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Global differential invariants of nondegenerate hypersurfaces

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Abstract: Let $\{g_{ij}(x)\}_{i,j=1}^{n}$ and $\{L_{ij}(x)\}_{i,j=1}^{n}$ be the sets of all coefficients of the first and second fundamental forms of a hypersurface x in R^{n+1}. For a connected open subset $U \subset R^n$ and a C^∞-mapping $x : U \to R^{n+1}$ the hypersurface x is said to be d-*nondegenerate*, where $d \in \{1, 2, \dots n\}$, if $L_{dd}(x) \neq 0$ for all $u \in U$. Let $M(n) = \{F : R^n \longrightarrow R^n \mid Fx = gx + b,\ g \in O(n),\ b \in R^n\}$, where $O(n)$ is the group of all real orthogonal $n \times n$-matrices, and $SM(n) = \{F \in M(n) \mid g \in SO(n)\}$, where $SO(n) = \{g \in O(n) \mid \det(g) = 1\}$. In the present paper, it is proved that the set $\{g_{ij}(x), L_{dj}(x), i, j = 1, 2, \dots, n\}$ is a complete system of a $SM(n+1)$-invariants of a d-non-degenerate hypersurface in R^{n+1}. A similar result has obtained for the group $M(n+1)$.

Key words: Hypersurface, Bonnet's theorem, differential invariant

1. Introduction

Let R be the field of real numbers, $n > 1$ a natural number and U a connected open subset of R^n. In what follows, a C^∞-mapping $x : U \to R^{n+1}$ will be called a *parametric* U-*hypersurface* (*hypersurface*, for short) in R^{n+1}. Let $g(x) = \sum_{i,j=1}^{n} g_{ij}(x) du_i du_j$ and $L(x) = \sum_{i,j=1}^{n} L_{ij}(x) du_i du_j$ be the first and second fundamental forms of a hypersurface $x(u) = x(u_1, u_2, \dots, u_n)$. The hypersurface x is said to be *regular* if $\delta_x := \det \|g_{ij}(x(u))\|_{i,j=1}^{n} \neq 0$ for all $u \in U$; let $H_{reg}(n)$ be the set of regular hypersurfaces in R^{n+1}. The hypersurface x is said to be d-*nondegenerate*, where $d \in \{1, 2, \dots n\}$, if $L_{dd}(x) \neq 0$ for all $u \in U$. Every d-nondegenerate hypersurface is regular for all $d \in \{1, 2, \dots n\}$, see Proposition 3.39 below.

Let $M(n) = \{F : R^n \longrightarrow R^n \mid Fx = gx + b,\ g \in O(n),\ b \in R^n\}$, where $O(n)$ is the group of all real orthogonal $n \times n$-matrices, and $SM(n) = \{F \in M(n) \mid g \in SO(n)\}$, where $SO(n) = \{g \in O(n) \mid \det(g) = 1\}$.

Let N be the set of all non-negative integers and

$$T = \{(i, j) \in N \times N \mid 1 \leq i \leq j \leq n\}.$$

By Bonnet's theorem ([2], p.49; [13], p.151, [11], p.71), if $x(u)$ and $y(u)$ are regular hypersurfaces such that

$$g_{ij}(x)(u) = g_{ij}(y)(u) \quad \text{and} \quad L_{ij}(x)(u) = L_{ij}(y)(u)$$

for all $(i, j) \in T$ and $u \in U$ then there exists an $F \in SM(n+1)$ such that $y(u) = Fx(u)$ for all $u \in U$.

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Bonnet's theorem is local; below we prove the following global Theorem 4.6: *For* $d \in \{1, 2, \dots n\}$ *fixed, let* $x(u), y(u)$ *be* d*-nondegenerate* U*-hypersurfaces in* R^{n+1} *such that equalities* $g_{ij}(x) = g_{ij}(y)$ *and* $L_{ds}(x) = L_{ds}(y)$ *hold for all* i, j, s *such that* $1 \leq i, j, s \leq n$, $i \leq j$ *and* $u \in U$. *Then there exist the unique* $g \in SO(n+1)$ *and* $b \in R^{n+1}$ *such that* $y = gx + b$.

Remark 1.1 *The number of elements in the complete system in Theorem 4.6 is* $\frac{1}{2}n(n+1) + n$, *whereas the number of elements in the complete system in Bonnet's theorem is* $n(n+1)$.

It is well-known that the coefficients of the first and second fundamental forms are not independent and their relations are subject to the Gauss-Codacci equations. Therefore the following problem is natural (see [3], p.21): *Let* $x, y \in H_{reg}(n)$. *Is there a proper subset* T_1 *of* T *such that equalities* $g_{ij}(x)(u) = g_{ij}(y)(u)$ *and* $L_{ij}(x)(u) = L_{ij}(y)(u)$ *for all* $(i,j) \in T_1$ *and* $\in U$ *imply existence of an* $F \in SM(n+1)$ *such that* $y(u) = Fx(u)$ *for all* $u \in U$? If the answer is negative the system $\{g_{ij}, L_{ij} \mid (i,j) \in T\}$ is called a *minimal complete system of* $SM(n+1)$*-invariants.*

In this paper we also give other complete systems of G-invariants of d-nondegenerate hypersurfaces for $G = SM(n+1)$ and complete systems of G-invariants of d-nondegenerate hypersurfaces for $G = M(n+1)$. Still other complete systems of invariants of hypersurfaces are investigated in works [1]; [3], p.21; [11]. If $n = 2$ and parameters $u_i = k_i$ for $i = 1, 2$ are principal curvatures of x, then there exists a complete system of differential invariants of a hypersurface $x(u_1, u_2)$ with 4 elements (see [1], p.39). Lemma 15.6 in [5], p.347, implies that the system of all coefficients of the first fundamental form of the hypersurface is not complete.

The paper is organized as follows. In Section 2, we give evident forms of coefficients L_{ij} of the second fundamental form L of the regular hypersurface in R^{n+1} (Corollary 2.5); this is used later on.

Let $G = SM(n+1)$ or $G = M(n+1)$. In Section 3, we give the definition of the differential field $R < x >^G$ of all G-invariant differential rational functions of the hypersurface x and the definition of the differential algebra $R\left\{x, \Delta_d^{-1}\right\}^G$ of all G-invariant differential polynomial functions of the hypersurface x and the function Δ_d^{-1}, where $\Delta_d := \det \|(y_i, z_j)\|_{i,j=1}^{n+1}$, and $y_1 = z_1 = \frac{\partial x}{\partial u_1}$, ..., $y_n = z_n = \frac{\partial x}{\partial u_n}$, $y_{n+1} = z_{n+1} = \frac{\partial^2 x}{\partial u_d^2}$.

In Theorems 3.5–3.40, we obtain descriptions of some generating systems of the differential field $R < x >^G$ and the differential algebra $R\left\{x, \Delta_d^{-1}\right\}^G$. These generating systems of G-invariants are useful for a description of complete systems of G-invariants of a hypersurface.

In Section 4, we obtain complete systems of G-invariant differential rational functions of the d-nondegenerate hypersurface for groups $G = SM(n+1)$ and $M(n+1)$, see Theorems 4.2–4.6.

Formulations of theorems and proofs of results in Section 3 and Section 4 are given for the case $d = 1$: for $d \in \{2, \dots, n\}$ they are similar. In what follows, $n > 1$. The case $n = 1$ is easily considered. Proofs that directly follow from definitions are omitted.

The results of the present paper give rise to the following problems:

(1) Which of the systems Eq. (4.1), Eq. (4.3), Eq. (4.4) is a minimal complete system?

(2) Describe a complete system of relations between the elements of every complete system of Eq. (4.1), Eq. (4.3), Eq. (4.4).

2. Coefficients of the second fundamental form of a given regular hypersurface

Let $(x,y)=x_1y_1+\cdots+x_{n+1}y_{n+1}$ be the inner product of two vectors in R^{n+1}. Denote by $\det Gr(a_1,\dots,a_n)$ the determinant of the Gram matrix $\|(a_k,a_l)\|_{k,l=1}^{n}$ of the vectors $a_i \in R^{n+1}$. Let $\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{n+1}\}$, where $\varepsilon_1=(1,0,\dots,0,0),\dots,\varepsilon_{n+1}=(0,0,\dots,0,1)$, be an orthonormal basis in R^{n+1}; let $\{a_1,a_2,\dots,a_n\}$ be a set of vectors in R^{n+1}. We consider $a_j=(a_{1j},a_{2j},\dots,a_{n+1,j})^\top$ and $\varepsilon=\{\varepsilon_1,\dots,\varepsilon_{n+1}\}^\top$, where $^\top$ is the transposition operation, as column-vectors. Set $P_k:=\det\|a_{ij}\|_{i=1,\dots,k-1,k+1,\dots,n+1;j=1,2,\dots,n}$ and $A_k=(-1)^{1+k}P_k$ for $k=1,2,\dots,n+1$. Let $[\varepsilon a_1a_2\dots a_n]:=A_1\varepsilon_1+A_2\varepsilon_2+\cdots+A_{n+1}\varepsilon_{n+1}\in R^{n+1}$.

Proposition 2.1 *We have*

$$([\varepsilon a_1a_2\dots a_n],[\varepsilon a_1a_2\dots a_n])=\det Gr(a_1,a_2,\dots,a_n). \tag{2.1}$$

Remark 2.2 *For $n=2$, the equality Eq.(2.1) follows from the Extended Lagrange Identity (see [6], p. 148).*

For any set of vectors $\{b,a_1,a_2,\dots,a_n\}$ in R^{n+1}, denote $[ba_1a_2\dots a_n]:=\det\|ba_1a_2\dots a_n\|$.

Proposition 2.3 *Let $\{a_1,a_2,\dots,a_n\}$ be a set of linearly independent vectors in R^{n+1}. Then the vector $\overline{n}=\frac{[\varepsilon a_1a_2\dots a_n]}{\sqrt{\det Gr(a_1,a_2,\dots,a_n)}}$ is a unit vector and*

$$([\varepsilon a_1a_2\dots a_n],a_j)=0$$

for all $j=1,2,\dots,n$.

Proof. Since the vectors $\{a_1,a_2,\dots,a_n\}$ are linearly independent, we obtain $\det Gr(a_1,a_2,\dots,a_n)\neq 0$. Hence, by Proposition 2.1, $\overline{n}$ is a unit vector. The equality $([\varepsilon a_1a_2\dots a_n],a_j)=[a_ja_1a_2\dots a_n]$ is obvious. Since $\|a_ja_1a_2\dots a_n\|$ has two equal columns, $[a_ja_1a_2\dots a_n]=0$. □

Proposition 2.4 *For any set $\{a_1,\dots,a_n\}$ of linearly independent vectors in R^{n+1} and $b\in R^{n+1}$, we have $(\overline{n},b)=\frac{[ba_1a_2\dots a_n]}{\sqrt{\det Gr(a_1,a_2,\dots,a_n)}}$.*

Corollary 2.5 *Let $x(u_1,u_2,\dots,u_n)$ be a regular hypersurface in R^{n+1}. Then the coefficients of the second fundamental form of x are*

$$L_{ij}(x)=\left[\frac{\partial^2 x}{\partial u_i\partial u_j}\frac{\partial x}{\partial u_1}\ \frac{\partial x}{\partial u_2}\ \dots\frac{\partial x}{\partial u_n}\right]\delta_x^{-\frac{1}{2}} \quad \textit{for any}\ \ i,j=1,2,\dots,n. \tag{2.2}$$

Proof. It follows from the definition of the coefficients $L_{ij}(x)=(\overline{n},\frac{\partial^2 x}{\partial u_i u_j})$ (see [2], p. 32]) and Proposition 2.4. □

Remark 2.6 *For $n=2$, Eq.(2.2) is known ([12], p. 80). For hypersurfaces given explicitly $x=x(u_1,\dots,u_n)$, where $n\geq 2$, Eq.(2.2) is given in ([2], p. 36).*

3. Generating systems of some differential algebras of G-invariant differential rational functions of the nondegenerate hypersurface for groups $G = M(n+1)$ and $G = SM(n+1)$

Below we use some notions and notation from the differential algebra, see [7–10].

Definition 3.1 *(See [7, 10]) Let $x(u) = x(u_1, u_2, \dots, u_n)$ be a U-hypersurface in R^{n+1}. For any $m_i \in N$ and $i = 1, 2, \dots, n$, we set*

$$x^{(0,0,\dots,0)} = x, \quad x^{(m_1,m_2,\dots,m_n)} = \frac{\partial^{m_1+m_2+\cdots+m_n} x}{\partial u_1^{m_1} u_2^{m_2} \cdots \partial u_n^{m_n}}.$$

Any polynomial $p(x, x^{(1,0,\dots,0,0)}, x^{(0,1,\dots,0,0)}, \dots, x^{(m_1,m_2,\dots,m_{n-1},m_n)})$ of x and a finite number of partial derivatives of x with coefficients in R is called a differential polynomial of x and briefly denoted by $p\{x\}$.

The set of all differential polynomials of x will be denoted by $R\{x\}$. It is a differential R-algebra with respect to the derivations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}$. This differential R-algebra is also an integral domain. The quotient field of it will be denoted by $R < x >$. It is a differential field with respect to the derivations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}$. An element h of $R < x >$ will be called a *differential rational function* of x and denoted by $h < x >$.

Let $x = x(u), y = y(u), \dots z = z(u)$ be a finite number of U-hypersurfaces in R^{n+1} and $f_1, f_2, \dots, f_m \in R < x, y, \dots, z >$. A *differential polynomial* of $x, y, \dots z, f_1, f_2, \dots, f_m$ is similarly defined; and hence it will be denoted by $p\{x, y, \dots, z, f_1, \dots, f_m\}$. The *differential R-algebra of all differential polynomials* of $x, y, \dots z, f_1, \dots, f_m$ is denoted by $R\{x, y, \dots z, f_1, \dots, f_m\}$. The *differential field of all differential rational functions* of $x, y, \dots z, f_1, f_2, \dots, f_m$ is denoted by $R < x, y, \dots z, f_1, \dots, f_m >$.

Clearly, the set $Fx(u)$ is a U-hypersurface in R^{n+1} for any U-hypersurface $x(u)$ in R^{n+1} and $F \in M(n+1)$.

Definition 3.2 *A differential rational function $h < x, y, \dots, z, f_1, \dots, f_m >$ will be called G-invariant, where G is a subgroup of $M(n+1)$, if for all $g \in G$ we have*

$$\begin{aligned} &h < gx, gy, \dots, gz, f_1 < gx, gy, \dots, gz >, \dots, f_m < gx, gy, \dots, gz >>= \\ &h < x, y, \dots, z, f_1 < x, y, \dots, z >, \dots, f_m < x, y, \dots, z >> . \end{aligned}$$

The set of all G-invariant differential rational functions of hypersurfaces
$x, y, \dots, z$ and functions $f_1, \dots, f_m$ will be denoted by

$$R < x, y, \dots, z, f_1, \dots, f_m >^G$$

It is a differential subfield of $R < x, y, \dots, z, f_1, \dots, f_m >$. The set of all G-invariant differential polynomial functions of $x, y, \dots, z$ and $f_1, \dots, f_m$ will be denoted by $R\{x, y, \dots, z, f_1, \dots, f_m\}^G$. It is a differential subalgebra of the differential algebra $R\{x, y, \dots, z, f_1, \dots, f_m\}$ and the differential field $R < x, y, \dots, z, f_1, \dots, f_m >^G$.

Definition 3.3 *Let K be a differential subfield of $R < x, y, \dots, z >$. A subset S of K is a generating system of the differential field K if the smallest differential subfield of it containing S is K.*

Definition 3.4 *Let $f_1, \dots, f_m \in R < x, y, \dots, z >$ and K be a differential R-subalgebra of $R\{x, y, \dots, z, f_1, \dots, f_m\}$. A subset S of K will be called a generating system of the differential algebra K if the smallest differential subalgebra of it containing S is K.*

Let $R\left\{x, \Delta_d^{-1}\right\}^G$ be the differential algebra of all G-invariant differential polynomial functions of a hypersurface x and the function Δ_d^{-1}. We note that $(x^{(m_1,m_2,\ldots,m_n)}, x^{(p_1,p_2,\ldots,p_n)})$ are $M(n+1)$-invariant functions. Hence functions Δ_d and Δ_d^{-1} are $M(n+1)$-invariant. In what follows, $\Delta := \Delta_1$; we investigate properties of the differential algebra $R\left\{x, \Delta_d^{-1}\right\}^G$ for $d = 1$; the other d being similar.

Theorem 3.5 *The set of elements*

$$\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right) \text{ for } 1 \le i \le j \le n; \quad \Delta^{-1}; \quad \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_r}\right) \text{ for } 1 \le r \le n, \tag{3.1}$$

is a generating system of the differential algebra $R\left\{x, \Delta^{-1}\right\}^{M(n+1)}$.

Proof. Let $R\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}, \Delta^{-1}\right\}$ be the differential algebra of all differential polynomial functions of $\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}$, Δ^{-1} and $R\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}, \Delta^{-1}\right\}^G$ be the differential algebra of all G-invariant differential polynomial functions of $\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}$, Δ^{-1}.

Lemma 3.6 $R\left\{x, \Delta^{-1}\right\}^{M(n+1)} = R\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}, \Delta^{-1}\right\}^{O(n+1)}$.

Proof. It is similar to the proof of Lemma 1 in [8]. □

Lemma 3.7 *The set of elements*

$$\left\{(x^{(m_1,\ldots,m_n)}, x^{(p_1,\ldots,p_n)}) \mid \sum_{i=1}^{n} m_i \ge 1, \sum_{i=1}^{n} p_i \ge 1, m_i, p_i \in \mathbb{N}\right\} \tag{3.2}$$

is a generating system of the differential algebra $R\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}\right\}^{O(n+1)}$.

Proof. It is similar to the proof of Lemma 3 in [8]. □

Lemma 3.8 *The set*
$\left\{(x^{(m_1,\ldots,m_n)}, x^{(p_1,\ldots,p_n)}), \Delta^{-1} \mid \sum_{i=1}^{n} m_i \ge 1, \sum_{i=1}^{n} p_i \ge 1, m_i, p_i \in \mathbb{N}\right\}$
is a generating system of the differential algebra $R\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}, \Delta^{-1}\right\}^{O(n+1)}$.

Proof. Let $f \in R\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}, \Delta^{-1}\right\}^{O(n+1)}$. Then f can be written in the form $f = \frac{h\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}\right\}}{\Delta^m}$, where $h\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}\right\} \in R\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}\right\}$ and $m \in N$. Let $g \in O(n+1)$. Since f is $O(n+1)$-invariant, we have $\frac{h\left\{\frac{\partial (gx)}{\partial u_1}, \ldots, \frac{\partial (gx)}{\partial u_n}\right\}}{\Delta(gx)^m} = \frac{h\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}\right\}}{\Delta(x)^m}$. Since $\Delta(x)$ is $M(n+1)$-invariant, we have $h\left\{\frac{\partial (gx)}{\partial u_1}, \ldots, \frac{\partial (gx)}{\partial u_n}\right\} = h\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n}\right\}$ for all $g \in O(n+1)$ that is $h \in R\left\{\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_n}\right\}^{O(n+1)}$. Now Lemma 3.7 implies Lemma 3.8. □

Let $V := \left\{ \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right) \text{ for } 1 \le i \le j \le n; \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_r} \right) \text{ for } 1 \le r \le n \right\}$ and $R\{V\}$ be the differential R-subalgebra of $R\left\{ \frac{\partial x}{\partial u_1}, \dots \frac{\partial x}{\partial u_n}, \Delta^{-1} \right\}^{O(n+1)}$ generated by V. Denote by $R\{V, \Delta^{-1}\}$ the differential R-subalgebra of

$R\left\{ \frac{\partial x}{\partial u_1}, \dots, \frac{\partial x}{\partial u_n}, \Delta^{-1} \right\}^{O(n+1)}$ generated by elements of V and the function Δ^{-1}. According to Lemma 3.6 and Lemma 3.8, for a proof of our theorem, it is enough to prove that $(x^{(m_1, m_2, \dots, m_n)}, x^{(p_1, p_2, \dots, p_n)}) \in R\{V, \Delta^{-1}\}$ for all $m_i, p_i \in N$ such that $m_1 + m_2 + \dots + m_n \ge 1$ and $p_1 + p_2 + \dots + p_n \ge 1$.

Let $V_0 := \left\{ \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right) \mid i, j \text{ such that } 1 \le i \le j \le n \right\}$, and $R\{V_0\}$ be the differential R-subalgebra of $R\left\{ \frac{\partial x}{\partial u_1}, \dots, \frac{\partial x}{\partial u_n}, \Delta^{-1} \right\}^{O(n+1)}$ generated by elements of V_0. Since $V_0 \subset V$, it follows that $R\{V_0\} \subset R\{V\}$.

Lemma 3.9 *We have* $(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_l}) \in R\{V_0\}$ *for all* $i, j, l \in \{1, 2, \dots, n\}$.

Proof. For all $i, j \in \{1, \dots, n\}$, we have $\frac{\partial}{\partial u_j} \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_i} \right) = 2 \left(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_i} \right)$. This equality implies that $(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_i}) \in R\{V_0\}$ for all i, j such that $1 \le i, j \le n$. Using the fact that $(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_i}) \in R\{V_0\}$ and the equality $\frac{\partial}{\partial u_i}(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}) = (\frac{\partial^2 x}{\partial^2 u_i}, \frac{\partial x}{\partial u_j}) + (\frac{\partial x}{\partial u_i}, \frac{\partial^2 x}{\partial u_i \partial u_j})$, we obtain $(\frac{\partial^2 x}{\partial u_i^2}, \frac{\partial x}{\partial u_j}) \in R\{V_0\}$ for all i, j such that $1 \le i, j \le n$. Assume that $i \ne j, i \ne l, j \ne l$. We have

$$\begin{cases} \frac{\partial}{\partial u_j} \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_l} \right) = \left(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_l} \right) + \left(\frac{\partial x}{\partial u_i}, \frac{\partial^2 x}{\partial u_j \partial u_l} \right), \\ \frac{\partial}{\partial u_i} \left(\frac{\partial x}{\partial u_j}, \frac{\partial x}{\partial u_l} \right) = \left(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_l} \right) + \left(\frac{\partial x}{\partial u_j}, \frac{\partial^2 x}{\partial u_i \partial u_l} \right), \\ \frac{\partial}{\partial u_l} \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right) = \left(\frac{\partial^2 x}{\partial u_i \partial u_l}, \frac{\partial x}{\partial u_j} \right) + \left(\frac{\partial x}{\partial u_i}, \frac{\partial^2 x}{\partial u_j \partial u_l} \right). \end{cases} \tag{3.3}$$

Put $\frac{\partial}{\partial u_j}(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_l}) = b_1$, $\frac{\partial}{\partial u_i}(\frac{\partial x}{\partial u_j}, \frac{\partial x}{\partial u_l}) = b_2$, $\frac{\partial}{\partial u_l}(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}) = b_3$, $(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_l}) = w_1$, $(\frac{\partial^2 x}{\partial u_j \partial u_l}, \frac{\partial x}{\partial u_i}) = w_2$, $(\frac{\partial^2 x}{\partial u_i \partial u_l}, \frac{\partial x}{\partial u_j}) = w_3$. Then the system Eq.(3.2) takes the following form

$$w_1 + w_2 = b_1, \quad w_1 + w_3 = b_2, \quad w_2 + w_3 = b_3.$$

As the system of equations for w_1, w_2, w_3 this system has the unique solution (w_1, w_2, w_3), where $w_1 = \frac{1}{2}(b_1 + b_2 - b_3) \in R\{V_0\}$, $w_2 = \frac{1}{2}(b_1 + b_3 - b_2) \in R\{V_0\}$, $w_3 = \frac{1}{2}(b_2 + b_3 - b_1) \in R\{V_0\}$. □

Lemma 3.10 $\Delta \in R\{V\}$.

Proof. By the definition of Δ, we have $\Delta = \det \|(y_i, z_j)\|_{i,j=1}^{n+1}$, see Introduction. The definition of V implies that $(y_i, z_j) = \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right) \in V$ for all $1 \le i, j \le n$ and $(y_{n+1}, z_{n+1}) = (\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2}) \in V$. For $i = n+1$ and any $1 \le j \le n$ or $j = n+1$ and any $1 \le i \le n$, we have $(y_i, z_j) = (\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_j})$ or $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_i})$, respectively. By Lemma 3.9, $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_i}) \in R\{V\}$ for all $1 \le i \le n$. Hence $\Delta \in R\{V\}$. □

Let $\det Gr(y_1,\dots,y_m;z_1,\dots,z_m) := \det ||(y_i,z_j)||_{i,j=1}^{m}$, where $y_1,\dots,y_m \in R^{n+1}$ and $z_1,\dots,z_m \in R^{n+1}$.

Lemma 3.11 ([14], p. 75) *For all vectors* $y_1,y_2,\dots,y_{n+2},z_1,z_2,\dots,z_{n+2}$ *in* R^{n+1}*, we have* $\det Gr(y_1,\dots,y_{n+2};z_1,\dots,z_{n+2}) = \det ||(y_i,z_j)||_{i,j=1}^{n+2} = 0$.

Lemma 3.12 *Let* $(m_1,m_2,\dots,m_n)\in N^n$ *and* $(p_1,p_2,\dots,p_n)\in N^n$ *be* n*-tuples such that*

$$\sum_{i=1}^{n} m_i \geq 1, \sum_{i=1}^{n} p_i \geq 1, (x^{(m_1,\dots,m_n)}, \tfrac{\partial^2 x}{\partial u_1^2}), (x^{(p_1,\dots,p_n)}, \tfrac{\partial^2 x}{\partial u_1^2}) \in R\left\{V,\Delta^{-1}\right\}$$

and for any i *such that* $1\leq i\leq n$ *we have*

$$\left(x^{(m_1,m_2,\dots,m_n)}, \frac{\partial x}{\partial u_i}\right), \quad \left(x^{(p_1,p_2,\dots,p_n)}, \frac{\partial x}{\partial u_i}\right) \in R\left\{V,\Delta^{-1}\right\}.$$

Then $(x^{(m_1,m_2,\dots,m_n)}, x^{(p_1,p_2,\dots,p_n)}) \in R\left\{V,\Delta^{-1}\right\}$.

Proof. Applying Lemma 3.11 to vectors

$$y_1 = z_1 = \frac{\partial x}{\partial u_1}, \cdots, y_n = z_n = \frac{\partial x}{\partial u_n}, y_{n+1} = z_{n+1} = \frac{\partial^2 x}{\partial u_1^2},$$
$$y_{n+2} = x^{(m_1,\dots,m_n)}, z_{n+2} = x^{(p_1,\dots,p_n)}$$

we obtain the equality $\det A = 0$, where $A = \|(y_i,z_j)\|_{i,j=1}^{n+2}$. Denote by $D_{n+2|j}$ the cofactor of the element (y_{n+2},z_j) of the matrix A, where $j=1,2,\dots,n+2$. The equality $\det A=0$ implies that

$$(y_{n+2},z_1)D_{n+2|1} + \cdots + (y_{n+2},z_{n+1})D_{n+2|n+1} + (y_{n+2},z_{n+2})D_{n+2|n+2} = 0. \tag{3.4}$$

Since $\Delta = D_{n+2|n+2} \neq 0$, Eq.(3.4) implies that

$$(y_{n+2},z_{n+2}) = (x^{(m_1,\dots,m_n)}, x^{(p_1,\dots,p_n)}) = \tag{3.5}$$
$$-\left((y_{n+2},z_1)D_{n+2|1} + \cdots + (y_{n+2},z_{n+1})D_{n+2|n+1}\right)\Delta^{-1}.$$

By the assumptions of our lemma, $(y_{n+2},z_{n+1}) = (x^{(m_1,m_2,\dots,m_n)}, \frac{\partial^2 x}{\partial u_1^2}) \in R\left\{V,\Delta^{-1}\right\}$ and $(y_{n+2},z_i) = (x^{(m_1,m_2,\dots,m_n)}, \frac{\partial x}{\partial u_i}) \in R\left\{V,\Delta^{-1}\right\}$ for all i such that $1\leq i\leq n$. We prove that $D_{n+2|s} \in R\left\{V,\Delta^{-1}\right\}$ for all s such that $1\leq s\leq n+1$. We have

$$D_{n+2|s} = (-1)^{n+2+s} detGr(y_1,\dots,y_{n+1};z_1,\dots,z_{s-1},z_{s+1},\dots,z_{n+1},z_{n+2})).$$

By the definition of V, we have $(y_{n+1},z_{n+1}) \in R\{V\}$ and $(y_i,z_j)\in R\{V\}$ for all i,j such that $1\leq i,j\leq n$. According to Lemma 3.9, we obtain $(y_{n+1},z_j)\in R\{V\}$ and $(y_i,z_{n+1})\in R\{V\}$ for all i,j such that $1\leq i,j\leq n$. By Lemma's assumptions, $(y_i,z_{n+2}) = (\frac{\partial x}{\partial u_i}, x^{(p_1,p_2,\dots,p_n)}) \in R\left\{V,\Delta^{-1}\right\}$ for all i such that $1\leq i\leq n$ and $(y_{n+1},z_{n+2}) = (\frac{\partial^2 x}{\partial u_1^2}, x^{(p_1,p_2,\dots,p_n)}) \in R\left\{V,\Delta^{-1}\right\}$. Hence $D_{n+2|s}\in R\left\{V,\Delta^{-1}\right\}$ for all s such that $1\leq s\leq n+1$ and Eq.(3.5) implies that $(y_{n+2},z_{n+2}) \in R\left\{V,\Delta^{-1}\right\}$. □

Lemma 3.13 $(\frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1 \partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ *for all* i, j *such that* $1 \leq i, j \leq n$.

Proof. By Lemma 3.9, $(\frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial x}{\partial u_j}) \in R\{V\}$ for all i, j such that $1 \leq i, j \leq n$. By the definition of V, $(\frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1^2}) \in V$ for all i such that $1 \leq i \leq n$. Hence, using Lemma 3.12, we obtain $(\frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1 \partial u_i}) \in R\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq n$. □

Lemma 3.14 $(\frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ *for all* i, j *such that* $1 \leq i, j \leq n$.

Proof. For all i, j such that $1 \leq i, j \leq n$, we have the following equality

$$\frac{\partial}{\partial u_1}\left(\frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial x}{\partial u_j}\right) = \left(\frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j}\right) + \left(\frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1 \partial u_j}\right). \tag{3.6}$$

By Lemma 3.9, we have $(\frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial x}{\partial u_j}) \in R\{V\}$, and by Lemma 3.13, we have $(\frac{\partial^2 x}{\partial u_1 \partial u_i}, \frac{\partial^2 x}{\partial u_1 \partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq n$. Hence Eq.(3.6) implies that $(\frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ for all such i, j. □

Lemma 3.15 $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_i \partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ *for all* i, j *such that* $1 \leq i, j \leq n$.

Proof. For all i, j such that $1 \leq i, j \leq n$, we have

$$\frac{\partial}{\partial u_i}\left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_j}\right) = \left(\frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j}\right) + \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_i \partial u_j}\right). \tag{3.7}$$

By Lemma 3.9, $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_j}) \in R\{V\}$. By Lemma 3.14, $(\frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq n$. Hence Eq.(3.7) implies that $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_i \partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ for all $1 \leq i, j \leq n$. □

Lemma 3.16 *For all* i *such that* $1 \leq i \leq n$, *we have* $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i}) \in R\{V\}$.

Proof. Since $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2}) \in V$, the equality $\frac{\partial}{\partial u_i}(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2}) = 2(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i})$ implies that $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i}) \in R\{V\}$ for all $1 \leq i \leq n$. □

Lemma 3.17 $(x^{(m_1, m_2, \ldots, m_n)}, \frac{\partial x}{\partial u_i}) \in R\left\{V, \Delta^{-1}\right\}$ *and* $(x^{(m_1, m_2, \ldots, m_n)}, \frac{\partial^2 x}{\partial u_1^2}) \in R\left\{V, \Delta^{-1}\right\}$ *for all* $(m_1, m_2, \ldots, m_n) \in N^n$ *such that* $m_1 + m_2 + \cdots + m_n \geq 1$.

Proof. We use induction on $r = m_1 + m_2 \cdots + m_n$. Let $r = 1$. Then

$$(x^{(m_1, m_2, \ldots, m_n)}, \frac{\partial x}{\partial u_i}) \in V \subset R\{V\}$$

by the definition of V, and

$$(x^{(m_1, m_2, \ldots, m_n)}, \frac{\partial^2 x}{\partial u_1^2}) \in R\{V\}$$

by Lemma 3.9. Hence our lemma holds for $r = 1$.

Assume that

$$(x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial x}{\partial u_i}) \in R\left\{V, \Delta^{-1}\right\}, (x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial^2 x}{\partial u_1^2}) \in R\left\{V, \Delta^{-1}\right\}$$

for all i such that $1 \leq i \leq n$ and $(m_1, m_2, \ldots, m_n) \in N^n$ such that $m_1 + m_2 \cdots + m_n = r$. Let us prove these properties for $r+1$. By the inductive hypothesis, we have

$$\begin{gathered}(x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial x}{\partial u_i}) \in R\left\{V, \Delta^{-1}\right\} \\ \left(x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial^2 x}{\partial u_1^2}\right) \in R\left\{V, \Delta^{-1}\right\}\end{gathered} \tag{3.8}$$

for all i such that $1 \leq i \leq n$ and $(m_1, m_2, \ldots, m_n) \in N^n$ such that $m_1 + m_2 \cdots + m_n = r$. By Lemma 3.9, $(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_l}) \in R\{V\}$ for all $i, j, l \in \{1, \ldots, n\}$, and by Lemma 3.15, $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_i \partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq n$. Hence, applying Lemma 3.12 to differential polynomials $x^{(m_1,m_2,\ldots,m_n)}$ and $\frac{\partial^2 x}{\partial u_i \partial u_j}$, we see that $(x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial^2 x}{\partial u_i \partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq n$. We have

$$\frac{\partial}{\partial u_i}\left(x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial x}{\partial u_j}\right) = \left(\frac{\partial x^{(m_1,\ldots,m_n)}}{\partial u_i}, \frac{\partial x}{\partial u_j}\right) + \left(x^{(m_1,\ldots,m_n)}, \frac{\partial^2 x}{\partial u_i \partial u_j}\right).$$

Since

$$(x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial x}{\partial u_i}) \in R\left\{V, \Delta^{-1}\right\}, (x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial^2 x}{\partial u_i \partial u_j}) \in R\left\{V, \Delta^{-1}\right\},$$

this equality implies that $(\frac{\partial x^{(m_1,m_2,\ldots,m_n)}}{\partial u_i}, \frac{\partial x}{\partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq n$.

By the inductive hypothesis (3.8) and Lemma 3.14, $(\frac{\partial^3 x}{\partial u_1^2 \partial u_i}, \frac{\partial x}{\partial u_j}) \in R\left\{V, \Delta^{-1}\right\}$ for all i, j such that $1 \leq i, j \leq n$ and by Lemma 3.16, $(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i}) \in R\{V\}$ for all i such that $1 \leq i \leq n$. Hence, applying Lemma 3.12 to $x^{(m_1,m_2,\ldots,m_n)}$ and $\frac{\partial^3 x}{\partial u_1^2 \partial u_i}$, we see that $(x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i}) \in R\left\{V, \Delta^{-1}\right\}$. Since

$$(x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial^2 x}{\partial u_1^2}) \in R\left\{V, \Delta^{-1}\right\}, (x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i}) \in R\left\{V, \Delta^{-1}\right\},$$

the equality

$$\frac{\partial}{\partial u_i}\left(x^{(m_1,m_2,\ldots,m_n)}, \frac{\partial^2 x}{\partial u_1^2}\right) = \left(\frac{\partial x^{(m_1,\ldots,m_n)}}{\partial u_i}, \frac{\partial^2 x}{\partial u_1^2}\right) + \left(x^{(m_1,\ldots,m_n)}, \frac{\partial^3 x}{\partial u_1^2 \partial u_i}\right)$$

implies that $(\frac{\partial x^{(m_1,m_2,\ldots,m_n)}}{\partial u_i}, \frac{\partial^2 x}{\partial u_1^2}) \in R\left\{V, \Delta^{-1}\right\}$. Thus we have

$$(\frac{\partial x^{(m_1,m_2,\ldots,m_n)}}{\partial u_i}, \frac{\partial x}{\partial u_j}) \in R\left\{V, \Delta^{-1}\right\}, (\frac{\partial x^{(m_1,m_2,\ldots,m_n)}}{\partial u_i}, \frac{\partial^2 x}{\partial u_1^2}) \in R\left\{V, \Delta^{-1}\right\}$$

for all i, j such that $1 \leq i, j \leq n$. Lemma is proved. □

Lemma 3.18 *$(x^{(m_1,m_2,\ldots,m_n)}, x^{(p_1,p_2,\ldots,p_n)}) \in R\left\{V,\Delta^{-1}\right\}$ for all $(m_1,\ldots,m_n),(p_1,\ldots,p_n) \in N^n$ such that $m_1+\cdots+m_n \geq 1$, $p_1+\cdots+p_n \geq 1$, and $R\left\{V,\Delta^{-1}\right\} = R\left\{x,\Delta^{-1}\right\}^{M(n+1)}$.*

Proof. By Lemma 3.17 and Lemma 3.12, $(x^{(m_1,m_2,\ldots,m_n)}, x^{(p_1,p_2,\ldots,p_n)}) \in R\left\{V,\Delta^{-1}\right\} \subset R\left\{x,\Delta^{-1}\right\}^{M(n+1)}$ for all $(m_1,m_2,\ldots,m_n) \in N^n, (p_1,p_2,\ldots,p_n) \in N^n$ such that $m_1+m_2+\cdots+m_n \geq 1$ and $p_1+p_2+\cdots+p_n \geq 1$. By Lemma 3.8, the system of all elements $(x^{(m_1,m_2,\ldots,m_n)}, x^{(p_1,p_2,\ldots,p_n)})$, where $(m_1,m_2,\ldots,m_n) \in N^n, (p_1,p_2,\ldots,p_n) \in N^n$, $m_1+m_2+\cdots+m_n \geq 1$ and $p_1+p_2+\cdots+p_n \geq 1$, is a generating system of $R\left\{x,\Delta^{-1}\right\}^{M(n+1)}$ as an R-algebra. Hence $R\left\{V,\Delta^{-1}\right\} = R\left\{x,\Delta^{-1}\right\}^{M(n+1)}$. □

The proof of Theorem 3.5 is completed. □

Theorem 3.19 *The set of elements*

$$\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right), \quad \textit{where } 1 \leq i \leq j \leq n; \quad \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_r}\right), \tag{3.9}$$

where $1 \leq r \leq n$, is a generating system of the differential field $R < x >^{M(n+1)}$.

Proof. Let $R < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_n} >$ be the differential field of all differential rational functions of $\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_n}$ and $R < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_n} >^G$ be the differential field of all G-invariant differential rational functions of $\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_n}$.

Lemma 3.20 *$R < x >^{M(n+1)} = R < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_n} >^{O(n+1)}$.*

Proof. It is similar to the proof of Lemma 1 in [8]. □

Lemma 3.21 *Let $f \in R < \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_n} >^{O(n+1)}$. Then there exist $O(n+1)$-invariant differential polynomials f_1, f_2 such that $f = f_1/f_2$.*

Proof. It is similar to the proof of Proposition 1 in ([4], p.7). □

Lemma 3.22 *The set*

$$\left\{(x^{(m_1,\ldots,m_n)}, x^{(p_1,\ldots,p_n)}) \mid \sum_{i=1}^{n} m_i \geq 1, \quad \sum_{i=1}^{n} p_i \geq 1, \quad m_i, p_i \in N\right\}$$

is a generating system of the differential field $R < \frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n} >^{O(n+1)}$.

Proof. It is similar to the proof of Lemma 3 in [8]. □

Let V be the system (3.9). By Lemma 3.10, $\Delta \in R\{V\} \subseteq R < V >$. Hence

$$R\left\{V,\Delta^{-1}\right\} \subseteq R < V > \subseteq R < \frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n} >^{O(n+1)}.$$

Lemma 3.18 and Lemma 3.22 imply $R<V>=R<\frac{\partial x}{\partial u_1},\ldots,\frac{\partial x}{\partial u_n}>^{O(n+1)}$, so $R<V>=R<x>^{M(n+1)}$. The proof of Theorem 3.19 is completed. □

For any set of vectors $\{a_1,a_2,\ldots,a_{n+1}\}$ in R^{n+1}, where the vector $a_j=(a_{1j},a_{2j},\ldots,a_{n+1j})^\top$ is a column-vector, let $[a_1a_2\ldots a_{n+1}]:=\det\|a_{ij}\|_{i,j=1}^{n+1}$. For any hypersurface $x(u)$ in R^{n+1}, consider

$$\left[x^{(m_{11},m_{12},\cdots,m_{1n})}\ \ldots x^{(m_{n+1,1},m_{n+1,2},\cdots,m_{n+1,n})}\right]$$

and set $\delta=\delta_x:=\det Gr(y_1,\ldots,y_n;z_1,\ldots,z_n)$, where $y_1=z_1=\frac{\partial x}{\partial u_1},y_2=z_2=\frac{\partial x}{\partial u_2},\cdots,y_n=z_n=\frac{\partial x}{\partial u_n}$.

Theorem 3.23 *The set of elements*

$$\begin{cases}\left(\dfrac{\partial x}{\partial u_i},\dfrac{\partial x}{\partial u_j}\right)\ \textit{for}\ 1\le i\le j\le n;\left(\dfrac{\partial^2 x}{\partial u_1^2},\dfrac{\partial^2 x}{\partial u_1\partial u_s}\right)\quad \textit{for}\ 2\le s\le n;\\ \qquad\qquad\qquad\qquad \delta^{-1},\ \Delta^{-1},\ \left[\dfrac{\partial x}{\partial u_1}\quad\dfrac{\partial x}{\partial u_2}\cdots\dfrac{\partial x}{\partial u_n}\quad\dfrac{\partial^2 x}{\partial u_1^2}\right]\end{cases}\tag{3.10}$$

is a generating system of the differential algebra $R\left\{x,\delta^{-1},\Delta^{-1}\right\}^{SM(n+1)}$.

Proof.

Lemma 3.24 *We have*

$$R\left\{x,\delta^{-1},\Delta^{-1}\right\}^{SM(n+1)}=R\left\{\frac{\partial x}{\partial u_1},\ldots,\frac{\partial x}{\partial u_n},\delta^{-1},\Delta^{-1}\right\}^{SO(n+1)}.$$

Proof. It is similar to the proof of Lemma 1 in [8]. □

Lemma 3.25 *The set of elements*

$$\begin{gathered}\delta^{-1},\Delta^{-1},\left[x^{(m_{11},\cdots,m_{1n})}x^{(m_{21},\cdots,m_{2n})}\cdots x^{(m_{n+11},\cdots,m_{n+1n})}\right],\\ (x^{(p_1,\ldots,p_n)},x^{(q_1,\cdots,q_n)}),\end{gathered}\tag{3.11}$$

where $m_{i1}+\cdots+m_{in}\ge 1,\ p_1+\cdots+p_n\ge 1,\ q_1+\cdots+q_n\ge 1$, *is a generating system of the differential algebra* $R\left\{\frac{\partial x}{\partial u_1},\ldots,\frac{\partial x}{\partial u_n},\delta^{-1},\Delta^{-1}\right\}^{SO(n+1)}$.

Proof. Let $R[x^{(m_1,m_2,\ldots,m_n)},m_1+m_2+\cdots+m_n\ge 1]^{SO(n+1)}$ be the R-algebra of all $SO(n+1)$-invariant polynomials of all $x^{(m_1,m_2,\ldots,m_n)}$, where $m_1+m_2+\cdots+m_n\ge 1$. By the First Main Theorem for $SO(n+1)$ (see [14], p.45), the system

$$\begin{gathered}\left[x^{(m_{11},m_{12},\cdots,m_{1n})}x^{(m_{21},m_{22},\cdots,m_{2n})}\cdots x^{(m_{n+11},m_{n+12},\cdots,m_{n+1n})}\right],\\ (x^{(p_1,p_2,\ldots,p_n)},x^{(q_1,q_2,\cdots,q_n)}),\end{gathered}$$

where $m_{i1}+m_{i2}+\cdots+m_{in}\geq 1, p_1+p_2+\cdots+p_n\geq 1, q_1+q_2+\cdots+q_n\geq 1$, is a generating system of $R[x^{(m_1,m_2,\dots,m_n)}, m_1+m_2+\cdots+m_n\geq 1]^{SO(n+1)}$. This implies, as in Lemma 3.8, that the system Eq.(3.11) is a generating system of $R\left\{x,\delta^{-1},\Delta^{-1}\right\}^{SM(n+1)}$. □

Denote by Z the set of elements

$$\left(\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_j}\right)\quad \text{for}\ \ 1\leq i\leq j\leq n;$$
$$\left(\frac{\partial^2 x}{\partial u_1^2},\frac{\partial^2 x}{\partial u_1\partial u_s}\right)\quad \text{for}\ \ 2\leq s\leq n;$$
$$\left[\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}\cdots\frac{\partial x}{\partial u_n}\frac{\partial^2 x}{\partial u_1^2}\right].$$

Then the system (3.10) has the form $\left\{Z,\delta^{-1},\Delta^{-1}\right\}$. Let $R\left\{Z\right\}$ be the differential subalgebra of $R\left\{\frac{\partial x}{\partial u_1},\dots,\frac{\partial x}{\partial u_n},\delta^{-1},\Delta^{-1}\right\}^{SO(n+1)}$ generated by the system Z. Denote by $R\left\{Z,\delta^{-1},\Delta^{-1}\right\}$ the differential R-subalgebra of the differential algebra $R<\frac{\partial x}{\partial u_1},\dots,\frac{\partial x}{\partial u_n}>^{SO(n+1)}$ generated by the system $\left\{Z,\delta^{-1},\Delta^{-1}\right\}$.

Lemma 3.26 $\delta\in R\left\{Z\right\}$.

Proof. Since $(y_i,z_j)=\left(\frac{\partial x}{\partial u_i},\frac{\partial x}{\partial u_j}\right)\in Z$ for all $1\leq i,j\leq n$, we see that $\delta\in R\left\{Z\right\}$. □

Lemma 3.27 *([14], p.53) The equality*

$$[y_1\dots y_{n+1}][z_1\dots z_{n+1}]=\det||(y_i,z_j)||_{i,j=1}^{n+1}$$

holds for all vectors $y_1,\dots,y_{n+1},z_1,\dots,z_{n+1}$ *in* R^{n+1}.

Lemma 3.28 $\Delta\in R\left\{Z\right\}$.

Proof. Applying Lemma 3.27 to vectors $y_1=z_1=\frac{\partial x}{\partial u_1},\ \cdots,y_n=z_n=\frac{\partial x}{\partial u_n},y_{n+1}=z_{n+1}=\frac{\partial^2 x}{\partial u_1^2}$, we obtain

$$\left[\frac{\partial x}{\partial u_1}\quad\frac{\partial x}{\partial u_2}\cdots\frac{\partial x}{\partial u_n}\quad\frac{\partial^2 x}{\partial u_1^2}\right]^2=\det\|(y_i,z_j)\|_{i,j=1}^{n+1}=\Delta. \tag{3.12}$$

Since $\left[\frac{\partial x}{\partial u_1}\quad\frac{\partial x}{\partial u_2}\cdots\frac{\partial x}{\partial u_n}\quad\frac{\partial^2 x}{\partial u_1^2}\right]\in Z$, we have $\Delta\in R\left\{Z\right\}$. □

By Lemma 3.24 and Lemma 3.25, to prove Theorem 3.23 it suffices to prove that

$$\left[x^{(m_{11},\cdots,m_{1n})}\dots x^{(m_{n+11},\cdots,m_{n+1n})}\right]\in R\left\{Z,\delta^{-1},\Delta^{-1}\right\},$$
$$(x^{(p_1,\dots,p_n)},x^{(q_1,\cdots,q_n)})\in R\left\{Z,\delta^{-1},\Delta^{-1}\right\}$$

for all $m_{ij},p_i,q_i\in N$ such that $m_{i1}+m_{i2}+\cdots+m_{in}\geq 1,p_1+p_2+\cdots+p_n\geq 1,q_1+q_2+\cdots+q_n\geq 1$.

Lemma 3.29 *$(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2}) \in R\{Z, \delta^{-1}\}$ and $V \subset R\{Z, \delta^{-1}\}$, where V is the system used in the proof of Theorem 3.5.*

Proof. Denote by $D_{n+1|j}$, where $j = 1, 2, \ldots, n+1$, the cofactor of the element (y_{n+1}, z_j) of the matrix $\|(y_i, z_j)\|_{i,j=1}^{n+1}$ in Eq.(3.12). Then Eq.(3.12) implies the equality

$$\Delta = (y_{n+1}, z_1)D_{n+1|1} + \cdots + (y_{n+1}, z_n)D_{n+1|n} + (y_{n+1}, z_{n+1})D_{n+1|n+1}. \tag{3.13}$$

Since $\delta = D_{n+1|n+1} \neq 0$, Eq.(3.13) implies that

$$\begin{aligned}(y_{n+1}, z_{n+1}) &= \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1^2}\right) \\ &= \Delta\delta^{-1} - (y_{n+1}, z_1)D_{n+1|1}\delta^{-1} - \cdots - (y_{n+1}, z_n)D_{n+1|n}\delta^{-1}.\end{aligned} \tag{3.14}$$

Since $V_0 \subset Z$, by Lemma 3.9, $(y_{n+1}, z_j) = (\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_j}) \in R\{V_0\} \subset R\{Z\}$ for all $j = 1, \ldots n$. We prove that $D_{n+1|s} \in R\{Z\}$ for all $s = 1, \ldots n$. Since

$$D_{n+1|s} = (-1)^{(n+1)+s} \det Gr(y_1, \ldots, y_n; z_1, \ldots, \quad z_{s-1}, z_{s+1}, \ldots, z_{n+1}),$$

the elements of $D_{n+1|s}$ have the forms (y_i, z_j), where $i, j \leq n$, and (y_k, z_{n+1}) for $k \leq n$. By the definition of Z, $(y_i, z_j) \in Z \subset R\{Z\}$ for all $i, j \leq n$. By Lemma 3.9, for all $k \leq n$ we have $(y_k, z_{n+1}) = (\frac{\partial x}{\partial u_k}, \frac{\partial^2 x}{\partial u_1^2}) \in R\{V_0\} \subset R\{Z\}$. Hence Eq.(3.14) implies that $(y_{n+1}, z_{n+1}) \in R\{Z, \delta^{-1}\}$. Since $V \subset Z \cup \{(y_{n+1}, z_{n+1})\}$, we obtain $V \subset R\{Z, \delta^{-1}\}$. □

Lemma 3.30 *We have $(x^{(p_1,p_2,\cdots,p_n)}, x^{(r_1,r_2,\cdots,r_n)}) \in R\{Z, \delta^{-1}, \Delta^{-1}\}$ for all $p_i, r_i \in N$ such that $p_1 + p_2 + \cdots + p_n \geq 1$ and $r_1 + r_2 + \cdots + r_n \geq 1$.*

Proof. By Lemma 3.29, $V \subseteq R\{Z, \delta^{-1}\}$. This implies that $R\{V, \Delta^{-1}\} \subseteq R\{Z, \delta^{-1}, \Delta^{-1}\}$. Hence, by Lemma 3.18,
$(x^{(p_1,p_2,\cdots,p_n)}, x^{(r_1,r_2,\cdots,r_n)}) \in R\{Z, \delta^{-1}, \Delta^{-1}\}$ for all $p_i, r_i \in N$ such that $p_1 + p_2 + \cdots + p_n \geq 1$ and $r_1 + r_2 + \cdots + r_n \geq 1$. □

Lemma 3.31 *$[x^{(m_{11},m_{12},\cdots,m_{1n})} x^{(m_{21},m_{22},\cdots,m_{2n})} \ldots x^{(m_{n+11},m_{n+12},\cdots,m_{n+1n})}] \in R\{Z, \delta^{-1}, \Delta^{-1}\}$ for all $m_{ij} \in N$ such that $m_{i1} + m_{i2} + \cdots + m_{in} \geq 1$ and $i = 1, 2, \ldots, n+1$.*

Proof. Applying Lemma 3.27 to vectors $y_1 = \frac{\partial x}{\partial u_1}, y_2 = \frac{\partial x}{\partial u_2}, \cdots, y_n = \frac{\partial x}{\partial u_n}, y_{n+1} = \frac{\partial^2 x}{\partial u_1^2}$; $z_1 = x^{(m_{11},m_{12},\cdots,m_{1n})}, z_2 = x^{(m_{21},m_{22},\cdots,m_{2n})}, \ldots, z_{n+1} = x^{(m_{n+11},m_{n+12},\cdots,m_{n+1n})}$, we obtain

$$[y_1 \ldots y_{n+1}][z_1 \ldots z_{n+1}] = \det ||(y_i, z_j)||_{i,j=1}^{n+1} \tag{3.15}$$

By Eq.(3.12), $\Delta = [y_1 \ldots y_{n+1}]^2$. Using this equality and Eq.(3.15), we obtain

$$[z_1 \ldots z_{n+1}] = \Delta^{-1}[y_1 \ldots y_{n+1}] \det ||(y_i, z_j)||_{i,j=1}^{n+1} \tag{3.16}$$

By Lemma 3.30, $(y_i, z_j) \in R\left\{Z, \delta^{-1}\Delta^{-1}\right\}$ for all $1 \leq i, j \leq n+1$. Eq.(3.16) implies, since $[y_1 \dots y_{n+1}] \in Z \subset R\left\{Z, \delta^{-1}\Delta^{-1}\right\}$, $[z_1 \dots z_{n+1}] \in R\left\{Z, \delta^{-1}\Delta^{-1}\right\}$. □

Let us finish the proof of our theorem. By Lemma 3.30, $(x^{(p_1,p_2,\cdots,p_n)}, x^{(r_1,r_2,\cdots,r_n)}) \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $p_i, r_i \in N$ such that $p_1 + p_2 + \cdots + p_n \geq 1$ and $r_1 + r_2 + \cdots + r_n \geq 1$. By Lemma 3.31,

$$\left[x^{(m_{11},\cdots,m_{1n})} x^{(m_{21},\cdots,m_{2n})} \cdots x^{(m_{n+11},\cdots,m_{n+1n})}\right] \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$$

for all $m_{ij} \in N$ such that $m_{i1} + \cdots + m_{in} \geq 1$, where $i = 1, 2, \dots, n+1$. Hence Lemma 3.24 and Lemma 3.25 imply that $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\} = R\left\{x(u), \Delta^{-1}\right\}^{SM(n+1)}$. The proof of Theorem 3.23 is completed. □

Theorem 3.32 *The set of elements*

$$\begin{aligned} &\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right) \quad \text{for } 1 \leq i \leq j \leq n; \\ &\left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_s}\right) \quad \text{for } 2 \leq s \leq n; \\ &\left[\frac{\partial x}{\partial u_1} \cdots \frac{\partial x}{\partial u_n} \quad \frac{\partial^2 x}{\partial u_1^2}\right] \end{aligned} \tag{3.17}$$

is a generating system of the differential field $R < x(u) >^{SM(n+1)}$.

Proof.

Lemma 3.33 $R < x >^{SM(n+1)} = R < \frac{\partial x}{\partial u_1}, \dots, \frac{\partial x}{\partial u_n} >^{SO(n+1)}$.

Proof. It is similar to the proof of Lemma 1 in [8]. □

Lemma 3.34 *Let* $f \in R < \frac{\partial x}{\partial u_1}, \dots, \frac{\partial x}{\partial u_n} >^{SO(n+1)}$. *Then there exist* $SO(n+1)$*-invariant differential polynomials* f_1, f_2 *such that* $f = f_1/f_2$.

Proof. It is similar to the proof of Proposition 1 in ([4], p. 7). □

Lemma 3.35 *The set of all elements*

$$\begin{aligned} &\left[x^{(m_{11},m_{12},\cdots,m_{1n})} x^{(m_{21},m_{22},\cdots,m_{2n})} \cdots x^{(m_{n+11},m_{n+12},\cdots,m_{n+1n})}\right], \\ &(x^{(p_1,p_2,\dots,p_n)}, x^{(q_1,q_2,\cdots,q_n)}), \end{aligned} \tag{3.18}$$

where $m_{i1} + m_{i2} + \cdots + m_{in} \geq 1, p_1 + p_2 + \cdots + p_n \geq 1, q_1 + q_2 + \cdots + q_n \geq 1$, *is a generating system of the differential field* $R < \frac{\partial x}{\partial u_1}, \dots, \frac{\partial x}{\partial u_n} >^{SO(n+1)}$.

Proof. Let $B := R[x^{(m_1,m_2,\dots,m_n)} \mid m_1 + m_2 + \cdots + m_n \geq 1]^{SO(n+1)}$. By the First Main Theorem for $SO(n+1)$ (see [14], p.45), the system Eq.(3.18) is a generating system of B. Lemma 3.34 implies that Eq.(3.18) is a generating system of $R < \frac{\partial x}{\partial u_1}, \dots, \frac{\partial x}{\partial u_n} >^{SO(n+1)}$. □

Let Z be the system Eq.(3.17). By Lemma 3.26 and Lemma 3.28, $\delta, \Delta \in R\{Z\} \subseteq R < Z >$. Hence $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\} \subseteq R < Z > \subseteq R < \frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n} >^{SO(n+1)}$. Lemmas 3.30, Lemma 3.31 and Lemma 3.35 imply that $R < Z >= R < \frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n} >^{SO(n+1)}$. Using Lemma 3.33, we get $R < Z >= R < x >^{SM(n+1)}$. Theorem 3.32 is completed. □

Theorem 3.36 *The set (where $i, j, s = 1, \ldots, n,\ 1 \leq i \leq j \leq n$)*

$$\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right),\ \delta^{-1},\ \Delta^{-1},\ \left[\frac{\partial^2 x}{\partial u_1 \partial u_s} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \cdots \frac{\partial x}{\partial u_n}\right] \tag{3.19}$$

is a generating system of the differential algebra $R\left\{x, \delta^{-1}, \Delta^{-1}\right\}^{SM(n+1)}$.

Proof. Let

$$W = \left\{\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right), \left[\frac{\partial^2 x}{\partial u_1 \partial u_s} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \cdots \frac{\partial x}{\partial u_n}\right] \mid 1 \leq i \leq j \leq n\right\}.$$

Denote by $R\{W\}$ the differential R-subalgebra of $R < \frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n} >^{O(n+1)}$ generated by elements of W and by $R\left\{W, \delta^{-1}, \Delta^{-1}\right\}$ the differential R -subalgebra of $R < \frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_n} >^{O(n+1)}$ generated by functions δ^{-1}, Δ^{-1}, and elements of W.

Consider the set $V_0 = \left\{\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right), 1 \leq i \leq j \leq n\right\}$ from the proof of Theorem 3.5. Since $V_0 \subset W$ and $\delta \in R\{V_0\}$, we have $\delta \in R\{V_0\} \subset R\{W\}$ and $R\left\{V_0, \delta^{-1}\right\} \subset R\left\{W, \delta^{-1}, \Delta^{-1}\right\}$.

By Lemma 3.9, $(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_l} x) \in R\{V_0\}$ for all $i, j, l \in \{1, 2, \ldots, n\}$. Hence

$$\left(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial x}{\partial u_l} x\right) \in R\{W\} \quad \text{for all } i, j, l \in \{1, 2, \ldots, n\}. \tag{3.20}$$

Lemma 3.37 $\left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_i}\right) \in R\left\{W, \delta^{-1}, \Delta^{-1}\right\}$ *for all $i = 1, 2, \ldots, n$.*

Proof. Applying Lemma 3.27 to vectors $y_1 = \frac{\partial x}{\partial u_1},\ y_2 = \frac{\partial x}{\partial u_2},\ \cdots, y_n = \frac{\partial x}{\partial u_n},\ y_{n+1} = \frac{\partial^2 x}{\partial u_1^2};\ z_1 = \frac{\partial x}{\partial u_1},\ z_2 = \frac{\partial x}{\partial u_2},\ \cdots, z_n = \frac{\partial x}{\partial u_n},\ z_{n+1} = \frac{\partial^2 x}{\partial u_1 \partial u_i}$, we obtain

$$[y_1 \ldots y_{n+1}][z_1 \ldots z_{n+1}] = \det ||(y_i, z_j)||_{i,j=1}^{n+1} \tag{3.21}$$

For $j = 1, 2, \ldots, n+1$, let $D_{n+1|j}$ be the cofactor of the element (y_{n+1}, z_j) of the matrix $\|(y_i, z_j)\|_{i,j}^{n+1}$. Eq.(3.21) implies that

$$[y_1 \ldots y_{n+1}][z_1 \ldots z_{n+1}] = (y_{n+1}, z_1)D_{n+1|1} + \cdots + (y_{n+1}, z_{n+1})D_{n+1|n+1}.$$

This equality implies

$$(y_{n+1}, z_{n+1})D_{n+1|n+1} = [y_1 \ldots y_{n+1}][z_1 \ldots z_{n+1}] - (y_{n+1}, z_1)D_{n+1|1} - \cdots - (y_{n+1}, z_n)D_{n+1|n}. \tag{3.22}$$

Since $D_{n+1|n+1} = \delta$, Eq.(3.22) implies that

$$(y_{n+1}, z_{n+1}) = \delta^{-1}([y_1 \dots y_{n+1}] [z_1 \dots z_{n+1}] - (y_{n+1}, z_1)D_{n+1|1} - \dots - (y_{n+1}, z_n)D_{n+1|n}). \tag{3.23}$$

Since $[y_1 \dots y_{n+1}] \in R\{W\}$ and $[z_1 \dots z_{n+1}] \in R\{W\}$, we obtain

$$[y_1 \dots y_{n+1}] [z_1 \dots z_{n+1}] \in R\{W\}.$$

By Lemma 3.9, $(y_{n+1}, z_i) = (\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_i}) \in R\{V_0\}$ for all $i = 1, 2, \dots n$. Hence $(y_{n+1}, z_i) = (\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial x}{\partial u_i}) \in R\{W\}$ for all $i = 1, 2, \dots, n$.

We prove that $D_{n+1|s} \in R\left\{W, \delta^{-1}, \Delta^{-1}\right\}$ for all $s = 1, 2, \dots n$. Since

$$D_{n+1|s} = (-1)^{(n+1)+s} \det Gr(y_1, y_2, \dots, y_n; z_1, z_2, \dots, z_{s-1}, z_{s+1}, \dots, z_{n+1}),$$

elements of $D_{n+1|s}$ have the forms (y_i, z_j), where $i, j \leq n$, and (y_k, z_{n+1}), where $k \leq n$. By the definition of W, $(y_i, z_j) \in W \subset R\{W\}$ for all $i, j \leq n$. By Lemma 3.9,

$$(y_k, z_{n+1}) = \left(\frac{\partial x}{\partial u_k}, \frac{\partial^2 x}{\partial u_1 \partial u_i}\right) \in R\{V_0\} \subset R\{W\}$$

for all $k \leq n$. Hence Eq.(3.23) implies that $(y_{n+1}, z_{n+1}) \in R\left\{W, \delta^{-1}, \Delta^{-1}\right\}$. □

Lemma 3.37 implies that $Z \subset R\left\{W, \delta^{-1}, \Delta^{-1}\right\}$, where Z is the system (3.17). By Theorem 3.23 $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\} = R\left\{x, \delta^{-1}, \Delta^{-1}\right\}^{SM(n+1)}$. Hence

$$R\left\{W, \delta^{-1}, \Delta^{-1}\right\} = R\left\{x, \delta^{-1}, \Delta^{-1}\right\}^{SM(n+1)}.$$

The proof of Theorem 3.36 is completed. □

Theorem 3.38 *The set of elements*

$$\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right), \quad \textit{where } 1 \leq i \leq j \leq n; \quad \left[\frac{\partial^2 x}{\partial u_1 \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \dots \frac{\partial x}{\partial u_n}\right] \tag{3.24}$$

is a generating system of the differential field $R < x >^{SM(n+1)}$.

Proof. Let W be the system Eq.(3.24). Since $\Delta = \left[\frac{\partial^2 x}{\partial u_1^2} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \dots \frac{\partial x}{\partial u_n}\right]^2$, we have $\Delta \in R\{W\}$. Hence $\Delta^{-1} \in R < W >$. Since $\delta \in R\{W\}$, we obtain that $\delta^{-1} \in R < W >$. So $R\left\{W, \delta^{-1}, \Delta^{-1}\right\} \subseteq R < W >$. Lemma 3.37 implies that

$$Z \subset R\left\{W, \delta^{-1}, \Delta^{-1}\right\} \subseteq R < W > \subseteq R < x >^{SM(n+1)}.$$

In the proof of Theorem 3.32, it is proved that $R < Z >= R < x >^{SM(n+1)}$. Hence, using the equality $R < Z >= R < x >^{SM(n+1)}$, we obtain $R < W >= R < x >^{SM(n+1)}$. Theorem 3.38 is proved. □

Proposition 3.39 *Let $d \in \{1, 2, \ldots, n\}$ and x be a d-nondegenerate U-hypersurface. Then x is a regular hypersurface and $\delta_x(u) > 0$ for all $u \in U$.*

Proof. Let x be a d-nondegenerate U-hypersurface. Then $L_{dd}(x(u)) \neq 0$ for all $u \in U$. This implies that $[a_1(x)a_2(x)\ldots a_{n+1}(x)] \neq 0$, where $a_i(x)$ are column-vectors, $a_i(x) = \frac{\partial x}{\partial u_i}$ for $1 \leq i \leq n$ and $a_{n+1}(x) = \frac{\partial^2 x}{\partial u_d^2}$. Hence the vectors $a_1(x), \ldots, a_{n+1}(x)$ are linearly independent for all $u \in U$. Then $a_1(x), a_2(x), \ldots, a_n(x)$ are also linearly independent. This implies that for all $u \in U$, $\det \|(a_i(x), a_j(x))\|_{i,j=1}^{n} = \delta_x(u) \neq 0$. In this case, it is known that $\delta_x(u) > 0$. □

Let $\{g_{ij}(x), L_{ij}(x) \mid i, j = 1, \ldots, n\}$ be the set of all coefficients of the first and second fundamental forms of a U-hypersurface $x(u)$ in R^{n+1}. Assume that $x(u)$ is a d-nondegenerate U-hypersurface in R^{n+1}. Then $\Delta_d \neq 0$ for all $u \in U$. Hence the function Δ_d^{-1} exists. By Proposition 3.39, $\delta_x(u) > 0$. Hence the function $\delta_x(u)^{-\frac{1}{2}}$ exists.

Theorem 3.40 *Let $d \in \{1, 2, \ldots, n\}$ and $x(u)$ be a d-nondegenerate U- hypersurface in R^{n+1}. Then the set*

$$\left\{g_{ij}(x), \Delta_d^{-1}, \delta^{-\frac{1}{2}}, L_{dr}(x) \mid i, j, r = 1, 2, \ldots, n;\ i \leq j\right\}$$

is a generating system of the differential algebra

$$R\left\{g_{ij}(x), \Delta_d^{-1}, \delta^{-\frac{1}{2}}, L_{ij}(x) \mid i, j = 1, \ldots, n; i \leq j\right\}.$$

Proof. For $d = 1$, let $W_1 := \{g_{ij}(x), L_{1r}(x) \mid i, j, r = 1, \ldots, n;\ i \leq j\}$ and $R\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$ be the differential R-subalgebra of

$$R\left\{g_{ij}(x), \Delta^{-1}, \delta^{-\frac{1}{2}}, L_{ij}(x) \mid i, j = 1, \ldots, n; i \leq j\right\}$$

generated by elements of the system W_1 and functions $\Delta^{-1}, \delta^{-\frac{1}{2}}$.

Using Eq.(2.2), we obtain $\left[\frac{\partial^2 x}{\partial u_1 \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \cdots \frac{\partial x}{\partial u_n}\right] = \delta^{-\frac{1}{2}} L_{1j}(x)$ for all $j = 1, \ldots, n$. Hence we have $\left[\frac{\partial^2 x}{\partial u_1 \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \cdots \frac{\partial x}{\partial u_n}\right] \in R\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$ for all $j = 1, \ldots, n$. This implies $W \subseteq R\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$, where W is the system Eq.(3.24). Hence $\left\{W, \Delta^{-1}, \delta^{-1}\right\} \subseteq R\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$. By Theorem 3.36 $\left[\frac{\partial^2 x}{\partial u_i \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \cdots \frac{\partial x}{\partial u_n}\right] \in \left\{W, \Delta^{-1}, \delta^{-1}\right\} \subseteq R\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$ for all $i, j = 1, \ldots, n$. Eq.(2.2) implies that

$$L_{ij} = \delta^{-\frac{1}{2}} \left[\frac{\partial^2 x}{\partial u_i \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \cdots \frac{\partial x}{\partial u_n}\right] \in R\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\}$$

for all $i, j = 1, \ldots, n$. Hence

$$R\left\{W_1, \Delta^{-1}, \delta^{-\frac{1}{2}}\right\} = R\left\{g_{ij}(x), \Delta^{-1}, \delta^{-\frac{1}{2}}, L_{ij}(x) \mid i, j = 1, \ldots, n; i \leq j\right\}.$$

The proof of Theorem 3.40 is completed. □

4. Complete systems of G-invariants of hypersurfaces

Let G be any subgroup of $M(n+1)$.

Definition 4.1 *Two U-hypersurfaces $x(u)$ and $y(u)$ in R^{n+1} will be called G-equivalent if there exists $F \in G$ such that $y(u) = Fx(u)$ for all $u \in U$. In this case, it will be denoted by $x \overset{G}{\sim} y$.*

In this section, $A(x) := \|a_1(x)a_2(x)\dots a_{n+1}(x)\|$ is the matrix with column-vectors $a_i(x) = \frac{\partial x}{\partial u_i}$ for all i such that $1 \le i \le n$, and $a_{n+1}(x) = \frac{\partial^2 x}{\partial u_1^2}$. Denote $[a_1(x)a_2(x)\dots a_{n+1}(x)] := \det A(x)$.

Any 1-nondegenerate U-hypersurface in R^{n+1} will be briefly called a *nondegenerate* U-hypersurface. Let x be a nondegenerate U-hypersurface in R^{n+1}. Since x is a nondegenerate hypersurface, we have

$$\Delta_x = [a_1(x)a_2(x)\dots a_{n+1}(x)]^2 \ne 0$$

for all $u \in U$. Hence $[a_1(x)a_2(x)\dots a_{n+1}(x)] \ne 0$ for all $u \in U$ and $A(x)^{-1}$ is well-defined.

Theorem 4.2 *Let $x(u), y(u)$ be nondegenerate U-hypersurfaces in R^{n+1}.*

(1) *Let $x \overset{M(n+1)}{\sim} y$. Then for all i,j,s such that $1 \le i,j,s \le n$ and for all $u \in U$, we have*

$$\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right) = \left(\frac{\partial y}{\partial u_i}, \frac{\partial y}{\partial u_j}\right), \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_s}\right) = \left(\frac{\partial^2 y}{\partial u_1^2}, \frac{\partial^2 y}{\partial u_1 \partial u_s}\right). \tag{4.1}$$

(2). *Conversely, assume that equalities Eq.(4.1) hold. Then $x \overset{M(n+1)}{\sim} y$. Moreover, the unique $g \in O(n+1)$ and the unique $b \in R^{n+1}$ exist such that $y(u) = gx(u) + b$ for all $u \in U$. Explicitly: $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$.*

Proof. (1) Assume that $x \overset{M(n+1)}{\sim} y$. The functions

$$\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right), \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_s}\right)$$

are $M(n+1)$-invariant, so equalities Eq.(4.1) hold.

(2) Assume that equalities Eq.(4.1) hold. Eq. (4.1) and Lemma 3.10 imply that $\Delta_x(u) = \Delta_y(u)$ for all $u \in U$. Since x, y are nondegenerate hypersurfaces, it follows that $\Delta_x(u) \ne 0$ and $\Delta_y(u) \ne 0$ for all $u \in U$. Hence $\Delta_x(u)^{-1} = \Delta_y(u)^{-1}$ for all $u \in U$. Let V be the system used in the proof of Theorem 3.5 and $f\{x\} \in R\{V, \Delta^{-1}\}$. Then Theorem 3.5, Eq.(4.1) and the equality $\Delta_x(u)^{-1} = \Delta_y(u)^{-1}$ imply that

$$f\{x(u)\} = f\{y(u)\} \quad \text{for all } u \in U. \tag{4.2}$$

For any s such that $1 \le s \le n$, we set $\frac{\partial A(x)}{\partial u_s} := \left\|\frac{\partial a_1(x)}{\partial u_s} \frac{\partial a_2(x)}{\partial u_s} \dots \frac{\partial a_{n+1}(x)}{\partial u_s}\right\|$. Consider the matrix $A(x)^{-1}\frac{\partial A(x)}{\partial u_s} = \left\|p_{ij}^s(x)\right\|$.

Lemma 4.3 $p_{ij}^s(x) \in R\{V, \Delta^{-1}\}$ *for all i,j,s such that $1 \le i,j \le n+1, 1 \le s \le n$.*

Proof. The equality $A(x)^{-1}\frac{\partial A(x)}{\partial u_s} = \left\|p^s_{ij}(x)\right\|$ implies that $A(x)\left\|p^s_{ij}(x)\right\| = \frac{\partial A(x)}{\partial u_s}$. Since x is a nondegenerate hypersurface, $\Delta_x(u) = (\det A(x)(u))^2 \neq 0$ for all $u \in U$. Since $\det A(x)(u) \neq 0$, the system $A(x)\left\|p^s_{ij}\{x\}\right\| = \frac{\partial A(x)}{\partial u_s}$ of linear equations has the following solution

$$p^s_{ij}(x) = \left[a_1(x)\dots a_{i-1}(x)\frac{\partial a_j(x)}{\partial u_s}a_{i+1}(x)\dots a_{n+1}(x))\right][a_1(x)\dots a_{n+1}(x)]^{-1}$$

where i,j,s such that $1 \le i,j \le n+1$ and $1 \le s \le n$. This equality implies that

$$p^s_{ij}(x) = \left[a_1(x)\dots a_{i-1}(x)\frac{\partial a_j(x)}{\partial u_s}a_{i+1}(x)\dots a_{n+1}(x)\right][a_1(x)\dots a_{n+1}(x)]\Delta^{-1}$$

for all i,j,s such that $1 \le i,j \le n+1$ and $1 \le s \le n$. Using Lemma 3.27 and Theorem 3.5, it is obtained that

$$\left[a_1(x)\dots a_{i-1}(x)\frac{\partial a_j(x)}{\partial u_s}a_{i+1}(x)\dots a_{n+1}(x))\right][a_1(x)\dots a_{n+1}(x)]$$

is an element of $R\left\{V,\Delta^{-1}\right\}$. Since $\Delta^{-1} \in R\left\{V,\Delta^{-1}\right\}$, it follows that $p^s_{ij}(x) \in R\left\{V,\Delta^{-1}\right\}$ for all i,j,s such that $1 \le i,j \le n+1$ and $1 \le s \le n$. □

Lemma 4.4 *$A(x(u))^{-1}\frac{\partial A(x(u))}{\partial u_s} = A(y(u))^{-1}\frac{\partial A(y(u))}{\partial u_s}$ for all s such that $1 \le s \le n$ and $u \in U$.*

Proof. Using Eq.(4.1), Eq.(4.2) and Lemma 4.3, we have $p^s_{ij}(x(u)) = p^s_{ij}(y(u))$ for all $u \in U$ and i,j,s such that $1 \le i,j \le n+1$ and $1 \le s \le n$. Hence the equality $A(x)^{-1}\frac{\partial A(x)}{\partial u_s} = \left\|p^s_{ij}(x)\right\|$ implies that $A(x(u))^{-1}\frac{\partial A(x(u))}{\partial u_s} = A(y(u))^{-1}\frac{\partial A(y(u))}{\partial u_s}$ for all s such that $1 \le s \le n$ and $u \in U$. □

Now we complete the proof of our theorem. We have the following equality

$$\begin{aligned}\frac{\partial(A(y)A(x)^{-1})}{\partial u_s} &= \frac{\partial A(y)}{\partial u_s}A(x)^{-1} + A(y)\frac{\partial A(x)^{-1}}{\partial u_s}\\ &= \frac{\partial A(y)}{\partial u_s}A(x)^{-1} - A(y)A(x)^{-1}\frac{\partial A(x)}{\partial u_s}A(x)^{-1}\\ &= A(y)\left(A(y)^{-1}\frac{\partial A(y)}{\partial u_s} - A(x)^{-1}\frac{\partial A(x)}{\partial u_s}\right)A(x)^{-1}\end{aligned}$$

for all s such that $1 \le s \le n$ and $u \in U$. Using this equality and the equality in Lemma 4.4, we see that $\frac{\partial(A(y)A(x)^{-1})}{\partial u_s} = 0$ for all s such that $1 \le s \le n$. Since U is a connected open subset of R^n, using this equality for all s such that $1 \le s \le n$, we see that $A(y(u))A(x(u))^{-1}$ does not depend on $u \in U$. Put $g = A(y)A(x)^{-1}$. Because $\det A_x(u) \neq 0$ and $\det A_y(u) \neq 0$ for all $u \in U$, we have $\det g \neq 0$ and $A(y) = gA(x)$ for all $u \in U$.

Let us prove that $g \in O(n+1)$. Lemma 3.17, Eq.(4.2) and the equality $A(x)^\top A(x) = \|(a_i(x),a_j(x))\|^{n+1}_{i,j=1}$ imply that $A(x)^\top A(x) = A(y)^\top A(y)$. This and the equality $A(y) = gA(x)$ imply that $g^\top g = I$, where I is the unit matrix. Hence $g \in O(n+1)$.

The equality $A_y(u) = gA_x(u)$ implies that $\frac{\partial y(u)}{\partial u_s} = g\frac{\partial x(u)}{\partial u_s}$ for all s such that $1 \le s \le n$ and $u \in U$. These equalities imply existence of a vector $b \in R^{n+1}$ such that $y(u) = gx(u) + b$ for all $u \in U$.

Let $y(u) = Dx(u) + c$ for certain $c \in R^{n+1}$ and $D \in O(n+1)$ and all $u \in U$. Then $\frac{\partial y(u)}{\partial u_i} = D\frac{\partial x(u)}{\partial u_i}$ for all $i = 1, 2, \ldots, n$ and $u \in U$. Using these equalities, we see that $A(y(u)) = DA(x(u))$ for all $u \in U$. Hence $D = A(y)A(x)^{-1} = g$. The uniqueness of g is proved. The equalities $y(u) = Dx(u) + c$ and $D = A(y)A(x)^{-1}$ imply that $c = y - A(y)A(x)^{-1}x = b$. Proof of Theorem 4.2 is completed. □

Theorem 4.2 means that the system Eq.(3.9) is a complete system of $M(n)$-invariants on the set of all nondegenerate U-hypersurfaces in R^{n+1}.

Theorem 4.5 *Let $x(u)$ and $y(u)$ be nondegenerate U-hypersurfaces in R^{n+1}. Then*

(1) *Let $x \overset{SM(n+1)}{\sim} y$. Then for all i, j such that $1 \leq i, j \leq n$ and s such that $2 \leq s \leq n$, and any $u \in U$, we have*

$$\begin{cases} \left(\dfrac{\partial x}{\partial u_i}, \dfrac{\partial x}{\partial u_j}\right) = \left(\dfrac{\partial y}{\partial u_i}, \dfrac{\partial y}{\partial u_j}\right), \left(\dfrac{\partial^2 x}{\partial u_1^2}, \dfrac{\partial^2 x}{\partial u_1 \partial u_s}\right) = \left(\dfrac{\partial^2 y}{\partial u_1^2}, \dfrac{\partial^2 y}{\partial u_1 \partial u_s}\right), \\ \left[\dfrac{\partial x}{\partial u_1} \quad \dfrac{\partial x}{\partial u_2} \cdots \dfrac{\partial x}{\partial u_n} \quad \dfrac{\partial^2 x}{\partial u_1^2}\right] = \left[\dfrac{\partial y}{\partial u_1} \quad \dfrac{\partial y}{\partial u_2} \cdots \dfrac{\partial y}{\partial u_n} \quad \dfrac{\partial^2 y}{\partial u_1^2}\right]. \end{cases} \tag{4.3}$$

(2) *Conversely, assume that equalities Eq.(4.3) hold. Then $x \overset{SM(n+1)}{\sim} y$. Moreover, the unique $g \in SO(n+1)$ and the unique $b \in R^{n+1}$ exist such that $y = gx + b$. Explicitly, we have $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$.*

Proof. (1) Assume that $x \overset{SM(n+1)}{\sim} y$. The functions

$$\left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right), \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_1 \partial u_s}\right), \left[\frac{\partial x}{\partial u_1} \quad \cdots \frac{\partial x}{\partial u_n} \quad \frac{\partial^2 x}{\partial u_1^2}\right]$$

are $SM(n+1)$-invariant, so equalities Eq.(4.3) hold.

(2) Assume that equalities Eq. (4.3) hold. Let Z be the system Eq.(3.17), $R\{Z\}$ be the differential R-subalgebra in Theorem 3.23. Let

$$\delta = \delta_x := \det Gr(v_1, v_2, \ldots, v_n; z_1, z_2, \ldots, z_n),$$

where $v_1 = z_1 = \frac{\partial x}{\partial u_1}, v_2 = z_2 = \frac{\partial x}{\partial u_2}, \cdots, v_n = z_n = \frac{\partial x}{\partial u_n}$. By Lemma 3.26 and Lemma 3.28, $\delta_x, \Delta_x \in R\{Z\}$. Hence Eq. (4.3) implies that $\delta_x = \delta_y, \Delta_x = \Delta_y$ for all $u \in U$. Since $x(u), y(u)$ are nondegenerate hypersurfaces, we have $\Delta_x(u) \neq 0$ and $\Delta_y(u) \neq 0$ for all $u \in U$. By Proposition 3.39, $\delta_x(u) > 0$ and $\delta_y(u) > 0$ for all $u \in U$.

The equalities $\delta_x = \delta_y$ and $\Delta_x = \Delta_y$ for all $u \in U$ and Proposition 3.39 imply that $\delta_x^{-1} = \delta_y^{-1}$ and $\Delta_x^{-1} = \Delta_y^{-1}$ for all $u \in U$. Let $f\{x\} \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$, where $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ is the differential algebra used in the proof of Theorem 3.23. Then equalities $\delta_x^{-1} = \delta_y^{-1}$ and $\Delta_x^{-1} = \Delta_y^{-1}$ and Eq.(4.3) imply $f(x) = f(y)$ for all $u \in U$. Using Lemma 3.29, Eq. (4.3) and the equality $f(x) = f(y)$, we obtain equalities Eq. (4.1). Hence by Theorem 4.2 there exist the unique $g \in O(n+1)$ and $b \in R^{n+1}$ such that $y(u) = gx(u) + b$ for all $u \in U$. This equality and Eq.(4.3) imply that

$$\left[\frac{\partial x}{\partial u_1} \quad \frac{\partial x}{\partial u_2} \cdots \frac{\partial x}{\partial u_n} \quad \frac{\partial^2 x}{\partial u_1^2}\right] = \det(g) \left[\frac{\partial y}{\partial u_1} \quad \frac{\partial y}{\partial u_2} \cdots \frac{\partial y}{\partial u_n} \quad \frac{\partial^2 y}{\partial u_1^2}\right].$$

Since $\Delta_x(u) = \left[\frac{\partial x}{\partial u_1} \quad \frac{\partial x}{\partial u_2} \cdots \frac{\partial x}{\partial u_n} \quad \frac{\partial^2 x}{\partial u_1^2}\right]^2 \neq 0$ for all $u \in U$, we see that $\det(g) = 1$. By Theorem 4.2, $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$. Proof of Theorem 4.5 is completed. □

Theorem 4.5 means that the system Eq. (3.17) is a complete system of $SM(n)$-invariants on the set of all nondegenerate U-hypersurfaces in R^{n+1}.

Theorem 4.6 *Let $d \in \{1, 2, \dots, n\}$ and $x(u), y(u)$ be d-nondegenerate U-hypersurfaces in R^{n+1}.*

(1) Assume that $x \overset{SM(n+1)}{\sim} y$. Then for all i, j, s such that $1 \leq i, j, s \leq n$, where $i \leq j$, and all $u \in U$, we have

$$g_{ij}(x) = g_{ij}(y), \qquad L_{ds}(x) = L_{ds}(y). \tag{4.4}$$

(2) Conversely, assume that equalities Eq. (4.4) hold. Then $x \overset{SM(n+1)}{\sim} y$. Moreover, the unique $g \in SO(n+1)$ and $b \in R^{n+1}$ exist such that $y = gx + b$. Here $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$.

Proof. (1) Assume that $x \overset{SM(n+1)}{\sim} y$. The functions $g_{ij}(x)$ and $L_{ds}(x)$, are $SM(n+1)$-invariant for all $1 \leq i, j, s \leq n$. So equalities Eq. (4.4) hold.

(2) Assume that equalities Eq. (4.4) hold. We prove the theorem for the case $d = 1$. Let W_1 be the set and $R\{W_1\}$ be the differential R-algebra defined in the proof of Theorem 3.40. Let $\delta = \delta_x$ be the function used in the proof of Theorem 3.23. Since $\delta = \det \|g_{ij}\|_{i,j=1}^n$, we have $\delta \in R\{W_1\}$. Using Eq. (2.2), we obtain $\Delta = \delta(L_{11})^2$. Hence $\Delta \in R\{W_1\}$. Since $x(u), y(u)$ are nondegenerate hypersurfaces, we have $\Delta_x(u) \neq 0$ and $\Delta_y(u) \neq 0$ for all $u \in U$. By Proposition 3.39, $\delta_x(u) > 0$ and $\delta_y(u) > 0$.

Let $R\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$ be the differential algebra used in the proof of Theorem 3.40. By Theorem 3.40, $L_{ij} \in R\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$ for all $i, j = 1, 2, \dots, n$. Using Eq. (2.2), we obtain $\left[\frac{\partial^2 x}{\partial u_i \partial u_j} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \cdots \frac{\partial x}{\partial u_n}\right] = \delta^{-\frac{1}{2}} L_{ij} \in R\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$ for all $i, j = 1, \dots n$. This implies $W \subset R\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\})$, where W is the set defined in the proof of Theorem 3.36. Hence $R\left\{W, \delta^{-1}, \Delta^{-1}\right\} \subseteq R\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$. Lemma 3.37 implies that $Z \subset R\left\{W, \delta^{-1}, \Delta^{-1}\right\}$, where Z is the system Eq. (3.17). Hence $R\{Z\} \subseteq R\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$.

The equalities $\delta_x = \delta_y$ and $\Delta_x = \Delta_y$ for all $u \in U$ imply that $\delta_x^{-1} = \delta_y^{-1}$ and $\Delta_x^{-1} = \Delta_y^{-1}$ for all $u \in U$. Let $f\{x\} \in R\{Z\} \subseteq R\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\}$. Then equalities $\delta_x^{-1} = \delta_y^{-1}$, $\Delta_x^{-1} = \Delta_y^{-1}$, and Eq. (4.4) imply that

$$f\{x(u)\} = f\{y(u)\} \tag{4.5}$$

for all $u \in U$. Since $R\{Z\} \subseteq R\left\{W_1, \delta^{-\frac{1}{2}}, \Delta^{-1}\right\})$, Eq. (4.5) implies Eq. (4.3). Then, by Theorem 4.5, $x \overset{SM(n+1)}{\sim} y$. Moreover, by Theorem 4.5, the unique $g \in SO(n+1)$ and $b \in R^{n+1}$ exist such that $y = gx + b$, namely $g = A(y)A(x)^{-1}$ and $b = y - A(y)A(x)^{-1}x$. Proof of Theorem 4.6 is completed. □

5. Conclusion and future work

Bonnet's theorem is well known for regular surfaces in R^3, and ensures that whenever the first and second fundamental forms of two parametric surfaces coincide, the surfaces are related by means of a rigid motion, i.e. they correspond to the same object up to a change in their position. The present paper provides more relaxed conditions to guarantee this property for nondegenerate hypersurfaces in R^{n+1}. In particular, our paper proves that one does not need to check that all the elements of both fundamental forms need to coincide. In fact, under certain hypotheses of nondegeneracy of the second fundamental form, it is enough to ensure the equality between some elements of the first and second fundamental forms, made precise in Theorem 4.6. It is known that, from Gauss-Codazzi equations, the elements of the first and second fundamental forms are not independent. Additionally, we showed that one does not need to assume regularity on the surface, since regularity is a consequence of the d-nondegeneracy of the parametrization (Proposition 3.39).

Besides, the results provided in our paper enable us to present the following questions arising naturally:

1. Which of the systems Eq. (4.1), Eq. (4.3), Eq. (4.4) is a minimal complete system?

2. Can we describe a complete system of relations between the elements of every complete system of Eq. (4.1), Eq. (4.3), Eq. (4.4)?

For future studies, we will consider the problem of devising a constructive method to identify when two parametric surfaces or hypersurfaces are the same. In more detail, if we take a parametric (hyper)surface $x(u)$, and apply a change of parameters $h(u)$, then $y(u) = (x \circ h)(u)$ and $x(u)$ define the same thing. But if $h(u)$ is unknown, it is not at all obvious, how to use Bonnet's theorem or the complete system provided in the paper, to detect that the images of $x(u)$ and $y(u)$ coincide.

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