

Inverse nodal problem for Sturm–Liouville operator on a star graph with nonequal edges

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Received: 28.02.2022

Accepted/Published Online: 21.04.2022

Final Version: 04.07.2022

Abstract: In this study, Sturm–Liouville operator was investigated on a star graph with nonequal edges. First, the behaviors of sufficiently large eigenvalues were learned, then the solution of the inverse problem was given to determine the potential functions and parameters of the boundary condition on the star graph with the help of a dense set of nodal points and obtain a constructive solution to the inverse problems of this class.

Key words: Sturm–Liouville operator, nodal problem, star graph

1. Introduction

In this work we consider a star graph G with vertex set $V = \{v_0, v_1, \dots, v_\nu\}$ and edge set $E = \{e_1, e_2, \dots, e_\nu\}$, where v_1, v_2, \dots, v_ν are the boundary vertices, v_0 is the interior vertex and $e_j = [v_j, v_0]$ for $j = \overline{1, \nu}$, positive integer $\nu \geq 2$. We suppose that the length of edge $e_j \in E$ is equal to ℓ_j , where $\ell_j = \ell > 0$, $j = \overline{1, \nu - p}$, and $\ell_j = 2\ell$, $j = \overline{\nu - p + 1, \nu}$. We introduce a parameter x for each edge $e_j \in E$, $x \in [0, \ell_j]$. The following choice of orientation is convenient for us: $x = 0$ corresponds to the boundary vertices v_1, v_2, \dots, v_ν and $x = \ell_j$ to the interior vertex v_0 .

Let $y = \{y_j(x)\}_{j=\overline{1, \nu}}$ be a vector function on G . Consider a second order differential expression

$$(L_j y_j)(x) := -y_j''(x) + q_j(x)y_j(x), \quad x \in [0, \ell_j], j = \overline{1, \nu},$$

$q_j(x)$, $j = \overline{1, \nu}$, are real valued functions from $L_2[0, \ell_j]$. The domain of expressions L_j

$$D(L_j) := \left\{ y_j \in W_2^1[0, \ell_j] : y_j' \in AC[0, \ell_j], L_j y_j \in L_2[0, \ell_j] \right\}, j = \overline{1, \nu}.$$

We study the boundary value problem $L = L(q, p, h)$ for the Sturm–Liouville equations on a star graph G :

$$(L_j y_j)(x) = \lambda y_j(x), \quad x \in [0, \ell_j], j = \overline{1, \nu}, \lambda = s^2 \tag{1.1}$$

with the matching conditions

$$y_1(\ell_1) = y_j(\ell_j), j = \overline{2, \nu}, \sum_{j=1}^{\nu} y_j'(\ell_j) = 0 \tag{1.2}$$

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2000 AMS Mathematics Subject Classification: 34A55, 34B24, 34L05

in the interior vertex v_0 , and the boundary conditions

$$y'_j(0) - h_j y_j(0) = 0, j = \overline{1, \nu} \quad (1.3)$$

in the boundary vertices v_1, \dots, v_ν , where h_j are real, $h = \{h_j\}_{j=\overline{1, \nu}}$, and $q = \{q_j(x)\}_{j=\overline{1, \nu}}$ are called potential functions on a star graph G .

The solution of many problems in different fields of mathematics, mechanics, physics, geophysics, electronics, nanotechnology, natural sciences and engineering is reduced to the problems of studying differential operators on graphs (trees) (see [2, 8, 14–17, 22–24, 27] references there in). Spectral inverse problems for differential operators given on graphs began to be studied more rapidly in the twenty-first century.

In 1988, by bringing a different perspective to this problem, it was reduced to the solution of inverse nodal problems for Sturm–Liouville operators by Hald and McLaughlin [9–12, 21]. In recent years, many studies have been conducted on nodal inverse problems, which is one of the important topics of inverse problems theory, and some of these studies are studies on graphs. The articles [2, 4, 5, 12, 18, 19, 25, 26, 28, 34] can be shown as the basis for such studies.

Inverse nodal problems on a graphs for the Sturm–Liouville operators and Dirac operators and for the diffusion operators are studied in sufficient detail in [5, 29]. The works [6, 33, 34] deal with inverse nodal problems for the Sturm–Liouville operators on star-shaped graphs with equal edges. The work [30] is related to inverse nodal problems on a equal edge graph with loops. The work [29] is concerned with inverse nodal problems for diffusion operators on a star-shaped graph with equal edges. In [13] authors consider inverse nodal problems for Dirac-type integro-differential operators on a star graph with equal edges. If the lengths of edges of graphs are nonequal, the problems will become more difficult (see [3, 20]). In particular, the asymptotic expressions of eigenvalues are difficult to find. For inverse nodal problems on graphs in [7] proved that the specification of the spectrum and the set of all nodal points uniquely determine the potential.

In the present paper unlike other publications we consider inverse nodal problems on a special kind of star graph G with nonequal edges and want to reconstruct the potential only by the dense subset of nodal points.

This paper is organized as follows: in Section 2, we find asymptotical formulas of eigenvalues of the problem L , Section 3 is concerned with the asymptotical formulas of the nodal points and an inverse nodal problem.

2. Properties of the spectrum

Let $C_j(x, s), S_j(x, s), j = \overline{1, \nu}$, be solutions of equations (1.1) under the initial conditions $C_j(0, s) = S'_j(0, s) = 1, C'_j(0, s) = S_j(0, s) = 0$, and denote by $\varphi_j(x, s), j = \overline{1, \nu}$, the solutions of equations (1.1) satisfying initial conditions $\varphi_j(0, s) = 1, \varphi'_j(0, s) = h_j$. For each fixed $x \in [0, \ell_j]$, the functions

$$C_j(x, s), S_j(x, s), C'_j(x, s), S'_j(x, s), \varphi_j(x, s)$$

and $\varphi'_j(x, s), j = \overline{1, \nu}$, are entire in s and

$$\varphi_j(x, s) = C_j(x, s) + h_j S_j(x, s).$$

From [31] one gets the following asymptotical formulas as $|s| \rightarrow \infty$,

$$\varphi_j(x, s) = \cos sx + A_j(x) \frac{\sin sx}{s} + O\left(\frac{e^{|\tau|x}}{s}\right), \tag{2.1}$$

$$\varphi'_j(x, s) = -s \sin sx + A_j(x) \cos sx + O\left(e^{|\tau|x}\right), \tag{2.2}$$

where $\tau = \text{Im } s$, and

$$A_j(x) = h_j + \frac{1}{2} \int_0^x q_j(t) dt \tag{2.3}$$

for $j = \overline{1, \nu}$.

Then the solutions of equations (1.1) satisfying the boundary conditions (1.3) are

$$y_j(x, s) = H_j(s) \varphi_j(x, s), \tag{2.4}$$

where $H_j(s)$ are functions independent of x . Substituting (2.4) into the marching conditions (1.2), we obtain the characteristic function of the problem L .

$$\begin{aligned} \Delta(s) &= \sum_{j=1}^{\nu} \varphi'_j(\ell_j, s) \prod_{j \neq k \in \{1, \dots, \nu\}} \varphi_k(\ell_j, s) = \\ &= \left\{ \sum_{j=1}^{\nu-p} \varphi'_j(\ell, s) \times \prod_{j \neq k \in \{1, \dots, \nu-p\}} \varphi_k(\ell, s) \right\} \times \prod_{k=\nu-p+1}^{\nu} \varphi_k(2\ell, s) + \\ &+ \left\{ \sum_{j=\nu-p+1}^{\nu} \varphi'_j(2\ell, s) \times \prod_{k=1}^p \varphi_k(\ell, s) \right\} \times \prod_{j \neq k \in \{p+1, \dots, \nu\}} \varphi_k(2\ell, s). \end{aligned} \tag{2.5}$$

As defined in [33] if λ_0 is a zero of $\Delta(\lambda)$ ($\lambda = s^2$) then the function $y(x, \lambda_0) = y(x, s_0^2)$ of the form $y(x, \lambda) = \{y_i(x, \lambda)\}_{i=\overline{1, \nu}}$, $y_i(x, \lambda) = A_i(\lambda) \varphi_i(x, \lambda)$ is an eigenfunction and $\lambda = s_0^2$ is an eigenvalue of problem (1.1)–(1.3). Conversely, if λ_0 is an eigenvalue then the corresponding eigenfunction is of the form $y(x, \lambda) = \{y_i(x, \lambda)\}_{i=\overline{1, \nu}}$, $y_i(x, \lambda) = A_i(\lambda) \varphi_i(x, \lambda)$ with $\lambda = \lambda_0$. Since $y(x, \lambda_0) \neq 0$, the algebraic system has a nontrivial solution: consequently, $\Delta(\lambda_0) = 0$.

Let us introduce the auxiliary function

$$\begin{aligned} \Delta_0(s) &= \sum_{j=1}^{\nu} (-s \sin s\ell_j) \prod_{j \neq k \in \{1, \dots, \nu\}} \cos s\ell_k = \\ &= (-s \sin s\ell) (\cos s\ell)^{\nu-p-1} (\cos 2s\ell)^{p-1} \{2\nu \cos^2 s\ell - (\nu - p)\}. \end{aligned} \tag{2.6}$$

Using (2.6) and Rouché method (see [31]), we deduce that the function of $\Delta_0(s)$ has countably many eigenvalues

$$\{t_n^\theta\}_{n \in \mathbb{Z}} = \left(\left\{ s_n^{(01)} \right\}_{n=-\infty}^{n=-0} \cup \left\{ s_n^{(01)} \right\}_{n=0}^{+\infty} \right) \left(\bigcup_{j=2}^{\nu-p} \left\{ s_n^{(0j)} \right\}_{n \in \mathbb{Z}} \right) \left(\bigcup_{j=\nu-p+1}^{\nu+1} \left\{ s_n^{(0j)} \right\}_{n \in \mathbb{Z}} \right),$$

we arrange the zeros in the following way:

$$\begin{aligned} s_{-0}^{(01)} &= s_0^{(01)} = 0, s_n^{(01)} = \frac{n\pi}{\ell}, n \in \mathbb{Z} \setminus \{0\}; \\ s_n^{(02)} &= \frac{n\pi + \theta}{\ell}, s_n^{(03)} = \frac{n\pi - \theta}{\ell}, n \in \mathbb{Z}; \\ s_n^{(0j)} &= \frac{(n + \frac{1}{2})\pi}{\ell}, j = \overline{1, \nu - p}, n \in \mathbb{Z}; \\ s_n^{(0j)} &= \frac{(n + \frac{1}{2})\pi}{2\ell}, j = \overline{\nu - p + 1, \nu + 1}, n \in \mathbb{Z}, \end{aligned}$$

where $\theta = \arccos \sqrt{\frac{\nu - p}{2\nu}}$. The set of zeros of $\Delta(s)$ denote by

$$\{t_n\}_{n \in \mathbb{Z}} = \left(\left\{ s_n^{(1)} \right\}_{n=-\infty}^{-0} \cup \left\{ s_n^{(1)} \right\}_{n=0}^{+\infty} \right) \left(\bigcup_{j=2}^{\nu-p} \left\{ s_n^{(j)} \right\}_{n \in \mathbb{Z}} \right) \left(\bigcup_{j=\nu-p+1}^{\nu+1} \left\{ s_n^{(j)} \right\}_{n \in \mathbb{Z}} \right),$$

we arrange the zeros in the following usual way: (1) $s_{-n}^{(j)} = -s_n^{(j)}$ for all not pure real $s_n^{(j)}$, (2) $\operatorname{Re} s_{n+1}^{(j)} \geq \operatorname{Re} s_n^{(j)}$. Using Lemma 2.1 in [29] we can obtain the following proposition.

Lemma 2.1 *The zeros of $\Delta(s)$ can be enumerated as follows:*

$$t_n = t_n^{(0)} + o(1). \tag{2.7}$$

Moreover, using Lemma 2.1 we can obtain the following result.

Lemma 2.2 *The following asymptotic behavior for the set $\{t_n\}$ of zeros of the function $\Delta(s)$ is true:*

(a)

$$s_n^{(1)} = \frac{n\pi}{\ell} + \frac{\sum_{j=1}^{\nu-p} \eta_j + \sum_{j=\nu-p+1}^{\nu} \eta_j}{(\nu + p)\pi n} + O\left(\frac{1}{n^2}\right), \tag{2.8}$$

(b)

$$s_n^{(2)} = \frac{n\pi + \theta}{\ell} + \frac{1}{n\pi + \theta} \left\{ \frac{p}{\nu^2 - p^2} \left(\sum_{j=1}^{\nu-p} \eta_j \right) + \left[\frac{(p-1)(\nu-p)}{2p\nu} + \frac{\nu^2 - p(p-1)}{2\nu(\nu+p)} \right] \left(\sum_{j=\nu-p+1}^{\nu} \eta_j \right) \right\} + O\left(\frac{1}{n^2}\right), \tag{2.9}$$

(c)

$$s_n^{(2)} = \frac{n\pi - \theta}{\ell} + \frac{1}{n\pi - \theta} \left\{ \frac{p}{\nu^2 - p^2} \left(\sum_{j=1}^{\nu-p} \eta_j \right) + \left[\frac{(p-1)(\nu-p)}{2p\nu} + \frac{\nu^2 - p(p-1)}{2\nu(\nu+p)} \right] \left(\sum_{j=\nu-p+1}^{\nu} \eta_j \right) \right\} + O\left(\frac{1}{n^2}\right), \tag{2.10}$$

(d)

$$s_n^{(j)} = \frac{(n + \frac{1}{2})\pi}{\ell} + \frac{\tau_{j-3}^{(1)}}{(n + \frac{1}{2})\pi} + O\left(\frac{1}{n^2}\right), j = \overline{4, \nu - p + 1}, \tag{2.11}$$

(e)

$$s_n^{(j)} = \frac{(n + \frac{1}{2})\pi}{2\ell} + \frac{2\tau_j^{(2)}}{(n + \frac{1}{2})\pi} + O\left(\frac{1}{n^2}\right), j = \overline{\nu - p + 2, \nu + 1}, \tag{2.12}$$

where $\tau_{j-3}^{(1)}$ and $\tau_j^{(2)}$ are the solutions of the equations

$$P_1(x) := \sum_{i=1}^{\nu-p} \prod_{i \neq k \in \{1, \dots, \nu-p\}} (x - \eta_k), P_2(x) := \sum_{i=\nu-p+1}^{\nu} \prod_{i \neq k \in \{p+1, \dots, \nu\}} (x + \eta_k)$$

respectively, and $\eta_j = h_j + \frac{1}{2} \int_0^{\ell_j} q_j(t) dt$, i.e. $\eta_j = A_j(\ell_j)$ for $j = \overline{1, \nu}$.

Proof (a) Using Rouché’s theorem, for sufficiently large integer $|n|$, $\Delta(s)$ has exactly one zero in a suitable neighborhood of $s_n^{(0)} = \frac{n\pi}{\ell}$, and denote

$$s_n^{(1)} = \frac{n\pi}{\ell} + \beta_n^{(1)}, \tag{2.13}$$

where $\beta_n^{(1)} = o(1)$ as $|n| \rightarrow +\infty$. Substituting (2.13) into $\Delta(s) = 0$, then from (2.5), we have

$$\begin{aligned} \Delta\left(s_n^{(1)}\right) &= \sum_{j=1}^{\nu-p} \left[-\frac{n\pi}{\ell} \sin \beta_n^{(1)} \ell + \eta_j \cos \beta_n^{(1)} \ell + o(1) \right] \times \prod_{j \neq k \in \{1, \dots, \nu-p\}} \left[\cos \beta_n^{(1)} \ell + o\left(\frac{1}{n}\right) \right] \times \\ &\times \prod_{k=\nu-p+1}^{\nu} \left[\cos 2\beta_n^{(1)} \ell + o\left(\frac{1}{n}\right) \right] + \sum_{j=\nu-p+1}^{\nu} \left[-\frac{n\pi}{\ell} \sin 2\beta_n^{(1)} \ell + \eta_j \cos 2\beta_n^{(1)} \ell + o(1) \right] \times \\ &\times \prod_{k=1}^p \left[\cos \beta_n^{(1)} \ell + o\left(\frac{1}{n}\right) \right] \times \prod_{j \neq k \in \{p+1, \dots, \nu\}} \left[\cos 2\beta_n^{(1)} \ell + o\left(\frac{1}{n}\right) \right] = 0. \end{aligned} \tag{2.14}$$

From (2.14), we have $\sin \beta_n^{(1)} \ell = O\left(\frac{1}{n}\right)$. Using the inversion formula, we get

$$\beta_n^{(1)} = \frac{C^{(1)}}{n\ell} + O\left(\frac{1}{n^2}\right), \tag{2.15}$$

where $C^{(1)}$ is a constant. Substituting (2.15) into (2.14) and let $|n| \rightarrow +\infty$, we obtain

$$C^{(1)} = \frac{\ell \left\{ \sum_{j=1}^{\nu-p} \eta_j + \sum_{j=\nu-p+1}^{\nu} \eta_j \right\}}{(\nu+p)\pi}. \tag{2.16}$$

In this case, if we make use of (2.13), (2.15) and (2.16) we get (2.8).

(b) As in case (a) using Rouché's theorem, for sufficiently large integer $|n|$, $\Delta(s)$ has only one zero in a suitable neighborhood of $s_n^{(02)} = \frac{n\pi + \theta}{\ell}$, and denote

$$s_n^{(2)} = \frac{n\pi + \theta}{\ell} + \beta_n^{(2)}, \tag{2.17}$$

where $\beta_n^{(2)} = o(1)$ as $|n| \rightarrow +\infty$. Substituting (2.17) into $\Delta(s) = 0$, then from (2.5), we have

$$\begin{aligned} \Delta\left(s_n^{(2)}\right) &= \sum_{j=1}^{\nu-p} \left[\frac{n\pi + \theta}{\ell} A_1^{(n)} - \eta_j A_2^{(n)} + o(1) \right] \times \prod_{j \neq k \in \{1, \dots, \nu-p\}} \left[A_2^{(n)} + \frac{\ell \eta_k}{n\pi + \theta} A_1^{(n)} + o\left(\frac{1}{n}\right) \right] \times \\ &\times \prod_{k=\nu-p+1}^{\nu} \left[A_3^{(n)} - \frac{\ell \eta_k}{n\pi + \theta} A_4^{(n)} + o\left(\frac{1}{n}\right) \right] + \sum_{j=\nu-p+1}^{\nu} \left[-\frac{n\pi + \theta}{\ell} A_4^{(n)} + \eta_j A_3^{(n)} + o(1) \right] \times \\ &\times \prod_{k=1}^p \left[A_2^{(n)} + \frac{\ell \eta_k}{n\pi + \theta} A_1^{(n)} + o\left(\frac{1}{n}\right) \right] \times \prod_{j \neq k \in \{p+1, \dots, \nu\}} \left[A_3^{(n)} - \frac{2\ell \eta_k}{n\pi + \theta} A_4^{(n)} + o\left(\frac{1}{n}\right) \right] \\ &= 0, \end{aligned} \tag{2.18}$$

where

$$A_1^{(n)} = \sqrt{\frac{\nu+p}{2\nu}} \cos \beta_n^{(2)} \ell + \sqrt{\frac{\nu-p}{2\nu}} \sin \beta_n^{(2)} \ell,$$

$$A_2^{(n)} = \sqrt{\frac{\nu-p}{2\nu}} \cos \beta_n^{(2)} \ell - \sqrt{\frac{\nu+p}{2\nu}} \sin \beta_n^{(2)} \ell,$$

$$A_3^{(n)} = \frac{p}{\nu} \cos 2\beta_n^{(2)} \ell + \frac{\sqrt{\nu^2 - p^2}}{\nu} \sin 2\beta_n^{(2)} \ell,$$

$$A_4^{(n)} = \frac{\sqrt{\nu^2 - p^2}}{\nu} \cos 2\beta_n^{(1)} \ell - \frac{p}{\nu} \sin 2\beta_n^{(2)} \ell,$$

or

$$\begin{aligned}
 & \frac{n\pi + \theta}{\ell} A_2^{(n)} A_3^{(n)} \left[(\nu - p) A_1^{(n)} A_3^{(n)} - p A_2^{(n)} A_4^{(n)} \right] + \\
 & + \left(\sum_{j=1}^{\nu-p} \eta_j \right) A_3^{(n)} \left[(\nu - p - 1) (A_1^{(n)})^2 A_3^{(n)} - (A_2^{(n)})^2 A_3^{(n)} - p A_1^{(n)} A_2^{(n)} A_4^{(n)} \right] - \\
 & - \left(\sum_{j=\nu-p+1}^{\nu} \eta_j \right) A_2^{(n)} \left[(\nu - p) A_1^{(n)} A_3^{(n)} A_4^{(n)} + (p - 1) A_2^{(n)} (A_4^{(n)})^2 + A_2^{(n)} (A_3^{(n)})^2 \right] + o(1) \\
 & = 0.
 \end{aligned} \tag{2.19}$$

If we make use of the identity

$$\begin{aligned}
 (\nu - p) A_1^{(n)} A_3^{(n)} - p A_2^{(n)} A_4^{(n)} &= \sin \beta_n^{(2)} \ell \left[2\nu \sqrt{\frac{\nu - p}{2\nu}} \cos^2 \beta_n^{(2)} \ell + \right. \\
 & \left. + 2p \sqrt{\frac{\nu - p}{2\nu}} \cos 2\beta_n^{(2)} \ell + (\nu - 2p) \sqrt{\frac{\nu + p}{2\nu}} \sin 2\beta_n^{(2)} \ell \right]
 \end{aligned}$$

Equation (2.19) implies

$$\begin{aligned}
 & \frac{n\pi + \theta}{\ell} \left(\sin \beta_n^{(2)} \ell \right) A_2^{(n)} A_3^{(n)} A_5^{(n)} + \\
 & + \left(\sum_{j=1}^{\nu-p} \eta_j \right) A_3^{(n)} \left[(\nu - p - 1) (A_1^{(n)})^2 A_3^{(n)} - (A_2^{(n)})^2 A_3^{(n)} - p A_1^{(n)} A_2^{(n)} A_4^{(n)} \right] - \\
 & - \left(\sum_{j=\nu-p+1}^{\nu} \eta_j \right) A_2^{(n)} \left[(\nu - p) A_1^{(n)} A_3^{(n)} A_4^{(n)} + (p - 1) A_2^{(n)} (A_4^{(n)})^2 + A_2^{(n)} (A_3^{(n)})^2 \right] + o(1) \\
 & = 0
 \end{aligned} \tag{2.20}$$

where

$$A_5^{(n)} = \left[2\nu \sqrt{\frac{\nu - p}{2\nu}} \cos^2 \beta_n^{(2)} \ell + 2p \sqrt{\frac{\nu - p}{2\nu}} \cos 2\beta_n^{(2)} \ell + (\nu - 2p) \sqrt{\frac{\nu + p}{2\nu}} \sin 2\beta_n^{(2)} \ell \right].$$

Using the limits of expressions $A_i^{(n)}, i = \overline{1, 5}$, when $|n| \rightarrow +\infty$, we have

$$\sin \beta_n^{(2)} \ell = O\left(\frac{1}{n\pi + \theta}\right)$$

from (2.20). Using the inversion formula, we get

$$\beta_n^{(2)} = \frac{C^{(2)}}{(n\pi + \theta)\ell} + O\left(\frac{1}{n^2}\right), \tag{2.21}$$

where $C^{(2)}$ is a constant. Substituting (2.21) into (2.20) and let $|n| \rightarrow +\infty$, we obtain

$$C^{(2)} = \frac{p\ell}{\nu^2 - p^2} \left(\sum_{j=1}^{\nu-p} \eta_j \right) + \left[\frac{(p-1)(\nu-p)}{2p\nu} + \frac{\nu^2 - p(p-1)}{2\nu(\nu+p)} \right] \ell \left(\sum_{j=\nu-p+1}^{\nu} \eta_j \right). \tag{2.22}$$

In this case if we make use of (2.17), (2.21) and (2.22) we get (2.9). Similarly, in the case (c) the formula (2.10) can be proved.

(d) Using Rouché’s theorem, for sufficiently large integer $|n|$, there lie exactly $\nu - p - 1$ zeros of $\Delta(s)$ in a suitable neighborhood of

$$s_n^{(0j)} = \frac{(n + \frac{1}{2})\pi}{\ell},$$

and denote

$$s_n^{(j)} = \frac{(n + \frac{1}{2})\pi}{\ell} + \beta_n^{(j)}, \quad j = \overline{4, \nu - p + 1}, \tag{2.23}$$

where $\beta_n^{(j)} = o(1)$ as $|n| \rightarrow +\infty$. Substituting (2.23) into $\Delta(s) = 0$, then from (2.5), we have

$$\begin{aligned} \Delta(s_n^{(j)}) &= \sum_{i=1}^{\nu-p} \left[\left(\frac{(n + \frac{1}{2})\pi}{\ell} + \beta_n^{(j)} \right) \cos \beta_n^{(j)} \ell + \eta_i \sin \beta_n^{(j)} \ell + o(1) \right] \times \\ &\times \prod_{i \neq k \in \{1, \dots, \nu-p\}} \left[\sin \beta_n^{(j)} \ell - \eta_k \frac{\cos \beta_n^{(j)} \ell}{\frac{(n + \frac{1}{2})\pi}{\ell} + \beta_n^{(j)}} + o\left(\frac{1}{n}\right) \right] \times \\ &\times \prod_{k=\nu-p+1}^{\nu} \left[-\cos 2\beta_n^{(j)} \ell - \eta_k \frac{\sin 2\beta_n^{(j)} \ell}{\frac{(n + \frac{1}{2})\pi}{\ell} + \beta_n^{(j)}} + o\left(\frac{1}{n}\right) \right] + \\ &+ \sum_{i=\nu-p+1}^{\nu} \left[\left(\frac{(n + \frac{1}{2})\pi}{\ell} + \beta_n^{(j)} \right) \sin 2\beta_n^{(j)} \ell - \eta_i \cos 2\beta_n^{(j)} \ell + o(1) \right] \times \\ &\times \prod_{k=1}^p \left[-\sin \beta_n^{(j)} \ell - \eta_k \frac{\cos \beta_n^{(j)} \ell}{\frac{(n + \frac{1}{2})\pi}{\ell} + \beta_n^{(j)}} + o\left(\frac{1}{n}\right) \right] \times \\ &\times \prod_{i \neq k \in \{p+1, \dots, \nu\}} \left[-\cos 2\beta_n^{(j)} \ell - \eta_k \frac{\sin 2\beta_n^{(j)} \ell}{\frac{(n + \frac{1}{2})\pi}{\ell} + \beta_n^{(j)}} + o\left(\frac{1}{n}\right) \right] \\ &= 0. \end{aligned} \tag{2.24}$$

Acting as in case (a) from (2.24) we find

$$\sin \beta_n^{(j)} \ell = O\left(\frac{1}{(n + \frac{1}{2})\pi}\right)$$

as $|n| \rightarrow +\infty$. Using the inversion formula, we get

$$\beta_n^{(j)} = \frac{C^{(j)}}{(n + \frac{1}{2})\pi\ell} + O\left(\frac{1}{n^2}\right), \tag{2.25}$$

where $C^{(j)}, j = \overline{4, \nu - p + 1}$ are all constants. Next, substituting (2.25) into (2.24) we have

$$\begin{aligned} & \sum_{i=1}^{\nu-p} \left[1 + o\left(\frac{1}{n}\right)\right] \times \prod_{i \neq k \in \{1, \dots, \nu-p\}} \left[\frac{C^{(j)}}{(n + \frac{1}{2})\pi} - \frac{\ell\eta_k}{(n + \frac{1}{2})\pi} + o\left(\frac{1}{n}\right)\right] \times \prod_{k=\nu-p+1}^{\nu} \left[1 + o\left(\frac{1}{n}\right)\right] + \\ & + \sum_{i=\nu-p+1}^{\nu} \left[\frac{2C^{(j)}}{(n + \frac{1}{2})\pi} - \frac{\ell\eta_i}{(n + \frac{1}{2})\pi} + o\left(\frac{1}{n}\right)\right] \times \prod_{k=1}^p \left[\frac{C^{(j)}}{(n + \frac{1}{2})\pi} - \frac{\ell\eta_k}{(n + \frac{1}{2})\pi} + o\left(\frac{1}{n}\right)\right] \times \\ & \times \prod_{i \neq k \in \{p+1, \dots, \nu\}} \left[1 + o\left(\frac{1}{n}\right)\right] = 0. \end{aligned}$$

Letting $|n| \rightarrow +\infty$ yields

$$\sum_{i=1}^{\nu-p} \prod_{i \neq k \in \{1, \dots, \nu-p\}} (C^{(j)} - \ell\eta_k) = 0, \quad j = \overline{4, \nu - p + 1}. \tag{2.26}$$

Denote

$$\tau_{j-3} = \frac{C^{(j)}}{\ell}, \quad j = \overline{4, \nu - p + 1}, \tag{2.27}$$

$$P_1(x) = \sum_{i=1}^{\nu-p} \prod_{i \neq k \in \{1, \dots, \nu-p\}} (x - \eta_k) dx. \tag{2.28}$$

In view of (2.23), (2.25), (2.26) and (2.28) we arrive at (2.11). Similarly, in the case (e) the formula (2.12) can be proved. □

3. Inverse nodal problem

Let

$$y(x, s) = \{y_j(x, s)\}_{j=\overline{1, \nu}}, y_j(x, s) = D_j\varphi_j(x, s), \tag{3.1}$$

where $D_j, j = \overline{1, \nu}$ are constants depending on s , then the function $y(x, s)$ satisfies (1.1) and (1.3). In this we denote $s_n = s_n^{(2)}$. It follows from (2.9) that

$$s_n = \frac{n\pi + \theta}{\ell} + \frac{M(\nu, p)}{n\pi + \theta} + O\left(\frac{1}{n^2}\right), \tag{3.2}$$

where

$$M(\nu, p) = \frac{p}{\nu^2 - p^2} \left(\sum_{j=1}^{\nu-p} \eta_j\right) + \left[\frac{(p-1)(\nu-p)}{2p\nu} + \frac{\nu^2 - p(p-1)}{2\nu(\nu+p)}\right] \left(\sum_{j=\nu-p+1}^{\nu} \eta_j\right). \tag{3.3}$$

Using asymptotics formulas (2.1) and (3.2), we obtain as $|n| \rightarrow +\infty$, uniformly in $x \in [0, \ell_j]$,

$$\varphi_j(x, s_n) = \cos s_n x + A_j(x) \frac{\sin s_n x}{s_n} + o\left(\frac{1}{n}\right), j = \overline{1, \nu}, \tag{3.4}$$

where

$$A_j(x) = 2h_j - M(\nu, p)x + \int_0^x q_j(t) dt. \tag{3.5}$$

Fix $j = \overline{1, \nu}$ and assume that x_{nj}^k are the zeros (i.e. nodal points) of the functions $\varphi_j(x, s_n)$. Taking (3.4) into account, from equation $\varphi_j(x, s_n) = 0$, we obtain the following asymptotic formula, uniform in j , for the zeros i.e. nodal points as $|n| \rightarrow +\infty$

$$x_{nj}^i = \frac{(i - \frac{1}{2})\pi}{s_n} + \frac{A_j(x_{nj}^i)}{s_n^2} + o\left(\frac{1}{n^2}\right), i \in \mathbb{Z}. \tag{3.6}$$

From (3.2) we have

$$\frac{1}{s_n} = \frac{\ell}{n\pi + \theta} - \frac{\ell^2 M(\nu, p)}{(n\pi + \theta)^3} + o\left(\frac{1}{n^3}\right). \tag{3.7}$$

Putting (3.7) into (3.6), we obtain the following asymptotical formulas for nodes (i.e. nodal points) as $|n| \rightarrow +\infty$ uniformly in $i \in \mathbb{Z}$:

$$x_{nj}^i = \frac{(i - \frac{1}{2})\pi\ell}{n\pi + \theta} + \frac{\ell^2 A_j(x_{nj}^i)}{(n\pi + \theta)^2} - \frac{(i - \frac{1}{2})\pi\ell^2 M(\nu, p)}{(n\pi + \theta)^3} + o\left(\frac{1}{n^3}\right). \tag{3.8}$$

Fix $j = \overline{1, \nu - p}$. It is easy to see from (3.8) that there exists $N_1 \in \mathbb{N}$ such that for all $|n| > N_1$ the function $\varphi_j(x, s_n)$ has exactly $|n|$ (simple) zeros in the interval $(0, \ell)$, namely,

$$0 < x_{nj}^1 < x_{nj}^2 < \dots < x_{nj}^n < \ell \text{ for } n > 0,$$

$$0 < x_{nj}^0 < x_{nj}^{-1} < \dots < x_{nj}^{n+1} < \ell \text{ for } n < 0.$$

Analogously, for the fix $j = \overline{\nu - p + 1, \nu}$ there exists $N_2 \in \mathbb{N}$ such that for all $|n| > N_2$ the function $\varphi_j(x, s_n)$ has exactly $2|n|$ (simple) nodes in the interval $(0, 2\ell)$, i.e.

$$0 < x_{nj}^1 < x_{nj}^2 < \dots < x_{nj}^{2n} < 2\ell \text{ for } n > 0,$$

$$0 < x_{nj}^0 < x_{nj}^{-1} < \dots < x_{nj}^{2n+1} < 2\ell \text{ for } n < 0.$$

The sets

$$X_j = \{x_{nj}^i : n > N_1, i = \overline{1, n}\} \cup \{x_{nj}^i : n < -N_1, i = \overline{0, n+1}\}, j = \overline{1, \nu - p}$$

and

$$X_j = \{x_{nj}^i : n > N_2, i = \overline{1, 2n}\} \cup \{x_{nj}^i : n < -N_2, i = \overline{0, 2n+1}\}, j = \overline{\nu - p + 1, \nu}$$

are called nodes sets on the e_j ($j = \overline{1, \nu - p}$) and e_j ($j = \overline{\nu - p + 1, \nu}$), respectively. Clearly, the set X_j is dense in $(0, \ell_j)$ for $j = \overline{1, \nu}$.

Theorem 3.1 Fix $j = \overline{1, \nu}$ and $x \in [0, \ell_j]$ suppose that $\{x_{nj}^i\} \subset X_j$ are chosen such that $\lim_{|n| \rightarrow +\infty} x_{nj}^i = x$.

Then there exists a finite limit

$$g_j(x) := \lim_{|n| \rightarrow +\infty} \left(x_{nj}^i - \frac{(i - \frac{1}{2})\pi\ell}{n\pi + \theta} \right) \frac{(n\pi + \theta)^2}{\ell^2}, \tag{3.9}$$

and

$$g_j(x) = A_j(x) - \pi M(\nu, p)x, \tag{3.10}$$

where $A_j(x)$ and $M(\nu, p)$ are defined by (2.3) and (3.3), respectively.

Proof If we use the asymptotical formula (3.8), we get that

$$\left(x_{nj}^i - \frac{(i - \frac{1}{2})\pi\ell}{n\pi + \theta} \right) \frac{(n\pi + \theta)^2}{\ell^2} = A_j(x_{nj}^i) - \frac{(i - \frac{1}{2})\pi M(\nu, p)}{n\pi + \theta} + o(1). \tag{3.11}$$

Since,

$$\lim_{|n| \rightarrow +\infty} x_{nj}^i = x, \quad \lim_{|n| \rightarrow +\infty} \frac{(i - \frac{1}{2})\pi\ell}{n\pi + \theta} = x$$

and

$$\lim_{|n| \rightarrow +\infty} A_j(x_{nj}^i) = A_j(x),$$

from this and (3.11), we conclude that as $|n| \rightarrow +\infty$ the limit of left hand side of (3.11) exist, and (3.10) holds. Theorem 3.1 is proved. □

Let us now state a uniqueness theorem and present a constructive procedure for solving the inverse nodal problem.

Theorem 3.2 Let $X_j^0 \subset X_j$ be a subset of nodes which is dense $(0, \ell_j)$ for $j = \overline{1, \nu}$. Then, the specification of $\bigcup_{j=1}^{\nu} X_j^0$ uniquely determines the potential $q_j(x) - \langle q \rangle$ a.e. on $(0, \ell_j)$ and the coefficients $h_j, j = \overline{1, \nu}$ of the boundary conditions. The potentials $q_j(x) - \langle q \rangle$ and the numbers h_j can be constructed via the following algorithm:

1. For $j = \overline{1, \nu}$, and each $x \in [0, \ell_j]$, we choose a sequence $\{x_{nj}^i\} \subset X_j^0$ such that $\lim_{|n| \rightarrow +\infty} x_{nj}^i = x$:

2. From (3.9), we find the function $g_j(x)$ and calculate values for $g_j(x)$ at $x = 0$, i.e.

$$h_j = g_j(0), j = \overline{1, \nu}. \tag{3.12}$$

3. The functions $q_j(x)$ can be determined as

$$q_j(x) - \langle q \rangle = 2g'_j(x) + \frac{\pi p}{\nu^2 - p^2} \sum_{j=1}^{\nu-p} h_j + \frac{\pi}{2} \left[\frac{(p-1)(\nu-p)}{p\nu} + \frac{\nu^2 - p(p-1)}{\nu(\nu+p)} \right] \sum_{j=\nu-p+1}^{\nu} h_j, \tag{3.13}$$

where

$$\langle q \rangle := \frac{\pi p}{\nu^2 - p^2} \sum_{j=1}^{\nu-p} \int_0^\ell q_j(t) dt + \frac{\pi}{2} \left[\frac{(p-1)(\nu-p)}{p\nu} + \frac{\nu^2 - p(p-1)}{\nu(\nu+p)} \right] \sum_{j=\nu-p+1}^{\nu} \int_0^{2\ell} q_j(t) dt.$$

Proof Formulas (3.12) and (3.13) can be derived from (3.10) step by step. We obtain the following reconstruction procedure:

- i) Taking value for $g_j(x)$ at $x = 0$, then it yields $h_j = g_j(0)$, $j = \overline{1, \nu}$.
- ii) After h_j are reconstructed, on takes derivatives of the function $g_j(x)$ we have (3.13). □

4. Example

Let graph G with vertex set $V = \{v_0, v_1, v_2\}$ and edge set $E = \{e_1, e_2\}$, where v_1, v_2 are the boundary vertices, v_0 is the interior vertex and $e_j = [v_j, v_0]$ for $j = \overline{1, 2}$. We suppose that the length of edge $e_1, e_2 \in E$ is equal to ℓ_1 and ℓ_2 ($\ell_1 \neq \ell_2$) respectively. In this case,

$$\Delta_0(s) = -s \sin s(\ell_1 + \ell_2).$$

From here,

$$s_n^0 = \frac{\pi n}{\ell_1 + \ell_2}, n \in \mathbb{Z}.$$

Next, it follows from Equation (2.5) that,

$$\Delta(s) = -s \sin s(\ell_1 + \ell_2) + (A_1 + A_2) \cos s(\ell_1 + \ell_2) + o(1).$$

From $\Delta(s) = 0$, we have

$$s_n = \frac{\pi n}{\ell_1 + \ell_2} + \frac{(A_1 + A_2)(\ell_1 + \ell_2)}{n\pi} + O\left(\frac{1}{n^2}\right).$$

Fix $j = \overline{1, 2}$ and assume that x_{nj}^k are the zeros of the functions $\varphi_j(x, s_n)$. Taking (3.4) in to account, from equation $\varphi_j(x, s_n) = 0$, we obtain the following asymptotic formula, uniform in j , for the zeros as $|n| \rightarrow +\infty$

$$x_{nj}^k = \frac{(k - \frac{1}{2})(\ell_1 + \ell_2)}{n} + \frac{A_j(x_{nj}^k)(\ell_1 + \ell_2)^2}{n^2\pi^2} - \frac{(k - \frac{1}{2})(\ell_1 + \ell_2)^2(A_1 + A_2)}{n^3\pi^2} + o\left(\frac{1}{n^2}\right).$$

Denote

$$g_j(x) := \lim_{|n| \rightarrow +\infty} \left(x_{nj}^k - \frac{(k - \frac{1}{2})(\ell_1 + \ell_2)}{n} \right) \frac{n^2\pi^2}{(\ell_1 + \ell_2)^2}.$$

then

$$g_j(x) = A_j(x) - \frac{(A_1 + A_2)x}{(\ell_1 + \ell_2)}. \tag{4.1}$$

1. For $j = 1, 2$, and each $x \in [0, \ell_j]$, we choose a sequence $\{x_{nj}^k\} \subset X_j^0$ such that

$$\lim_{|n| \rightarrow +\infty} x_{nj}^k = x;$$

2. From (4.1), we find the function $g_j(x)$ and calculata values for $g_j(x)$ at $x = 0$, i.e.

$$h_j = g_j(0), \quad j = 1, 2.$$

3. The functions $q_j(x)$ can be determined as

$$q_j(x) - \langle q \rangle = 2g_j'(x) + \frac{2(h_1 + h_2)}{(\ell_1 + \ell_2)}, \quad j = 1, 2,$$

where

$$\langle q \rangle = \frac{1}{(\ell_1 + \ell_2)} \left[\int_0^{\ell_1} q_1(t) dt + \int_0^{\ell_2} q_2(t) dt \right].$$

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