İzmir, Turkey

Received: 28.07.2021 • **Accepted/Published Online:** 07.03.2022 • **Final Version:** 05.05.2022

Abstract: The main purpose of this paper is to examine what kind of information the smooth Riemannian map defined between two Riemannian manifolds provides about the character of the Riemannian map when a horizontal hyperelastic curve on the total manifold is carried to a hyperelastic curve on the base manifold. For the solution of the mentioned problem, firstly, the behavior of an arbitrary horizontal curve on the total manifold under a Riemannian map is investigated and the equations related to pullback connection are obtained. The necessary conditions are given for the Riemannian map to be h-isotropic or totally umbilical when a horizontal Frenet curve in the total manifold transforms to a hyperelastic curve on the base manifold. Then, the concept of the $\mathfrak{h}$-hyperelastic Riemannian map is defined and using these findings, the Riemannian map along horizontal hyperelastic curves is characterized.

Key words: Riemannian map, hyperelastic curve, second fundamental form, isotropic Riemannian map, umbilical Riemannian map

1. Introduction

Manifolds are an important topic that we see today in every field of research area and modeling applications. The characterization of manifolds through curves, which is the most basic concept in geometry, is a geometrically important method. The special curves were first considered on Riemannian manifolds by Nomizu and Yano in [7]. The authors described the concept of a circle and showed that if a circle on a submanifold is carried to the ambient manifold along an immersion, the submanifold is totally umbilical and the mean curvature vector field is parallel. Ikawa expressed a characterization of a helix by a differential equation in a Riemannian manifold and obtained the necessary and sufficient condition as depending upon the mean curvature vector field that a helix in a Riemannian submanifold corresponds to a helix in ambient manifold [5].

The basic properties of Riemannian submersions were firstly given by Gray [4] and O'Neill [8]. Riemannian submersions and isometric immersions are special maps between Riemannian manifolds. Therefore, Fischer [3] defined Riemannian maps which is a generalization of them in 1992 as follows. Assume that $F : (M, g_M) \to (\bar{M}, g_{\bar{M}})$ is a C^∞ map from the Riemannian manifold M with $\dim M = m$ to the Riemannian manifold $\bar{M}$ with $\dim \bar{M} = n$, where $0 < rankF < \min\{m, n\}$. Thus, we represent the kernel space of F_* by $\ker F_*$ and

*Correspondence: bayram.sahin@ege.edu.tr

2010 *AMS Mathematics Subject Classification:* 53B20, 53C42

$(\ker F_*)^\perp$ is orthogonal complementary space to $\ker F_*$. So, we have

$$TM = \ker F_* \oplus (\ker F_*)^\perp,$$

where TM is the tangent bundle of M, $rangeF_*$ denotes the range of F_*, and $(rangeF_*)^\perp$ denotes the orthogonal complementary space to $rangeF_*$ in $T\bar{M}$. The tangent bundle $T\bar{M}$ of $\bar{M}$ is given by

$$T\bar{M} = (rangeF_*) \oplus (rangeF_*)^\perp.$$

Now, a smooth map $F : (M^m, g_M) \to ((\bar{M})^n, g_{\bar{M}})$ is called Riemannian map if F_* satisfies $g_{\bar{M}}(F_*X_1, F_*X_2) = g_M(X_1, X_2)$, for X_1, X_2 vector fields tangent to $H = (\ker F_*)^\perp$ [3].

Hyperelastic curves are one of the generalizations of classical elastic curves that have been studied and developed extensively in differential geometry. The mathematical idealization of this problem is that of minimizing the following bending energy functional

$$\mathcal{F}_\gamma^r = \int (\kappa^r + \lambda)\, ds, \tag{1.1}$$

acting on a suitable space of curves in a Riemannian manifold, where κ denotes the curvature of γ [1, 9, 10]. Such curves are called free hyperelastic curves when $\lambda = 0$. The (free) hyperelastic curve has been employed to furnish reduction methods in constructing Chen-Willmore submanifolds (see, [1, 2, 4]). The functional $\mathcal{F}_\gamma^r$ is nothing but the classical Euler-Bernoulli's bending (or elastic) energy functional for $r = 2$ (see, [6, 11]). Recently, the effect of hyperelastic curves on the geometry of isometric immersions is studied in [13]. In this work, we extend our results for Riemannian maps taking into consideration the concept of h-isotropic Riemannian map developed in [14]. Thus we show that one can investigate the geometry of a Riemannian map itself, and base manifold and target manifold of a Riemannian map by using hyperelastic curves or classical elastic curves.

The paper is organized as follows. In Section 2, we collect the main results and notions for this paper. In the third section, first preparatory lemmas are given and then the conditions for the Riemannian map to be an isotropic map are found in case a horizontal Frenet curve on the total manifold is a hyperelastic curve on the base manifold (Proposition 3.1). In addition, if a horizontal hyperelastic curve on the total manifold is a hyperelastic curve on the base manifold along the Riemannian map, the geometric properties of the Riemannian map have been obtained in terms of the notions of umbilical Riemannian maps and the mean curvature vector field (Theorem 3.1). Moreover, if the curve is a classical elastic curve, the geometric properties of the Riemannian map have been found by the equation involving curvature and torsion (Theorem 3.2). Furthermore, some results deduced from the above proposition and theorems have been also expressed.

2. Preliminaries

Let (M, g_M) and $(\bar{M}, g_{\bar{M}})$ be Riemannian manifolds, $F : (M, g_M) \to (\bar{M}, g_{\bar{M}})$ a smooth map between them and γ a curve on M. γ is called a horizontal curve if $\dot{\gamma}(t) \in (kerF_*)^\perp$ for any $t \in I$. We suppose that $p_2 = F(p_1)$ for each $p_1 \in M$ and $\nabla^{\bar{M}}$ denotes the Levi-Civita connection on $(\bar{M}, g_{\bar{M}})$. Then the second fundamental form of F is given by

$$(\nabla F_*)(X, Y) = \overset{\bar{M}}{\nabla}{}^F_X F_*(Y) - F_*(\overset{M}{\nabla}_X Y), \tag{2.1}$$

for X, $Y \in \Gamma(TM)$, where $\overset{\bar{M}}{\nabla^F}$ is the pullback connection of $\nabla^{\bar{M}}$. It is known that the second fundamental form is symmetric. First note that the second fundamental form $(\nabla F_*)(X,Y)$, $\forall X,Y \in \Gamma((\ker F_{*p_1})^{\perp})$, of a Riemannian map has no components in $rangeF_*$. More precisely we have the following;

$$g_{\bar{M}}((\nabla F_*)(X,Y), F_*(Z)) = 0, \tag{2.2}$$

$\forall X,Y,Z \in \Gamma((\ker F_{*p_1})^{\perp})$ [12]. For $X,Y \in \Gamma((\ker F_{*p_1})^{\perp})$ and $V \in \Gamma((rangeF_*)^{\perp})$, we have

$$\overset{\bar{M}}{\nabla}_{F_*(X)} V = -S_V F_* X + \nabla_X^{F^{\perp}} V, \tag{2.3}$$

where $S_V F_* X$ is the tangential component (a vector field along F) of $\overset{\bar{M}}{\nabla}_{F_*(X)} V$ and

$$g_{\bar{M}}(S_V F_* X, F_* Y) = g_{\bar{M}}(V, (\nabla F_*)(X,Y)), \tag{2.4}$$

since (∇F_*) is symmetric, S_V is a symmetric linear transformation of $rangeF_*$. From (2.1) and (2.3), we have

$$\begin{aligned} R^{\bar{M}}(F_*X, F_*Y)F_*Z &= -S_{(\nabla F_*)(Y,Z)} F_*X + S_{(\nabla F_*)(X,Z)} F_*Y \\ &\quad + F_*(R^M(X,Y)Z) + (\tilde{\nabla}_X(\nabla F_*))(Y,Z) \\ &\quad - (\tilde{\nabla}_Y(\nabla F_*))(X,Z), \end{aligned} \tag{2.5}$$

where $\tilde{\nabla}$ is the covariant derivative of the second fundamental form. On the other hand, we have the following covariant derivatives

$$\left(\tilde{\nabla}_X(\nabla F_*)\right)(Y,Z) = \nabla_X^{F^{\perp}}(\nabla F_*)(Y,Z) - (\nabla F_*)(\overset{M}{\nabla}_X Y, Z) - (\nabla F_*)(Y, \overset{M}{\nabla}_X Z), \tag{2.6}$$

and

$$\left(\tilde{\nabla}_X S\right)_V F_*(Y) = F_*(\overset{M}{\nabla}_X {}^*F_*(S_V F_*(Y))) - S_{(\nabla_X^F)^{\perp} V} F_*(Y) - S_V P \overset{\bar{M}}{\nabla^F}_X F_*(Y), \tag{2.7}$$

where P denotes the projection morphism on $rangeF_*$ and ${}^*F_*$ is the adjoint map of F_* [12]. From (2.6) and (2.7), we can see the following formula for $X,Y \in \Gamma((\ker F_{*p_1})^{\perp})$ and $V \in \Gamma((rangeF_*)^{\perp})$

$$g_{\bar{M}}(\left(\tilde{\nabla}_X(\nabla F_*)\right)(Y,Z), V) = g_{\bar{M}}(\left(\tilde{\nabla}_X S\right)_V F_*(Y), F_*(Z)), \tag{2.8}$$

[14].

Let F be a Riemannian map from a connected Riemannian manifold (M, g_M), $dimM \geq 2$ to a Riemannian manifold $(\bar{M}, g_{\bar{M}})$. We know that F is an umbilical Riemannian map at $p_1 \in M$, if

$$S_V F_{*p_1}(X_{p_1}) = g_{\bar{M}}(H_2, V) F_{*p_1}(X_{p_1}), \tag{2.9}$$

for $X \in \Gamma(rangeF_*)$, $V \in \Gamma((rangeF_*)^{\perp})$ and H_2 is, nowhere zero, vector field on $(rangeF_*)^{\perp}$. If F is umbilical for $\forall p_1 \in M$ then we say that F is an umbilical Riemannian map [12]. A Riemannian map

$F : (M, g_M) \to (\bar{M}, g_{\bar{M}})$ is said to be h-isotropic at $p \in M$ if $\lambda(X) = \|(\nabla F_*)(X,X)\| / \|F_* X\|^2$ does not depend on the choice of $X \in \Gamma((\ker F_*)^\perp$. If the map is h-isotropic at every point, then the map is said to be h-isotropic. When the function $\lambda = \lambda(p)$ is constant along F, F is called a constant $(\lambda h-)$isotropic map. It is obvious from Lemma 3.1 by Ozkan Tukel et al. in 2021 that the map F is h-isotropic at $p \in M$ if and only if the second fundamental form ∇F_* satisfies

$$g_{\bar{M}}((\nabla F_*)(X,X), (\nabla F_*)(X,Y)) = 0, \tag{2.10}$$

for an arbitrary orthogonal pair $X, Y \in \Gamma((\ker F_{*p_1})^\perp)$. Furthermore, a totally umbilical Riemannian map F is a h-isotropic at the point p_1. Also, from Corollary 3.1 by Ozkan Tukel et al. in 2021, a h-isotropic Riemannian map F is a totally umbilical at p_1 if it satisfies following condition

$$(\nabla F_*)(X,Y) = 0, \tag{2.11}$$

for orthonormal vector fields X and Y at p_1 in $\Gamma((\ker F_{*p_1})^\perp)$. In addition, if α is a horizontal curve with curvature κ in M and $\gamma = F \circ \alpha$ a curve with curvature $\tilde{\kappa}$ in $\bar{M}$ along F, we have for $\forall t \in \mathbb{R}$

$$\tilde{\kappa} = \tilde{\kappa}(t) = \sqrt{\kappa(t)^2 + \|(\nabla F_*)(\dot{\alpha}(t), \dot{\alpha}(t))\|^2}, \tag{2.12}$$

[14].

3. Geometry of the Riemannian map by hyperelastic curves

We firstly remind a characterization of the hyperelastic curve in a Riemannian manifold M. We consider the family of C^∞ curves as follows

$$\begin{aligned} \Im = \quad & \{ \gamma |\, \gamma : [0,\ell] \subset \mathbb{R} \to M \\ & \gamma\,(i\ell) = p_i, \quad p_i \in M, \quad \gamma_s\,(i\ell) = v_i, \quad v_i \in T_{p_i}M \quad i = 0,1 \}. \end{aligned}$$

For a parametrized curve $\gamma \in \Im$, κ is the geodesic curvature of γ. Then a hyperelastic curve is a critical point of the functional $\mathcal{F}^r_\gamma : \Im \to [0, \infty)$ defined by (1.1) for a natural number $r \geq 2$ [1].

Let $\gamma : I \to M$ be an immersed unit speed curve in a $n-$dimensional Riemannian manifold M. We denote by T, W and U, the unit tangent vector field, the unit normal vector field, and the binormal vector field of γ, respectively. $\kappa = \|\nabla_T T\|$ is the geodesic curvature and $\tau = -< \nabla_T U, W >$ is the torsion of γ. γ has also curvatures $\kappa_1 = \kappa > 0$, $\kappa_2 = \tau$, κ_3, $\kappa_4, ..., \kappa_{n-1}$, and Frenet frame $W_0 = T$, $W_1 = W$, $W_2 = U$, W_3, $W_4, ..., W_{n-1}$. Then, the Frenet equations are given by

$$\nabla_T W_i = -\kappa_i W_{i-1} + \kappa_{i+1} W_{i+1}, \qquad 0 \leq i \leq n-1,$$

(defining $\kappa_0 = \kappa_n = 0$) [6]. In this case γ is called a Frenet curve of order n.

Critical points of the functional (1.1) are characterized by the Euler-Lagrange equation

$$\overset{M}{\nabla}{}^2_T \left(\kappa^{r-2} \nabla_T T\right) + \kappa^{r-2} R(\overset{M}{\nabla}_T T, T)T + \overset{M}{\nabla}_T (\lambda T) = 0, \tag{3.1}$$

where $\overset{M}{\nabla}$ is the Levi-Civita connection of M and

$$\lambda = \frac{2r-1}{r}\kappa^r + b, \tag{3.2}$$

for some constant $b \in \mathbb{R}$ [1, 9].

The following lemma will be crucial for our computations.

Lemma 3.1 *Let M and $\bar{M}$ be Riemannian manifolds and $F : M \to \bar{M}$ a Riemannian map such that γ is a horizontal curve on M. Assume that $\overline{\gamma}(s) = F \circ \gamma(s)$ is a curve with curvature $\overline{\kappa}$ in $\bar{M}$. Then we have in the following equations:*

$$
\begin{aligned}
i)\ (\overline{\nabla}^F_{X_s})^2 \left(\overline{\kappa}^{r-2}\overline{\nabla}_{X_s} X_s\right) = \ & F_*(\nabla^2_{X_s}\left(\xi \nabla_{X_s} X_s\right)) + (\nabla F_*)\left(X_s, \nabla_{X_s}\left(\xi \nabla_{X_s} X_s\right)\right) \\
& - S_{\xi(\nabla F_*)(X_s, \nabla_{X_s} X_s)} F_*(X_s) + \nabla^{F^\perp}_{X_s} \xi (\nabla F_*)(X_s, \nabla_{X_s} X_s) \\
& - F_*(\nabla_{X_s}{}^* F_* S_{\xi(\nabla F_*)(X_s, X_s)} F_*(X_s)) \\
& - (\nabla F_*)\left(S_{\xi(\nabla F_*)(X_s,X_s)} F_*(X_s), X_s\right) \\
& - S_{\nabla^{F^\perp}_{X_s} \xi(\nabla F_*)(X_s,X_s)} F_*(X_s) + (\nabla^{F^\perp}_{X_s})^2 \xi (\nabla F_*)(X_s, X_s),
\end{aligned}
$$

$$
\begin{aligned}
ii)\quad \overline{\kappa}^{r-2} \bar{R}\left(F_*(\bar{\nabla}_{X_s} X_s), F_*(X_s)\right) F_*(X_s) = & \ \xi F_*(R\left(\nabla_{X_s} X_s, X_s\right) X_s) \\
& - \xi S_{(\nabla F_*)(X_s,X_s)} F_*(\nabla_{X_s} X_s) + \xi S_{(\nabla F_*)(\nabla_{X_s} X_s, X_s)} F_*(X_s) \\
& + \xi \left(\bar{\nabla}_{\nabla_{X_s} X_s} (\nabla F_*)\right)(X_s, X_s) - \xi\left(\bar{\nabla}_{X_s}(\nabla F_*)\right)\left(\nabla_{X_s} X_s, X_s\right),
\end{aligned}
$$

where $\xi = \left(\kappa^2 + \|(\nabla F_*)(X_s, X_s)\|^2\right)^{\frac{r-2}{2}}$, κ is the curvature of γ in M, ∇ and $\bar{\nabla}$ are the connections of M and $\bar{M}$, respectively. Also, s is the arc length parameter, the vector field X_s is always the unit tangent vector field along γ and the vector field $F_*(X_s)$ along $\overline{\gamma}(s)$ represents $F_*(X)(s) = F_{*\gamma(s)} X_s$.

Proof Assume that $\gamma(s)$ is a horizontal curve with curvature $\kappa(s)$ on the total manifold M and $\overline{\gamma}(s) = (F \circ \gamma)(s)$ is a curve with $\overline{\kappa}(s)$ on $\bar{M}$. We can define a vector field $F_* X_s$ along $\overline{\gamma}(s)$ by

$$F_*(X)(s) = F_{*\gamma(s)} X_s,$$

for each vector field X_s along γ.

(i) From (2.1) and (2.3), we obtain

$$
\begin{aligned}
\overline{\nabla}^F_{X_s}\left(\overline{\kappa}^{r-2}\overline{\nabla}_{X_s} X_s\right) = & \ \overline{\nabla}^F_{X_s}\left(\xi F_*(\nabla_{X_s} X_s)\right) + \overline{\nabla}^F_{X_s}\left(\xi(\nabla F_*)(X_s, X_s)\right) \\
= & \ F_*(\nabla_{X_s}\left(\xi \nabla_{X_s} X_s\right)) + \xi(\nabla F_*)\left(X_s, \nabla_{X_s} X_s\right) \\
& - S_{\xi(\nabla F_*)(X_s,X_s)} F_*(X_s) + \nabla^{F^\perp}_{X_s} \xi(\nabla F_*)(X_s, X_s).
\end{aligned} \tag{3.3}
$$

Taking derivative of (3.3) and using (2.1) and (2.3), we get the desired equation.

(ii) From (2.5) and (2.1) we have

$$
\begin{aligned}
\bar{R}\left(F_*(\nabla_{X_s} X_s), F_*(X_s)\right) F_*(X_s) = & \ F_*(R\left(\nabla_{X_s} X_s, X_s\right) X_s) \\
& - S_{(\nabla F_*)(X_s.X_s)} F_*(\nabla_{X_s} X_s) + S_{(\nabla F_*)(\nabla_{X_s} X_s, X_s)} F_*(X_s) \\
& + \left(\bar{\nabla}_{\nabla_{X_s} X_s}(\nabla F_*)\right)(X_s, X_s) - \left(\bar{\nabla}_{X_s}(\nabla F_*)\right)\left(\nabla_{X_s} X_s, X_s\right).
\end{aligned} \tag{3.4}
$$

Multiplying (3.4) with ξ, the proof of ii is completed. □

By using the characterization of hyperelastic curve, we get the following lemma.

Lemma 3.2 *Let M and $\bar{M}$ be Riemannian manifolds and $F : M \to \bar{M}$ a Riemannian map such that γ is a horizontal curve on M. Assume that $\overline{\gamma}(s) = F \circ \gamma(s)$ is a hyperelastic curve in $\bar{M}$. Then we have the following* *equations:*

i) $\lambda_s - 3\xi\kappa\kappa_s - 2\kappa^2\xi_s = \xi((r-2)(\kappa^2 + \|(\nabla F_*)(X_s, X_s)\|^2)^{-1}$
$$\|(\nabla F_*)(X_s, X_s)\|^2 + \tfrac{3}{2})\nabla^{F^\perp}_{X_s}\|(\nabla F_*)(X_s, X_s)\|^2,$$

ii) $(\nabla F_*)(X_s, U_s) = 0.$

Proof Assume that $F : M \to \bar{M}$ is a Riemannian map and F transforms a horizontal Frenet curve γ to a hyperelastic curve $\overline{\gamma}$. From (3.1), we have the following equation

$$\begin{aligned}&\overline{\nabla}^2_{F_*(X_s)}\left(\overline{\kappa}^{r-2}\overline{\nabla}_{F_*(X_s)}F_*(X_s)\right)\\ &\quad +\overline{\kappa}^{r-2}\overline{R}(F_*(\nabla_{X_s}X_s), F_*(X_s))F_*(X_s) + \overline{\nabla}^F_{X_s}F_*(\lambda X_s) = 0,\end{aligned} \tag{3.5}$$

and

$$\lambda = \frac{2r-1}{r}\overline{\kappa}^r + b, \tag{3.6}$$

for some constant $b \in \mathbb{R}$. Taking into consideration Lemma 3.1 and using (3.5), we obtain

$$\begin{aligned}&F_*(\nabla^2_{X_s}\left(\xi\nabla_{X_s}X_s\right)) + (\nabla F_*)\left(X_s, \nabla_{X_s}\left(\xi\nabla_{X_s}X_s\right)\right) + \nabla^{F^\perp}_{X_s}\xi(\nabla F_*)(X_s, \nabla_{X_s}X_s)\\ &-F_*(\nabla_{X_s}{}^*F_*S_{\xi(\nabla F_*)(X_s,X_s)}F_*(X_s)) - (\nabla F_*)\left(S_{\xi(\nabla F_*)(X_s,X_s)}F_*(X_s), X_s\right)\\ &-S_{\nabla^{F^\perp}_{X_s}\xi(\nabla F_*)(X_s,X_s)}F_*(X_s) + (\nabla^{F^\perp}_{X_s})^2\xi(\nabla F_*)(X_s, X_s) + \xi F_*(R\left(\nabla_{X_s}X_s, X_s\right)X_s)\\ &-\xi S_{(\nabla F_*)(X_s,X_s)}F_*(\nabla_{X_s}X_s) + \xi\left(\bar{\nabla}_{\nabla_{X_s}X_s}(\nabla F_*)\right)(X_s, X_s)\\ &-\xi\left(\bar{\nabla}_{X_s}(\nabla F_*)\right)\left(\nabla_{X_s}X_s, X_s\right) + F_*(\nabla_{X_s}\lambda X_s) + \lambda(\nabla F_*)(X_s, X_s) = 0,\end{aligned} \tag{3.7}$$

where

$$\lambda = \frac{2r-1}{r}(\kappa^2 + \|(\nabla F_*)(X_s, X_s)\|^2)^{\frac{r}{2}} + b.$$

The $rangeF_*$ component of (3.7) is

$$\begin{aligned}&F_*(\nabla^2_{X_s}\left(\xi\nabla_{X_s}X_s\right)) - F_*(\nabla_{X_s}{}^*F_*S_{\xi(\nabla F_*)(X_s,X_s)}F_*(X_s))\\ &-S_{\nabla^{F^\perp}_{X_s}\xi(\nabla F_*)(X_s,X_s)}F_*(X_s) - \xi S_{(\nabla F_*)(X_s,X_s)}F_*(\nabla_{X_s}X_s)\\ &\quad +\xi F_*(R\left(\nabla_{X_s}X_s, X_s\right)X_s) + F_*(\nabla_{X_s}\lambda X_s) = 0.\end{aligned} \tag{3.8}$$

We have from (2.7) and (2.6)

$$\begin{aligned}&F_*(\nabla_{X_s}{}^*F_*S_{\xi(\nabla F_*)(X_s,X_s)}F_*(X_s)) = (\bar{\nabla}_{X_s}S)_{\xi(\nabla F_*)(X_s,X_s)}F_*(X_s)\\ &\quad +S_{\nabla^{F^\perp}_{X_s}\xi(\nabla F_*)(X_s,X_s)}F_*(X_s) + S_{\xi(\nabla F_*)(X_s,X_s)}P\nabla^F_{X_s}F_*(X_s),\end{aligned} \tag{3.9}$$

and

$$\xi\left(\bar{\nabla}_{X_s}(\nabla F_*)\right)(X_s, X_s) = \xi\nabla^{F^\perp}_{X_s}(\nabla F_*)(X_s, X_s) - 2\xi(\nabla F_*)(X_s, \nabla_{X_s}X_s), \tag{3.10}$$

respectively. Substituting (3.9) and (3.10) into (3.8), we have

$$\begin{aligned}&F_*(\nabla^2_{X_s}\left(\xi\nabla_{X_s}X_s\right)) - (\bar{\nabla}_{X_s}S)_{\xi(\nabla F_*)(X_s,X_s)}F_*(X_s) - 2S_{\nabla^{F^\perp}_{X_s}\xi(\nabla F_*)(X_s,X_s)}F_*(X_s)\\ &\quad -2\xi S_{(\bar{\nabla}_{X_s}(\nabla F_*))(X_s,X_s)}F_*(X_s) - 4\xi S_{(\nabla F_*)(X_s,\nabla_{X_s}X_s)}F_*(X_s)\\ &\quad -\xi S_{(\nabla F_*)(X_s,X_s)}P\nabla^F_{X_s}F_*(X_s) - \xi S_{(\nabla F_*)(X_s,X_s)}F_*(\nabla_{X_s}X_s)\\ &\quad -\xi F_*(R\left(\nabla_{X_s}X_s, X_s\right)X_s) + F_*(\nabla_{X_s}\lambda X_s) = 0.\end{aligned} \tag{3.11}$$

With aid of Frenet equations of γ, (3.11) reduces to

$$\begin{aligned}&\left(\lambda_s - 3\xi\kappa\kappa_s - 2\kappa^2\xi_s\right)F_*(X_s) + (\xi\kappa_{ss} - \kappa^3\xi + 2\kappa_s\xi_s + \xi_{ss}\kappa - \kappa\tau^2\xi\\&+\lambda\kappa)F_*(W_s) + \left(2\kappa_s\tau\xi + \kappa\tau_s\xi + 2\kappa\tau\xi_s\right)F_*(U_s) + \xi\kappa\tau\kappa_3 F_*(W_{3s})\\&-4\xi\kappa S_{(\nabla F_*)(W_s,X_s)}F_*(X_s) - 2\xi\kappa S_{(\nabla F_*)(X_s,X_s)}F_*(W_s)\\&= 2S_{\nabla^{F\perp}_{X_s}\xi(\nabla F_*)(X_s,X_s)}F_*(X_s) + 2\xi S_{(\bar{\nabla}_{X_s}(\nabla F_*))(X_s,X_s)}F_*(X_s)\\&+(\bar{\nabla}_{X_s}S)_{\xi(\nabla F_*)(X_s,X_s)}F_*(X_s) + \xi\kappa F_*(R(W_s,X_s)X_s).\end{aligned} \tag{3.12}$$

Taking inner product with $F_*(X_s)$ to (3.12) and using (3.10), we calculate

$$\begin{gathered}\lambda_s - 3\xi\kappa\kappa_s - 2\kappa^2\xi_s - 6\xi\kappa g_{\overline{M}}((\nabla F_*)(W_s,X_s),(\nabla F_*)(X_s,X_s)) =\\+2g_{\overline{M}}(S_{\nabla^{F\perp}_{X_s}\xi(\nabla F_*)(X_s,X_s)}F_*(X_s),F_*(X_s)) - 4\kappa\xi g_{\overline{M}}(S_{(\nabla F_*)(W_s,X_s)}F_*(X_s),F_*(X_s))\\+g_{\overline{M}}((\bar{\nabla}_{X_s}S)_{\xi(\nabla F_*)(X_s,X_s)}F_*(X_s),F_*(X_s)).\end{gathered}$$

From (2.4), (2.6), and (2.8), we obtain

$$\begin{aligned}\lambda_s - 3\xi\kappa\kappa_s - 2\kappa^2\xi_s &= 2g_{\overline{M}}(\nabla^{F\perp}_{X_s}\xi(\nabla F_*)(X_s,X_s),(\nabla F_*)(X_s,X_s))\\&+g_{\overline{M}}(\nabla^{F\perp}_{X_s}(\nabla F_*)(X_s,X_s),\xi(\nabla F_*)(X_s,X_s))\\&=2(\nabla^{F\perp}_{X_s}\xi)\|(\nabla F_*)(X_s,X_s)\|^2 + \tfrac{3}{2}\xi\nabla^{F\perp}_{X_s}\|(\nabla F_*)(X_s,X_s)\|^2\\&=\xi((r-2)(\kappa^2+\|(\nabla F_*)(X_s,X_s)\|^2)^{-1}\|(\nabla F_*)(X_s,X_s)\|^2\\&+\tfrac{3}{2})\nabla^{F\perp}_{X_s}\|(\nabla F_*)(X_s,X_s)\|^2.\end{aligned} \tag{3.13}$$

The $(rangeF_*)^\perp$ component of (3.7) is obtained as

$$\begin{aligned}&(\nabla F_*)\left(X_s,\nabla_{X_s}\left(\xi\nabla_{X_s}X_s\right)\right) + \nabla^{F\perp}_{X_s}\xi(\nabla F_*)(X_s,\nabla_{X_s}X_s)\\&-\xi(\nabla F_*)\left(S_{(\nabla F_*)(X_s,X_s)}F_*(X_s),X_s\right) + (\nabla^{F\perp}_{X_s})^2\xi(\nabla F_*)(X_s,X_s)\\&+\xi\left(\bar{\nabla}_{\nabla_{X_s}X_s}(\nabla F_*)\right)(X_s,X_s) - \xi\left(\bar{\nabla}_{X_s}(\nabla F_*)\right)\left(\nabla_{X_s}X_s,X_s\right)\\&+\lambda(\nabla F_*)(X_s,X_s) = 0.\end{aligned} \tag{3.14}$$

We have from (2.6)

$$\begin{aligned}\xi(\nabla^{F\perp}_{X_s})^2(\nabla F_*)(X_s,X_s) &= \xi\left(\bar{\nabla}^2_{X_s}(\nabla F_*)\right)(X_s,X_s) + 4\xi\nabla^{F\perp}_{X_s}(\nabla F_*)(X_s,\nabla_{X_s}X_s)\\&\quad-2\xi(\nabla F_*)(\nabla_{X_s}X_s,\nabla_{X_s}X_s) - 2\xi(\nabla F_*)(X_s,\nabla^2_{X_s}X_s).\end{aligned} \tag{3.15}$$

On the other hand we have

$$\begin{aligned}(\nabla^{F\perp}_{X_s})^2\xi(\nabla F_*)(X_s,X_s) &= ((\nabla^{F\perp}_{X_s})^2\xi)(\nabla F_*)(X_s,X_s) + 2(\nabla^{F\perp}_{X_s}\xi)(\nabla^{F\perp}_{X_s})(\nabla F_*)(X_s,X_s)\\&+\xi(\nabla^{F\perp}_{X_s})^2(\nabla F_*)(X_s,X_s).\end{aligned} \tag{3.16}$$

Substituting (3.15), (3.16) into (3.14) and using (2.6), we get

$$\begin{aligned}&(\nabla F_*)(X_s,\xi_s\nabla_{X_s}X_s) + \xi(\nabla F_*)(X_s,\nabla^2_{X_s}X_s) + (\nabla^{F\perp}_{X_s}\xi)(\nabla F_*)(X_s,\nabla_{X_s}X_s)\\&+\xi\nabla^{F\perp}_{X_s}(\nabla F_*)(X_s,\nabla_{X_s}X_s)) - \xi(\nabla F_*)\left(S_{(\nabla F_*)(X_s,X_s)}F_*(X_s),X_s\right)\\&+((\nabla^{F\perp}_{X_s})^2\xi)(\nabla F_*)(X_s,X_s) + 2(\nabla^{F\perp}_{X_s}\xi)(\nabla^{F\perp}_{X_s})(\nabla F_*)(X_s,X_s)\\&+\xi(\nabla^{F\perp}_{X_s})^2(\nabla F_*)(X_s,X_s) + \xi\left(\bar{\nabla}_{\nabla_{X_s}X_s}(\nabla F_*)\right)(X_s,X_s)\\&-\xi(\bar{\nabla}_{X_s}(\nabla F_*))(\nabla_{X_s}X_s,X_s) + \lambda(\nabla F_*)(X_s,X_s) = 0.\end{aligned}$$

The processes are arranged as follows

$$\begin{aligned}&(\nabla F_*)(X_s,\xi_s\nabla_{X_s}X_s)+4\xi(\nabla F_*)(X_s,\nabla^2_{X_s}X_s)+5(\nabla^{F^\perp}_{X_s}\xi)(\nabla F_*)(X_s,\nabla_{X_s}X_s)\\&+4\xi\left(\bar{\nabla}_{X_s}(\nabla F_*)\right)(X_s,\nabla_{X_s}X_s)+3\xi(\nabla F_*)(\nabla_{X_s}X_s,\nabla_{X_s}X_s)\\&-\xi(\nabla F_*)\left(S_{(\nabla F_*)(X_s,X_s)}F_*(X_s),X_s\right)+((\nabla^{F^\perp}_{X_s})^2\xi)(\nabla F_*)(X_s,X_s)\\&+2(\nabla^{F^\perp}_{X_s}(\nabla F_*))(\bar{\nabla}_{X_s}(\nabla F_*))(X_s,X_s)+\xi\left(\bar{\nabla}^2_{X_s}(\nabla F_*)\right)(X_s,X_s)\\&+\xi\left(\bar{\nabla}_{\nabla_{X_s}X_s}(\nabla F_*)\right)(X_s,X_s)+\lambda(\nabla F_*)(X_s,X_s)=0.\end{aligned}$$

Taking into consideration the Frenet equations, we have

$$\begin{aligned}&(\kappa\xi_s+4\xi\kappa_s+5(\nabla^{F^\perp}_{X_s}\xi)\kappa)(\nabla F_*)(X_s,W_s)+4\xi\kappa\tau(\nabla F_*)(X_s,U_s)\\&+4\xi\kappa\left(\bar{\nabla}_{X_s}(\nabla F_*)\right)(X_s,W_s)+3\xi\kappa^2(\nabla F_*)(W_s,W_s)+\xi\kappa\left(\bar{\nabla}_{W_s}(\nabla F_*)\right)(X_s,X_s)\\&+2(\nabla^{F^\perp}_{X_s}\xi)\left(\bar{\nabla}_{X_s}(\nabla F_*)\right)(X_s,X_s)=(4\xi\kappa^2-((\nabla^{F^\perp}_{X_s})^2\xi)+\lambda)(\nabla F_*)(X_s,X_s)\\&+\xi(\nabla F_*)\left(S_{(\nabla F_*)(X_s,X_s)}F_*(X_s),X_s\right)-\xi\left(\bar{\nabla}^2_{X_s}(\nabla F_*)\right)(X_s,X_s).\end{aligned}\tag{3.17}$$

Changing U_s into $-U_s$ in (3.17) and using (3.17), we get

$$(\nabla F_*)(X_s,U_s)=0.\tag{3.18}$$

In the following proposition, we give a characterization of the Riemannian map by using horizontal hyperelastic curves. □

Proposition 3.3 *Let M and $\bar{M}$ be Riemannian manifolds and $F:M\to\bar{M}$ a Riemannian map such that γ is a horizontal Frenet curve on M. Assume that $\bar{\gamma}(s)=F\circ\gamma(s)$ is a hyperelastic curve with curvature $\bar{\kappa}$ in $\bar{M}$. Then F is a $h-$isotropic Riemannian map if one of the following conditions are satisfied.*

(i) $r=2$, that is $\bar{\gamma}(s)$ is a classical elastic curve,

(ii) the curvature κ of γ is a constant when $\bar{\kappa}\neq 0$.

Proof If $\bar{\gamma}=\bar{\gamma}(s)$ is a hyperelastic curve with curvature $\bar{\kappa}$ and $\gamma(s)$ has the curvature κ, then we have the following cases.
(**i**): If $r=2$, then (3.13) reduces to

$$\nabla^{F^\perp}_{X_s}\|(\nabla F_*)(X_s,X_s)\|^2=0.\tag{3.19}$$

Thus $\|(\nabla F_*)(X_s,X_s)\|$ is equal to a constant value, that is, F is a h-isotropic Riemannian map. (**ii**): We suppose that κ is a constant value. From (3.13), we obtain

$$\begin{aligned}&\xi((r-2)(\kappa^2+\|(\nabla F_*)(X_s,X_s)\|^2)^{-1}\|(\nabla F_*)(X_s,X_s)\|^2\\&+\tfrac{3}{2})\nabla^{F^\perp}_{X_s}\|(\nabla F_*)(X_s,X_s)\|^2=0.\end{aligned}\tag{3.20}$$

On the other hand, since $\bar{\kappa}\neq 0$ and $r>2$, then we have (3.19). □

From (3.18) and Proposition 3.3, the proof of the following corollary is clear.

Corollary 3.4 *Let $F:M\to\bar{M}$ be a Riemannian map between Riemannian manifolds M and $\bar{M}$, $\gamma(s)$ a horizontal Frenet curve on M and $\bar{\gamma}(s)=F\circ\gamma(s)$ a hyperelastic curve with curvature $\bar{\kappa}$ in $\bar{M}$. In this case, F is totally umbilical Riemannian map if it is a $h-$isotropic Riemannian map.*

Now, we introduce the concept of $\mathfrak{h}$-hyperelastic Riemannian map.

Definition 3.5 *Let F be a Riemannian map from a Riemannian manifold M to a Riemannian manifold $\bar{M}$ such that γ is a horizontal hyperelastic curve on M. If the curve $\overline{\gamma} = F \circ \gamma$ is also a hyperelastic curve on $\bar{M}$, then the Riemannian map F is called a $\mathfrak{h}$-hyperelastic Riemannian map.*

The following theorem gives a result of $\mathfrak{h}$-hyperelastic Riemannian map in terms of the mean curvature vector field and totally umbilical Riemannian map.

Theorem 3.6 *Let $F : M \to \bar{M}$ be a $\mathfrak{h}$-hyperelastic Riemannian map between Riemannian manifolds M and $\bar{M}$. If γ is a horizontal hyperelastic curve with the constant curvature κ on M, then F is a totally umbilical Riemannian map and the mean curvature vector field H_2 satisfies*

$$(\nabla^{F^\perp}_{X_s})H_2 = CH_2, \tag{3.21}$$

where $C = \kappa^2 + \|H_2\|^2 + \frac{\lambda}{\xi} = const$. Conversely if $(\nabla F_*)(X_s, X_s) = 0$, where X_s is the tangent vector of γ, the Riemannian map F transports a horizontal hyperelastic curve on the total manifold to a hyperelastic curve on the target manifold.

Proof Let $F : M \to \bar{M}$ be a $\mathfrak{h}$-hyperelastic Riemannian map between Riemannian manifolds M and $\bar{M}$. Suppose that $\gamma(s)$ is a hyperelastic curve with a constant curvature κ and the tangent vector field X_s in M. Then γ satisfies the Euler-Lagrange equation (3.1) with (3.2). We can see from Proposition 3.3, F is a $h-$isotropic Riemannian map. Since $\overline{\gamma}$ is a hyperelastic curve in $\bar{M}$, we have (3.5) with (3.6). On the other hand Corollary 3.4 shows that F is totally umbilical map. On the other hand, (3.17) reduces to

$$\begin{aligned}
4\xi\kappa\left(\bar{\nabla}_{X_s}(\nabla F_*)\right)(X_s, W_s) + 3\xi\kappa^2(\nabla F_*)(W_s, W_s) + \xi\kappa\left(\bar{\nabla}_{W_s}(\nabla F_*)\right)(X_s, X_s)\\
= 4\xi\kappa^2(\nabla F_*)(X_s, X_s) + \xi(\nabla F_*)\left(S_{(\nabla F_*)(X_s,X_s)}F_*(X_s), X_s\right)\\
+\lambda(\nabla F_*)(X_s, X_s) - \xi\left(\bar{\nabla}^2_{X_s}(\nabla F_*)\right)(X_s, X_s).
\end{aligned} \tag{3.22}$$

Changing W_s into $-W_s$ in (3.22) and using (3.22), we have

$$4\xi\kappa\left(\bar{\nabla}_{X_s}(\nabla F_*)\right)(X_s, W_s) + \xi\kappa\left(\bar{\nabla}_{W_s}(\nabla F_*)\right)(X_s, X_s) = 0. \tag{3.23}$$

Substituting (3.23) into (3.22) and using totally umbilicity of the Riemannian map, we have

$$-\xi\kappa^2 H_2 = \xi\|H_2\|^2 H_2 + \lambda H_2 - \xi(\nabla^{F^\perp}_{X_s})^2 H_2.$$

Then, we obtain

$$(\nabla^{F^\perp}_{X_s})^2 H_2 = (\kappa^2 + \|H_2\|^2 + \frac{\lambda}{\xi})H_2.$$

Conversely, we assume that the $(\nabla F_*)(X_s, X_s) = 0$ and H_2 satisfies (3.21). Since γ is a horizontal hyperelastic curve with the curvature κ, $\overline{\gamma}$ satisfies the Euler-Lagrange equation (3.5) with (3.6). □

Then we have the following result.

Corollary 3.7 *If $F : M \to \bar{M}$ is a $\mathfrak{h}$-hyperelastic Riemannian map and γ is a hyperelastic curve with the constant curvature in M, then γ has constant torsion.*

Proof Let $F: M \to \bar{M}$ be a $\mathfrak{h}$-hyperelastic Riemannian map and γ a hyperelastic curve with the constant curvature κ in M. Then (3.12) can be written as

$$\begin{aligned}(-\kappa^2-\tau^2+\tfrac{\lambda}{\xi})F_*(W_s)+\tau_s F_*(U_s)\\ +\xi\kappa\tau\kappa_3 F_*(W_{3s}) = 4S_{(\nabla F_*)(W_s,X_s)}F_*(X_s)+2S_{(\nabla F_*)(X_s,X_s)}F_*(W_s).\end{aligned} \tag{3.24}$$

Taking inner product with $F_*(W_s)$ in (3.24), we get

$$\|H_2\|^2 = \frac{1}{2}\left(-\kappa^2-\tau^2+\frac{\lambda}{\xi}\right). \tag{3.25}$$

Because of $\|(\nabla F_*)(X_s,X_s)\| = const. = \|H_2\|$, one can see from (3.25), the torsion τ of γ is a constant, too. □

A unit speed curve γ in a Riemannian manifold M is called an elastic curve (or elastica) if it satisfies (3.1) with (3.2) for $r=2$ [6, 10]. The following theorem gives a characterization of totally umbilical Riemannian map by the behavior of elastic curves along a Riemannian map.

Theorem 3.8 *Let $F: M \to \bar{M}$ be a Riemannian map from a Riemannian manifold M to a Riemannian manifold $\bar{M}$ transporting each horizontal elastic curve with the tangent vector X_s, curvature κ and torsion τ to an elastic curve on $\bar{M}$. Then F is a totally umbilical Riemannian map and the mean curvature vector field H_2 satisfies*

$$(\nabla^{F^\perp}_{X_s})^2 H_2 = \frac{1}{2}\left(\kappa^2-\tau^2+3\lambda+\frac{\kappa_{ss}}{\kappa}\right)H_2, \qquad \|H_2\| = const.. \tag{3.26}$$

Conversely, if F is a totally umbilical Riemannian map, its mean curvature vector satisfies (3.26) and $\nabla^{F^\perp}_{\nabla_{X_s}X_s}H_2 = 2\lambda H_2$, then a horizontal elastic curve in M is an elastic curve in $\bar{M}$.

Proof Suppose that $\gamma(s)$ is a horizontal elastic curve with unit tangent vector field X_s on M. Then the following equation is satisfied

$$(\nabla_{X_s})^3 X_s + R(\nabla_{X_s}X_s, X_s)X_s + \nabla_{X_s}(\lambda X_s) = 0,$$

with (3.2) for $r=2$. Since $\overline{\gamma}$ is an elastic curve with curvature $\bar{\kappa}$ in $\bar{M}$, we have also (3.5) with (3.6) for $r=2$. Proposition 3.3 and Corollary 3.4 show that F is totally umbilical Riemannian map. If (3.12) is rewritten for $r=2$, and taking inner product with $F_*(W_s)$, we get

$$\|H_2\|^2 = \frac{1}{2}\left(\frac{\kappa_{ss}}{\kappa}-\tau^2+\lambda-\kappa^2\right). \tag{3.27}$$

Now we consider for the similar process for the normal part in case of $r=2$. If necessary calculations are taken, we obtain

$$(\nabla^{F^\perp}_{X_s})^2 H_2 = \left(\lambda+\kappa^2+\|H_2\|^2\right)H_2. \tag{3.28}$$

Combining (3.27) and (3.28), we find (3.26).
Conversely, we assume that F is a totally umbilical Riemannian map and H_2 satisfies (3.26). On the other hand, we have

$$(\nabla^F_{X_s})^3 F_*(X_s) = F_*(\nabla^3_{X_s}X_s) - F_*(\nabla_{X_s}\|H_2\|^2 F_*(X_s)) + \lambda H_2. \tag{3.29}$$

If we add $\overline{R}(F_*(\nabla_{X_s}X_s), F_*(X_s))F_*(X_s) + \nabla^F_{X_s}F_*(\lambda X_s)$ both sides of (3.29), then we have from (2.6), (3.4), and (3.21)

$$\begin{aligned}&(\nabla^F_{X_s})^3F_*(X_s)+\overline{R}(F_*(\nabla_{X_s}X_s), F_*(X_s))F_*(X_s) + \nabla^F_{X_s}F_*(\lambda X_s)\\&= F_*(\nabla^3_{X_s}X_s) - 2F_*(\nabla_{X_s}\|H_2\|^2 F_*(X_s)) + F_*(R(\nabla_{X_s}X_s, X_s)X_s)\\&+F_*(\nabla_{X_s}\lambda X_s) + 2\lambda H_2 - \nabla^{F^\perp}_{\nabla_{X_s}X_s}H_2.\end{aligned} \tag{3.30}$$

Since γ is a horizontal elastic curve in M, the tangent part of (3.30) is calculated as

$$F_*(\nabla^3_{X_s}X_s) + F_*(R(\nabla_{X_s}X_s, X_s)X_s) + F_*(\nabla_{X_s}(\lambda - 2\|H_2\|^2)F_*(X_s)) = 0,$$

where $\lambda = \frac{3}{2}\kappa^2 + \bar{b}$, for suitable choosing $\bar{b} = b + 2\|H_2\|^2$ and normal part of (3.30) is found zero if $\nabla^{F^\perp}_{\nabla_{X_s}X_s}H_2 = 2\lambda H_2$. □

Acknowledgment

This paper is supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) with the number 119F025.

References

[1] Arroyo J, Garay OJ, Barros M. Closed free hyperelastic curves in the hyperbolic plane and Chen-Willmore rotational hypersurfaces. Israel Journal of Mathematics 2003; 138: 171-187. doi: 10.1007/BF02783425

[2] Arroyo J, Garay OJ, Mencia JJ. Closed free hyperelastic curves in real space forms. Proceeding of the XII Fall Workshop on Geometry and Physics, Coimbra, 2003; 1-13.

[3] Fischer AE. Riemannian maps between Riemannian manifolds. Contemporary Mathematics 1992; 132: 331-366. doi:10.1090/conm/132/1188447

[4] Garay OJ. Riemannian submanifolds shaped by the bending energy and its allies. Proceedings of The Sixteenth International Workshop on Differential Geometry 2012; 16: 57-70.

[5] Ikawa T. On some curves in Riemannian geometry. Soochow Journal of Mathematics 1981; 7: 37-44.

[6] Langer J, Singer DA. The total squared curvature of closed curves. Journal of Differential Geometry 1984; 20 (1): 1-22. doi:10.4310/jdg/1214438990

[7] Nomizu K, Yano K. On circles and spheres in Riemannian geometry. Mathematische Annalen 1974; 210 (2): 163-170.

[8] O'Neill B. The fundamental equations of a submersion. Michigan Mathematical Journal 1966; 13 (4): 459-469. doi:10.1307/mmj/1028999604

[9] Özkan Tükel G, Turhan T, Yücesan A. Hyperelastic Lie quadratics. Honam Mathematical Journal 2019; 41 (2): 369-380. doi:10.5831/HMJ.2019.41.2.369

[10] Popiel T, Noakes L. Elastica in $SO(3)$. Journal of Australian Mathematical Society 2007; 83: 105-124. doi:10.1017/S1446788700036417

[11] Singer DA. Lectures on elastic curves and rods. In AIP Conference Proceedings American Institute of Physics 2008; 41 (2): 369-380.

[12] Şahin B. Riemannian Submersions, Riemannian Maps in Hermitian Geometry and Their Applications. Elsevier, 2017.

[13] Şahin B, Özkan Tükel G, Turhan T. Hyperelastic curves along immersions. Miskolc Mathematical Notes 2021; 22 (2): 915-927. doi:10.18514/MMN.2021.3501

[14] Özkan Tükel G, Turhan T, Şahin B. Isotropic Riemannian maps and helices along Riemannian maps, arXiv 2105.10119, 2021.