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On a class of nonlocal porous medium equations of Kirchhoff type

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Abstract: We study the Dirichlet problem for the degenerate parabolic equation of the Kirchhoff type

$$u_t - a\left(\|u\|^p_{L^p(\Omega)}\right)\sum_{i=1}^{n} D_i\left(|u|^{p-2}D_iu\right) + b(x,t,u) = f(x,t) \quad \text{in } Q_T = \Omega\times(0,T),$$

where $p\geq 2$, $T>0$, $\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a smooth bounded domain. The coefficient $a(\cdot)$ is real-valued function defined on $\mathbb{R}_+$ and $b(\cdot,\cdot,\tau)$ is a measurable function with variable nonlinearity in τ. We prove existence of weak solutions of the considered problem under appropriate and general conditions on a and b. Sufficient conditions for uniqueness are found and in the case $f\equiv 0$ the decay rates for $\|u\|_{L^2(\Omega)}$ are obtained.

Key words: Nonlocal diffusion, Kirchhoff-type problem, variable nonlinearity, porous medium equation

1. Introduction

This paper is concerned with the existence, uniqueness and behavior of the solution for a nonlocal nonlinear parabolic Dirichlet-type boundary value problem whose model example is the following:

$$\begin{cases} \frac{\partial u}{\partial t} - a\left(\|u\|^p_{L^p(\Omega)}\right)\sum\limits_{i=1}^{n} D_i\left(|u|^{p-2}D_iu\right) + b(x,t,u) = f(x,t), \\ u(x,0) = 0 = u_0(x), \quad u\mid_{\Gamma_T} = 0 \end{cases} \tag{1.1}$$

where $p\geq 2$ and $(x,t)\in Q_T := \Omega\times(0,T)$, $T>0$, $\Gamma_T := \partial\Omega\times[0,T]$, $\Omega\subset\mathbb{R}^n\ (n\geq 2)$ is a bounded domain with Lipschitz boundary, $D_i\equiv\partial/\partial x_i$, $a(\cdot)$ is real-valued function defined on $\mathbb{R}_+$ and $b:\Omega\times(0,T)\times\mathbb{R}\to\mathbb{R}$, $b(x,t,\tau)$ is a function with variable nonlinearity in τ, (for example, $b(x,t,\tau) = b_0(x,t)|\tau|^{\alpha(x,t)-2}\tau + b_1(x,t)$). Besides for the functions $u(t):(0,T)\mapsto L^p(\Omega)$ we denote

$$\|u(t)\|^p_{L^p(\Omega)} = \|u(t)\|^p_p = \int_\Omega |u(x,t)|^p\,dx.$$

Equation (1.1) is considered as the class of nonlocal evolution equations, in which the arguments of some terms are functionals of the unknown function. Such equations are often termed the Kirchhoff type equations. This is because an equation (of hyperbolic type) with one of the coefficients given by the Dirichlet energy integral of the unknown function was first proposed by Kirchhoff in 1883 as a model of the transversal oscillation of a string

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[24]. In this model, the change of the string length caused by oscillation was taken into account. Pohozaev worked on some early classical studies of these Kirchhoff of type equations in [27].

There are numerous nonlocal mathematical models of Kirchhoff type studied by many authors to express the processes in physics and engineering see, e.g., [1, 9, 11, 12, 15, 20, 35, 41] and references therein. For example, nonlocal PDEs arise in mathematical modelling of migration of a population to describe the density of some biological species are worked in [3, 13, 17, 21], nonlocal models obtained from combustion theory is considered in [5, 22] and in medicine [8].

There is a series of papers devoted to answer the questions of existence, uniqueness and asymptotic behavior of solutions of the initial and boundary-value problems for the equations

$$u_t - a(l(u))\Delta u = f, \quad u_t - a(\|\nabla u\|^2_{L^2(\Omega)})\Delta u = f, \quad u_t - a(\int_\Omega u\, dx)\Delta u = h$$

which were studied in [10–12, 14, 15, 41] with a continuous function a whose argument $l(u)$ was a linear continuous functional on $L^2(\Omega)$, or a continuously differentiable function a of the argument $\|\nabla u\|^2_{L^2(\Omega)}$. In these works, the equation is nondegenerate: there exist positive constants $0 < m_0 \le M_0 < \infty$ such that

$$m_0 \le a(s) \le M_0, \quad \forall s \in \mathbb{R}. \tag{1.2}$$

The nonlocal problems without condition (1.2) were studied in [1, 2, 28]. Paper [2] deals with the homogeneous Dirichlet problem for the degenerate nonlocal equation

$$u_t - \|u(t)\|^{2\gamma}_{L^2(\Omega)}\Delta u = f, \quad \gamma \in \mathbb{R}.$$

It is proven that for $\gamma \ge 0$ the solution exists globally in time, in the case $\gamma < 0$ local in time existence is established and is shown that if $f \equiv 0$ then every solution vanishes in finite time.

The class of nonlocal fractional equations of Kirchhoff type were studied in [37–40]. Paper [38] deals with the following problem:

$$u_t + M([u]_s^2)(-\Delta)^s u = |u|^{p-2}u, \qquad 0 < s < 1 < p < \infty.$$

The diffusion coefficient $M(\cdot)$ is a continuous function depending on the Gagliardo seminorm $[u]_s$ and could be zero at the origin. It is assumed that $M(s) \ge ms^{\theta-1}$ for all $s \ge 0$ and some constants $m > 0$, $\theta > 1$. The authors prove local in time solvability of the Dirichlet problem and find conditions of finite-time blow-up of nonnegative solutions.

The authors of [40] study existence and multiplicity of solutions for the Schrödinger–Kirchhoff type problems involving the fractional p–Laplacian of the form

$$\|u\|_\lambda^{(\theta-1)}[\lambda(-\Delta)^s_p u + V(x)|u|^{p-2}u] = |u|^{p_s^*-2}u + f(x,u) \text{ in } \mathbb{R}^n.$$

The main features of this paper are the facts that the Kirchhoff function is zero at zero and the potential function satisfies the critical frequency $\inf_{x\in\mathbb{R}} V(x) = 0$.

Degenerate nonlinear nonlocal evolution equations which can be understood as a porous medium equation whose pressure law is nonlinear and nonlocal was considered in [6, 7, 16] (and see references therein). In [16], authors studied the following evolution equation of diffusive type with nonlocal effects:

$$\partial_t u - \operatorname{div}\left(|u|^{m_1}\nabla(-\Delta)^{-s}\left[|u|^{m_2-1}u\right]\right) = f, \quad \text{in } \mathbb{R}^n \times (0,T)$$

which corresponds to the well-known porous medium equation $u_t = \operatorname{div}(u^{m_1}\nabla u)$ when $s = 0$ and $m_2 = 1$. They proved the existence of weak solutions for $s \in (0,1)$ and $m_1, m_2 > 0$. For the case $f \equiv 0$, decay estimates are obtained and conditions for finite time extinction of solution are found. This equation with $m_1 = m_2 = 1$ which reads as $u_t = \operatorname{div}(u\nabla(-\Delta)^{-s}u)$ was first introduced in the paper [7].

The boundary-value problems including the equations of type (1.1) is known as Newtonian filtration equation which can be given in the following general form:

$$u_t = \Delta\varphi(u) + h.$$

Equation (1.1) is a parabolic equation with implicit degeneracy which is so called the porous medium equation [23, 25, 36] i.e.

$$u_t = \Delta\left(|u|^{m-1}u\right) + h,$$

where $m > 1$. There exists abundant physical applications where this simple model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. This equation is parabolic for u different from 0 and degenerates when $u = 0$. Under condition $m > 1$, above equation describes the nonstationary flow of a compressible Newtonian fluid in a porous medium under polytropic conditions.

In the present paper, we generalize the results mentioned above to the Kirchhoff type porous medium equation by considering the equation with nonlocal diffusion

$$a(\|u\|^p_{L^p(\Omega)})\Delta\left(|u|^{p-2}u\right) \text{ for all } p \geq 2$$

and such an additional term h with variable nonlinearity.

We want to emphasize that if we rearrange the diffusion part of the equation (1.1), we arrive at

$$u_t = a\left(\|u\|^p_{L^p(\Omega)}\right)\Delta\left(|u|^{p-2}u\right) + F(x,t,u,f). \tag{1.3}$$

To the best of our knowledge, there has not been any studies on porous medium equations of the type (1.3) providing a nonlocal coefficient $a(\|u\|^p_{L^p(\Omega)})$ and forcing term F whose argument depends on $|u|$ with variable nonlinearity.

We apply the general solvability theorem [32], see Theorem 2.6, to prove the existence of weak solution of (1.1). We study problem (1.1) on the domain of the operator generated by addressed problem and verify the existence of sufficiently smooth solution of the problem under more general (weak) conditions. Essentially we show that problem (1.1) has a solution in the space

$$S_0 := L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T) \cap W^{1,q}\left(0,T;W^{-1,q}(\Omega)\right) \cap \{u : u(x,0) = 0\},$$

where

$$\mathring{S}_{1,(p-2)q,q}(\Omega) := \left\{u \in L^1(\Omega) : \sum_{i=1}^{n}\left(\int_\Omega |u|^{(p-2)q}|D_i u|^q\,dx\right) < \infty\right\} \cap \{u\,|_{\partial\Omega} \equiv 0\}.$$

Apart from linear boundary value problems, the sets generated by nonlinear problems are subsets of linear spaces which do not have the linear structure (see [29–34] and references therein).

In Section 4, we study a protopype of equation (1.1) by taking $b(x,t,u) = C_\alpha |\tau|^{\alpha(x,t)-2}\tau$ i.e.

$$u_t - a\left(\|u\|^p_{L^p(\Omega)}\right)\sum_{i=1}^{n} D_i\left(|u|^{p-2} D_i u\right) + C_\alpha |u|^{\alpha(x,t)-2} u = f.$$

We prove the uniqueness of this equation under more restrictive conditions and in the case $f=0$ the decay rates for $\|u(t)\|_{L^2(\Omega)}$ are derived.

2. Preliminaries

Although the forcing term in Equation (1.1) involves the variable power of the unknown function, the theory of the generalized Lebesgue spaces which are so called Orlicz-Lebesgue space as well is not used in this work, except for several basic facts which can be found in [4, 18].

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ such that $|\Omega| > 0$ (Throughout the paper, we denote by $|\Omega|$ the Lebesgue measure of Ω). Let $\alpha(x,t) \geq 1$ be a measurable bounded function defined on the cylinder $Q_T = \Omega \times (0,T)$ i.e.

$$1 \leq \alpha^- \equiv \underset{Q_T}{ess}\inf|\alpha(x,t)| \leq \underset{Q_T}{ess}\sup|\alpha(x,t)| \equiv \alpha^+ < \infty. \tag{2.1}$$

Then on the set of all functions on Q_T define the functional σ_α and $\|\cdot\|_{L^{\alpha(x,t)}(Q_T)}$ by

$$\sigma_\alpha(u) \equiv \int\limits_{Q_T} |u|^{\alpha(x,t)}\, dxdt$$

and

$$\|u\|_{L^{\alpha(x,t)}(Q_T)} \equiv \inf\left\{\lambda > 0 |\ \sigma_\alpha\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

The generalized Lebesgue space is defined as follows:

$$L^{\alpha(x,t)}(Q_T) := \{u : u \text{ is a measurable real-valued function in } Q_T,\ \sigma_\alpha(u) < \infty\}.$$

The space $L^{\alpha(x,t)}(Q_T)$ becomes a Banach space under the norm $\|.\|_{L^{\alpha(x,t)}(Q_T)}$ which is so-called Luxemburg norm.

Lemma 2.1 *Let* $0 < |\Omega| < \infty,\ \alpha_1,\ \alpha_2$ *fulfill* (2.1) *then*

$$L^{\alpha_1(x,t)}(Q_T) \subset L^{\alpha_2(x,t)}(Q_T) \iff \alpha_2(x,t) \leq \alpha_1(x,t) \text{ for a.e } (x,t) \in Q_T.$$

Lemma 2.2 *The dual space of* $L^{\alpha(x,t)}(Q_T)$ *is* $L^{\alpha^*(x,t)}(Q_T)$ *if and only if* $\alpha \in L^\infty(Q_T)$. *The space* $L^{\alpha(x,t)}(Q_T)$ *is reflexive if and only if*

$$1 < \alpha^- \leq \alpha^+ < \infty$$

here $\alpha^*(x,t) \equiv \frac{\alpha(x,t)}{\alpha(x,t)-1}$.

For $u \in L^{\alpha(x,t)}(Q_T)$ and $v \in L^{\alpha^*(x,t)}(Q_T)$ where $\alpha,\ \alpha^*$ satisfy (2.1) and $\frac{1}{\alpha(x,t)} + \frac{1}{\alpha^*(x,t)} = 1$, the following inequalities hold:

$$\int\limits_{Q_T} |uv|\ dxdt \leq 2\,\|u\|_{L^{\alpha(x,t)}(Q_T)}\,\|v\|_{L^{\alpha^*(x,t)}(Q_T)}$$

and

$$\min\{\|u\|^{\alpha^-}_{L^{\alpha(x,t)}(Q_T)}, \|u\|^{\alpha^+}_{L^{\alpha(x,t)}(Q_T)}\} \leq \sigma_\alpha(u) \leq \max\{\|u\|^{\alpha^-}_{L^{\alpha(x,t)}(Q_T)}, \|u\|^{\alpha^+}_{L^{\alpha(x,t)}(Q_T)}\}. \tag{2.2}$$

We introduce certain nonlinear function spaces (pn-spaces) which are complete metric spaces and directly connected to the problem under consideration. We also give some embedding results for these spaces [30–34] (see also references cited therein).

Definition 2.3 *Let $\gamma \geq 0,\ \beta \geq 1,\ \varrho = (\varrho_{1,..,}\varrho_n)$ is multiindex, $m \in \mathbb{Z}^+,\ \Omega \subset \mathbb{R}^n\,(n \geq 1)$ is bounded domain with Lipschitz boundary.*

$$S_{m,\gamma,\beta}(\Omega) \equiv \left\{ u \in L^1(\Omega) \mid [u]^{\gamma+\beta}_{S_{m,\gamma,\beta}(\Omega)} \equiv \sum_{0 \leq |\varrho| \leq m} \left(\int\limits_{\Omega} |u|^\gamma |D^\varrho u|^\beta\, dx \right) < \infty \right\}$$

in particularly,

$$\mathring{S}_{1,\gamma,\beta}(\Omega) \equiv \left\{ u \in L^1(\Omega) \mid [u]^{\gamma+\beta}_{\mathring{S}_{1,\gamma,\beta}(\Omega)} \equiv \sum_{i=1}^{n} \left(\int\limits_{\Omega} |u|^\gamma |D_i u|^\beta\, dx \right) < \infty \right\} \cap \{u\mid_{\partial\Omega} \equiv 0\}$$

and for $p \geq 1$,

$$L^p\left(0,T;\mathring{S}_{1,\gamma,\beta}(\Omega)\right) \equiv \left\{ u \in L^1(Q_T) \mid [u]^p_{L^p\left(0,T;\mathring{S}_{1,\gamma,\beta}(\Omega)\right)} \equiv \int\limits_0^T [u]^p_{\mathring{S}_{1,\gamma,\beta}(\Omega)}\, dt < \infty \right\}.$$

*These spaces are called pn-spaces.**

Theorem 2.4 *Let $\gamma \geq 0,\ \beta \geq 1$ then $\varphi : \mathbb{R} \mapsto \mathbb{R},\ \varphi(t) \equiv |t|^{\frac{\gamma}{\beta}}\, t$ is a homeomorphism between $S_{1,\gamma,\beta}(\Omega)$ and $W^{1,\beta}(\Omega)$.*

Theorem 2.5 *The following embeddings hold:*

(i) Let $\gamma,\ \gamma_1 \geq 0$ and $\beta_1 \geq 1,\ \beta \geq \beta_1,\ \frac{\gamma_1}{\beta_1} \geq \frac{\gamma}{\beta},\ \gamma_1 + \beta_1 \leq \gamma + \beta$ then we have

$$\mathring{S}_{1,\gamma,\beta}(\Omega) \subseteq \mathring{S}_{1,\gamma_1,\beta_1}(\Omega).$$

* $S_{1,\gamma,\beta}(\Omega)$ is a complete metric space with the following metric: $\forall u, v \in S_{1,\gamma,\beta}(\Omega)$

$$d_{S_{1,\gamma,\beta}}(u,v) = \left\| |u|^{\frac{\gamma}{\beta}}\, u - |v|^{\frac{\gamma}{\beta}}\, v \right\|_{W^{1,\beta}(\Omega)}$$

(ii) *Let* $\gamma \geq 0,\ \beta \geq 1,\ n > \beta$ *and* $\frac{n(\gamma+\beta)}{n-\beta} \geq r$ *then there is a continuous embedding*

$$\mathring{S}_{1,\gamma,\beta}(\Omega) \subset L^r(\Omega).$$

Furthermore for $\frac{n(\gamma+\beta)}{n-\beta} > r$ *the embedding is compact.*

(iii) *If* $\gamma \geq 0,\ \beta \geq 1$ *and* $p \geq \gamma+\beta$ *then*

$$W_0^{1,p}(\Omega) \subset \mathring{S}_{1,\gamma,\beta}(\Omega)$$

holds.

In the following, we present the general solvability theorem [32](see also for similar theorems [31, 34]). We will employ this theorem to demonstrate the existence of a weak solution of problem (1.1).

Theorem 2.6 *Let* X *and* Y *be Banach spaces with dual spaces* X^* *and* Y^* *respectively,* Y *be a reflexive Banach space,* $M_0 \subseteq X$ *be a weakly complete "reflexive" pn-space,* $X_0 \subseteq M_0 \cap Y$ *be a separable vector topological space. Let the following conditions be fulfilled:*

(i) $\xi : S_0 \longrightarrow L^q(0,T;Y)$ *is a weakly compact (weakly continuous) mapping, where*

$$S_0 := L^p(0,T;M_0) \cap W^{1,q}(0,T;Y) \cap \{x(t) : x(0) = 0\}$$

$1 < \max\{q, q'\} \leq p < \infty,\ q' = \frac{q}{q-1}$*;*

(ii) *there is a linear continuous operator* $A : W^{s,m}(0,T;X_0) \longrightarrow W^{s,m}(0,T;Y^*),\ s \geq 0,\ m \geq 1$ *such that* A *commutes with* $\frac{\partial}{\partial t}$ *and the conjugate operator* A^* *has* $ker(A^*) = 0$*;*

(iii) *operators* ξ *and* A *generate, in generalized sense, a coercive pair on space* $L^p(0,T;X_0)$*, i.e. there exist a number* $r > 0$ *and a function* $\Psi : \mathbb{R}^1_+ \longrightarrow \mathbb{R}^1_+$ *such that* $\Psi(\tau)/\tau \nearrow \infty$ *as* $\tau \nearrow \infty$ *and for any* $x \in L^p(0,T;X_0)$ *such that* $[x]_{L^p(M_0)} \geq r$ *following inequality holds:*

$$\int_0^T \langle \xi(t, x(t)), Ax(t) \rangle \, dt \geq \Psi\left([x]_{L^p(M_0)}\right);$$

(iv) *there exist some constants* $C_0 > 0,\ C_1, C_2 \geq 0$ *and* $\nu > 1$ *such that the inequalities*

$$\int_0^T \langle \eta(t), A\eta(t) \rangle \, dt \geq C_0 \|\eta\|^{\nu}_{L^q(0,T;Y)} - C_2,$$

$$\int_0^t \left\langle \frac{\partial x}{\partial \tau}, Ax(\tau) \right\rangle d\tau \geq C_1 \|x\|^{\nu}_Y(t) - C_2, \quad a.e.\ t \in [0,T]$$

hold for any $x \in W^{1,p}(0,T;X_0)$ *and* $\eta \in L^p(0,T;X_0)$*.*

Assume that that conditions (i)–(iv) are fulfilled. Then the Cauchy problem

$$\frac{dx}{d\tau} + \xi\left(t, x\left(t\right)\right) = y\left(t\right), \qquad y \in L^{q}\left(0, T; Y\right); \quad x\left(0\right) = 0$$

is solvable in S_0 *in the following sense*

$$\int\limits_{0}^{T} \left\langle \frac{dx}{d\tau} + \xi\left(t, x\left(t\right)\right), y^{*}\left(t\right) \right\rangle dt = \int\limits_{0}^{T} \left\langle y\left(t\right), y^{*}\left(t\right) \right\rangle, \quad \forall y^{*} \in L^{q'}\left(0, T; Y^{*}\right),$$

for any $y \in L^{q}\left(0, T; Y\right)$ *satisfying the inequality*

$$\sup \left\{ \frac{1}{[x]_{L^{p}\left(0,T;M_0\right)}} \int\limits_{0}^{T} \left\langle y\left(t\right), Ax\left(t\right) \right\rangle dt : x \in L^{p}\left(0, T; X_0\right) \right\} < \infty.$$

3. Assumptions and the main result

Assume that following conditions are fulfilled for problem (1.1):

(H.1) Let $p \geq 2,\ a(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function and there exist positive constants $0 < m \leq M < \infty$ such that

$$m \leq a(s) \leq M, \quad \forall s \in \mathbb{R}. \tag{3.1}$$

(H.2) $b : Q_T \times \mathbb{R} \to \mathbb{R}$, is a Carathédory function that fulfills the following conditions: There exists a measurable function $\alpha : \Omega \times (0, T) \longrightarrow \mathbb{R},\ 1 < \alpha^{-} \leq \alpha\left(x, t\right) \leq \alpha^{+} < \infty$ such that $b\left(x, t, \tau\right)$ satisfies the following inequalities a.e. $\left(x, t, \tau\right) \in Q_T \times \mathbb{R}$:

$$\left|b\left(x, t, \tau\right)\right| \leq b_0\left(x, t\right)\left|\tau\right|^{\alpha(x,t)-1} + b_1\left(x, t\right) \tag{3.2}$$

and

$$b\left(x, t, \tau\right)\tau \geq b_2\left(x, t\right)\left|\tau\right|^{\alpha(x,t)} - b_3\left(x, t\right). \tag{3.3}$$

Here $b_i,\ i = 0, 1, 2, 3$ are nonnegative, measurable functions defined on Q_T and $b_2\left(x, t\right) \geq B_0 > 0$ a.e. $\left(x, t\right) \in Q_T$.

We study problem (1.1) for the functions $f \in L^{q}\left(0, T; W^{-1,q}\left(\Omega\right)\right) + L^{\alpha^{*}(x,t)}\left(Q_T\right)$ where α^{*} is conjugate of α i.e. $\alpha^{*}\left(x, t\right) := \frac{\alpha(x,t)}{\alpha(x,t)-1}$ and the dual space $W^{-1,q}\left(\Omega\right) := \left(W_0^{1,p}\left(\Omega\right)\right)^{*},\ q := \frac{p}{p-1}$.

Let us denote S_0 by

$$S_0 := L^{p}\left(0, T; \mathring{S}_{1,(p-2)q,q}\left(\Omega\right)\right) \cap L^{\alpha(x,t)}\left(Q_T\right) \cap W^{1,q}\left(0, T; W^{-1,q}\left(\Omega\right)\right) \cap \{u : u\left(x, 0\right) = 0\}.$$

The solution of the problem (1.1) is understood in the following sense:

Definition 3.1 *A function $u : Q_T \to \mathbb{R}$ is called a weak solution of problem (1.1) if*

(i) $u \in S_0$;

(ii) for every test-function $\eta \in L^p\left(0, T; \mathring{S}_{1,(p-2)q,q}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T) \cap W^{1,q}\left(0, T; W^{-1,q}(\Omega)\right)$

$$\int_{Q_T} u_t \eta \, dz + \sum_{i=1}^{n} \int_{Q_T} \left(a(\|u\|_p^p) \, |u|^{p-2} D_i u \right) D_i \eta \, dz + \int_{Q_T} a(x, t, u) \, \eta \, dz = \int_{Q_T} f \eta \, dz; \tag{3.4}$$

Prior to formulating the main theorem of this section, we introduce following function

$$\beta(x,t) := \begin{cases} \frac{p\alpha^*(x,t)}{p-\alpha(x,t)} & \text{if } \alpha^+ < p, \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 3.2 *Let the conditions* **(H.1)–(H.2)** be fulfilled. If $b_0 \in L^{\beta(x,t)}(Q_T)$, $b_1 \in L^{\alpha^*(x,t)}(Q_T)$, $b_2 \in L^{\infty}(Q_T)$, $b_3 \in L^1(Q_T)$ then for all $f \in L^q\left(0, T; W^{-1,q}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$ problem (1.1) has a weak solution in the space S_0 and $\partial u / \partial t$ belongs to $L^q\left(0, T; W^{-1,q}(\Omega)\right)$.

We introduce the following mappings in order to apply Theorem 2.6 to prove Theorem 3.2.

$$\xi : S_0 \longrightarrow L^q\left(0, T; W^{-1,q}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T),$$

$$\xi(u) := -a(\|u\|_p^p) \sum_{i=1}^{n} D_i \left(|u|^{p-2} D_i u \right) + b(x, t, u),$$

$$A : L^p\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T) \subset S_0 \longrightarrow L^p\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T),$$

$$A(u) := u.$$

We prove several lemmas to show that all conditions of Theorem 2.6 are fulfilled under the conditions of Theorem 3.2.

Lemma 3.3 *Under the conditions of Theorem 3.2, ξ and A generate a "coercive pair" on $L^p\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T)$.*

Proof Since $A \equiv Id$, being "coercive pair" equals to order coercivity of ξ on $L^p\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T)$.

For $u \in L^p\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T)$, we have the following equation:

$$\langle \xi(u), u \rangle_{Q_T} = \sum_{i=1}^{n} \left(\int_0^T a(\|u\|_p^p) \int_\Omega |u|^{p-2} |D_i u|^2 \, dx dt \right) + \int_{Q_T} b(x, t, u) \, u \, dz.$$

By using (3.1) and (3.3), we have

$$\begin{aligned} \langle \xi(u), u \rangle_{Q_T} \geq m \sum_{i=1}^{n} \left(\int_0^T \int_\Omega |u|^{p-2} |D_i u|^2 \, dz \right) + \int_{Q_T} |b_2(x,t)| \, |u|^{\alpha(x,t)} \, dz \\ - \int_{Q_T} |b_3(x,t)| \, dz \end{aligned} \tag{3.5}$$

Using (3.3) and Definition 2.3 we obtain

$$\langle \xi(u), u \rangle_{Q_T} \geq m \, [u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2),2}(\Omega)\right)} + B_0 \int_{Q_T} |u|^{\alpha(x,t)} \, dz - \|b_3\|_{L^1(Q_T)} . \tag{3.6}$$

If we consider the embedding

$$\mathring{S}_{1,(p-2),2}(\Omega) \subset \mathring{S}_{1,(p-2)q,q}(\Omega)$$

to estimate pseudo-norm in (3.6), we get

$$\langle \xi(u), u \rangle_{Q_T} \geq mC \, [u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)} + B_0 \int_{Q_T} |u|^{\alpha(x,t)} \, dz - \|b_3\|_{L^1(Q_T)} . \tag{3.7}$$

Using (2.2) to estimate the integral right-hand side of (3.7), we obtain

$$\langle \xi(u), u \rangle_{Q_T} \geq C_0 \left([u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)} + \|u\|^{\alpha^-}_{L^{\alpha(x,t)}(Q_T)} \right) - C_1 . \tag{3.8}$$

Here, $C_1 = C_1\left(\|b_3\|_{L^1(Q_T)}, B_0\right)$, $C_0 = C_0(p, m, B_0, |\Omega|)$ are positive constants. So from (3.8) the proof is completed. □

Lemma 3.4 *Under the conditions of Theorem 3.2, ξ is bounded from S_0 into $L^q\left(0,T;W^{-1,q}(\Omega)\right)+L^{\alpha^*(x,t)}(Q_T)$.*

Proof We define the mappings

$$\begin{aligned} \xi_1(u) &:= -a(\|u\|_p^p) \sum_{i=1}^{n} D_i \left(|u|^{p-2} D_i u \right), \\ \xi_2(u) &:= b(x,t,u) . \end{aligned}$$

We need to show that, these mappings are both bounded from $L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T)$ into $L^q\left(0,T;W^{-1,q}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$.

Let us show that ξ_1 is bounded: For $u \in L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)$ and $v \in L^p\left(0,T;W_0^{1,p}(\Omega)\right)$

$$\left|\langle \xi_1(u), v\rangle_{Q_T}\right| \le \sum_{i=1}^{n}\left(\int_0^T a(\|u\|_p^p)\int_\Omega |u|^{p-2}|D_i u|\,|D_i v|\,dxdt\right)$$

Applying Hölder's inequality and by (3.1) we find,

$$\begin{aligned}\left|\langle \xi_1(u), v\rangle_{Q_T}\right| &\le M\left[\sum_{i=1}^{n}\left(\int_0^T\int_\Omega |u|^{(p-2)q}|D_i u|^q\,dxdt\right)\right]^{\frac{1}{q}}\left[\sum_{i=1}^{n}\left(\int_0^T\int_\Omega |D_i v|^p\,dxdt\right)\right]^{\frac{1}{p}}\\ &= M\,[u]^{p-1}_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)}\|v\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)}.\end{aligned}$$

By the last inequality, boundedness of ξ_1 is obtained.

Similarly, from (3.2) and Theorem 2.5, for all $u\in S_0$, we have the following estimate

$$\begin{aligned}\sigma_{\alpha^*}(\xi_2(u)) &= \sigma_{\alpha^*}(b(x,t,u))\\ &= \int_0^T\int_\Omega |b(x,t,u)|^{\alpha^*(x,t)}\,dz\\ &\le C_2\left(\sigma_\alpha(u)+[u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)}\right)+C_3,\end{aligned}$$

here $C_2 = C_2\left(\alpha^+,\alpha^-,\|b_0\|_{L^{\beta(x,t)}(Q_T)}\right)$, $C_3 = C_3(\sigma_\beta(b_0),\sigma_{\alpha^*}(b_1),|\Omega|)>0$ are constants. That yields $\xi_2 : L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)\cap L^{\alpha(x,t)}(Q_T)\to L^{\alpha^*(x,t)}(Q_T)$ is bounded. □

Lemma 3.5 *Under the conditions of Theorem 3.2, ξ is weakly compact from S_0 into $L^q\left(0,T;W^{-1,q}(\Omega)\right)+L^{\alpha^*(x,t)}(Q_T)$.*

Proof We first prove the weak compactness of ξ_1 where $\xi_1(u) := -a(\|u\|_p^p)\sum_{i=1}^n D_i\left(|u|^{p-2}D_i u\right)$. Let $\{u_m(x,t)\}_{m=1}^\infty\subset S_0$ be bounded and $u_m \overset{S_0}{\rightharpoonup} \tilde{u_0}$. It is sufficient to find a subsequence of $\left\{u_{m_j}\right\}_{m=1}^\infty\subset\{u_m\}_{m=1}^\infty$ which satisfies $\xi_1\left(u_{m_j}\right)\overset{L^q\left(0,T;W^{-1,q}(\Omega)\right)}{\rightharpoonup}\xi_1(\tilde{u_0})$.

For a.e. $t\in(0,T)$, $u_m(\cdot,t)\in\mathring{S}_{1,(p-2)q,q}(\Omega)$ and by using the one-to-one correspondence between the classes (Theorem 2.4)

$$\mathring{S}_{1,(p-2)q,q}(\Omega)\underset{\varphi^{-1}}{\overset{\varphi}{\longleftrightarrow}}W_0^{1,q}(\Omega)$$

with the homeomorphism

$$\varphi(\tau)\equiv|\tau|^{p-2}\tau,\quad \varphi^{-1}(\tau)\equiv|\tau|^{-\frac{p-2}{p-1}}\tau,$$

for all $m\ge1$ we have

$$|u_m|^{p-2}\,u_m \in L^q\left(0,T;W_0^{1,q}(\Omega)\right) \text{ is bounded.}$$

Due to the fact $L^q\left(0,T;W_0^{1,q}(\Omega)\right)$ is a reflexive space, there exists a subsequence $\{u_{m_j}\}_{m=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ such that

$$\left|u_{m_j}\right|^{p-2} u_{m_j} \overset{L^q\left(0,T;W_0^{1,q}(\Omega)\right)}{\rightharpoonup} \zeta. \tag{3.9}$$

Now, we show that $\zeta = |\tilde{u_0}|^{p-2}\,\tilde{u_0}$. According to compact embedding [34],

$$L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \cap W^{1,q}\left(0,T;W^{-1,q}(\Omega)\right) \hookrightarrow L^p(Q_T) \tag{3.10}$$

we have

$$\exists \left\{u_{m_{j_k}}\right\}_{m=1}^{\infty} \subset \left\{u_{m_j}\right\}_{m=1}^{\infty},\; u_{m_{j_k}} \overset{L^p(Q_T)}{\rightarrow} \tilde{u_0}$$

which implies

$$u_{m_{j_k}} \underset{a.e}{\overset{Q_T}{\rightarrow}} \tilde{u_0}$$

by the continuity of $\varphi(\cdot)$, we get

$$\left|u_{m_{j_k}}\right|^{p-2} u_{m_{j_k}} \underset{a.e}{\overset{Q_T}{\rightarrow}} |\tilde{u_0}|^{p-2}\,\tilde{u_0}$$

that yields $\zeta = |\tilde{u_0}|^{p-2}\,\tilde{u_0}$.

By the the compact embedding (3.10) and continuity of $a(\cdot)$ we have

$$a(\|u_{m_{j_k}}(t)\|_p^p) \to a(\|\tilde{u_0}(t)\|_p^p) \quad \text{a.e. in } (0,T). \tag{3.11}$$

Thus by using (3.9) and (3.11) together with the boundedness of ξ_1 from Lemma 3.4, we deduce that for each $v \in L^p\left(0,T;W_0^{1,p}(\Omega)\right)$

$$\begin{aligned}\langle \xi_1\left(u_{m_{j_k}}\right), v\rangle_{Q_T} &= \sum_{i=1}^{n}\int_0^T a(\|u_{m_{j_k}}\|_p^p)\langle -D_i\left(\left|u_{m_{j_k}}\right|^{p-2} D_i u_{m_{j_k}}\right), v\rangle_\Omega\, dt \\ &\underset{m_j\nearrow\infty}{\longrightarrow} \sum_{i=1}^{n}\int_0^T a(\|\tilde{u_0}(t)\|_p^p)\langle -D_i\left(|\tilde{u_0}|^{p-2} D_i\tilde{u_0}\right), v\rangle_\Omega\, dt = \langle \xi_1(\tilde{u_0}), v\rangle_{Q_T}\end{aligned}$$

whence, the result is obtained.

We shall show the weak compactness of ξ_2:

$$b : L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T) \to L^{\alpha^*(x,t)}(Q_T)$$

is bounded by Lemma 3.4, thus for $m \geq 1$, $\xi_2(u_m) = \{b(x,t,u_m)\}_{m=1}^{\infty} \subset L^{\alpha^*(x,t)}(Q_T)$.

From Lemma 2.2, $L^{\alpha^*(x,t)}(Q_T)$ $(1 < (\alpha^*)^- < \infty)$ is a reflexive space, so $\{u_m\}_{m=1}^{\infty}$ has a subsequence $\{u_{m_j}\}_{m=1}^{\infty}$ such that

$$b\left(x,t,u_{m_j}\right) \overset{L^{\alpha^*(x,t)}(Q_T)}{\rightharpoonup} \psi.$$

We deduce from the compact embedding (3.10) that

$$\exists \left\{u_{m_{j_k}}\right\}_{m=1}^{\infty} \subset \left\{u_{m_j}\right\}_{m=1}^{\infty}, \; u_{m_{j_k}} \overset{L^p(Q_T)}{\rightarrow} \tilde{u_0}$$

thus

$$u_{m_{j_k}} \underset{a.e}{\overset{Q_T}{\rightarrow}} \tilde{u_0}.$$

Accordingly, the continuity of $b(x,t,.)$ for almost $(x,t) \in Q_T$ implies that

$$b(x,t,u_{m_{j_k}}) \underset{a.e}{\overset{Q_T}{\rightarrow}} b\left(x,t,\tilde{u_0}\right),$$

so, we arrive at $\psi = b(x,t,\tilde{u_0})$ i.e. $\xi_2(u_{m_{j_k}}) \overset{L^q\left(0,T;W^{-1,q}(\Omega)\right)+L^{\alpha^*(x,t)}(Q_T)}{\rightharpoonup} \xi_2\left(\tilde{u_0}\right).$

As a conclusion, we show that ξ is weakly compact from S_0 into $L^q\left(0,T;W^{-1,q}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T).$ $\square$

Now, we give the proof of main theorem of this section.

Proof of Theorem 3.2. Since $A = Id$, obviously it is a linear bounded map and satisfies the conditions (ii) of Theorem 2.6. Furthermore for any $u \in W_0^{1,p}(Q_T)$ the following inequalities are valid:

$$\int_0^T \langle u,u\rangle_\Omega \, dt = \int_0^T \|u\|_{L^2(\Omega)}^2 \, dt \geq K \|u\|_{L^q(0,T;W^{-1,q}(\Omega))}^2$$

and

$$\int_0^t \left\langle \frac{\partial u}{\partial \tau}, u \right\rangle_\Omega d\tau = \frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2 \geq K\frac{1}{2}\|u(t)\|_{W^{-1,q}(\Omega)}^2,$$

a.e. $t \in [0,T]$ (constant $K > 0$ comes from embedding inequality). Thus condition (iv) of Theorem 2.6 is satisfied as well. Consequently from Lemmas 3.3–3.5, it follows that the mappings ξ and A fulfill all the conditions of Theorem 2.6. Employing this theorem to problem (1.1), we find that (1.1) is solvable in S_0 for any $f \in L^q\left(0,T;W^{-1,q}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$ satisfying the following inequality

$$\sup\left\{ \frac{1}{[u]_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)} + \|u\|_{L^{\alpha(x,t)}(Q_T)}} \int_0^T \langle f,u\rangle_\Omega \, dt : u \in Q_0 \right\} < \infty$$

where $Q_0 := L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T).$

Considering the norm definition of f in $L^q\left(0,T;W^{-1,q}(\Omega)\right)+L^{\alpha^*(x,t)}(Q_T)$, we conclude that (1.1) is solvable in S_0 for any $f\in L^q\left(0,T;W^{-1,q}(\Omega)\right)+L^{\alpha^*(x,t)}(Q_T)$. In order to complete the proof, it remains to remark that (1.1) can be written in the form

$$\frac{\partial u}{\partial t}=f(x,t)-F(x,t,u,D_iu),$$

and under the conditions of Theorem 3.2, right hand belongs to $L^q\left(0,T;W^{-1,q}(\Omega)\right)$ which implies $\partial u/\partial t\in L^q\left(0,T;W^{-1,q}(\Omega)\right)$.

Remark 3.6 *We note that if the function $\alpha(x,t)$ in (3.2) satisfies the inequality $\alpha^+<p$ then the existence of a solution of the problem (1.1) can be shown under more general (weaker) conditions. This is verified in the following theorem.*

Theorem 3.7 *Assume that* **(H.1)** and (3.2) are satisfied with $1<\alpha^-\le\alpha(x,t)\le\alpha^+<p$. If $b_0\in L^{\beta_1(x,t)}(Q_T)$, $b_1\in L^{\alpha^*(x,t)}(Q_T)$ where $\beta_1(x,t):=\frac{p\alpha^*(x,t)}{p-\alpha(x,t)}$ then for $f\in L^q\left(0,T;W^{-1,q}(\Omega)\right)$ problem (1.1) has a generalized solution in the space $L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)\cap W^{1,q}\left(0,T;W^{-1,q}(\Omega)\right)$.

Proof We deduce from inequality (3.2) that

$$\langle\xi(u),u\rangle_{Q_T}\ge\sum_{i=1}^{n}\left(\int_0^T a(\|u\|_p^p)\int_\Omega|u|^{p-2}|D_iu|^2\,dz\right)-\int_{Q_T}|b_0(x,t)|\,|u|^{\alpha(x,t)}\,dz$$
$$-\int_{Q_T}|b_1(x,t)|\,dz.$$

Estimating the second integral above by Young's inequality and using $L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)\subset L^p(Q_T)$, we attain the following inequality which gives the coercivity of ξ,

$$\langle\xi(u),u\rangle_{Q_T}\ge C\,[u]^p_{L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)}-\tilde{C}.$$

here $C=C(p,m,|\Omega|)$ and $\tilde{C}=\tilde{C}\left(\epsilon,\|b_0\|_{L^{\beta_1(x,t)}(Q_T)},\|b_1\|_{L^{\alpha^*(x,t)}(Q_T)}\right)$.

From the embedding

$$L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)\subset L^p(Q_T)\subset L^{\alpha(x,t)}(Q_T),$$

weak compactness and boundedness of $f:L^p\left(0,T;\mathring{S}_{1,(p-2)q,q}(\Omega)\right)\cap W^{1,q}\left(0,T;W^{-1,q}(\Omega)\right)\to L^q\left(0,T;W^{-1,q}(\Omega)\right)$ follows from Lemmas 3.4 and 3.5. Thus by the virtue of the proof of Theorem 3.2, we get the desired result. □

4. A model equation

Let assume that

$$b(x,t,\tau) = C_\alpha |\tau|^{\alpha(x,t)-2}\tau \text{ with } C_\alpha > 0 \text{ constant}$$

where $\alpha(x,t)$ is a measurable function in Q_T with values in an interval $[\alpha^-, \alpha^+] \subset (1,\infty)$.

We consider the model version of (1.1)

$$\begin{cases} u_t - a(\|u\|_p^p) \sum\limits_{i=1}^{n} D_i \left(|u|^{p-2} D_i u \right) + C_\alpha |u|^{\alpha(x,t)-2} u = f, \\ u\,(x,0) = 0 = u_0(x), \quad u\,|_{\Gamma_T} = 0 \end{cases} \tag{4.1}$$

Suppose that following conditions are fulfilled:

$$\begin{cases} p \equiv 2 \text{ and there exists a constant } L > 0 \text{ such that for all } s_1, s_2 \geq 0 \\ |a(s_1) - a(s_2)| \leq L|s_1 - s_2|. \end{cases} \tag{4.2}$$

Note that in the case of $p = 2$, the sum $\sum\limits_{i=1}^{n} D_i \left(|u|^{p-2} D_i u \right) \equiv \Delta u$ and $\mathring{S}_{1,(p-2)q,q}(\Omega) \equiv W_0^{1,2}(\Omega)$.

Theorem 4.1 *Assume that* **(H.1)** holds and $f \in L^2(0,T;W^{-1,2}(\Omega)) + L^{\alpha^*(x,t)}(Q_T)$. If (4.2) is satisfied then problem (4.1) has at most one weak solution $u \in S_0$.

Proof Theorem 3.2 provide that (4.1) has a solution in S_0. Suppose that $u_1, u_2 \in S_0$ are two different solutions of problem (4.1).

Let us take $w = u_1 - u_2$ for the test-function in identities (3.4) for u_i in the cylinder $Q_T \cap \{t \leq \tau \leq t+h\}$, $t, t+h \in [0,T]$. Subtracting the results and dividing by h we arrive at the equality

$$\begin{aligned} \frac{1}{2h} \int_t^{t+h} \frac{d}{dt} \left(\|w\|_2^2 \right) d\tau + \frac{1}{h} \int_t^{t+h} \int_\Omega \left[a(\|u_1\|_2^2) \nabla u_1 - a(\|u_2\|_2^2) \nabla u_2 \right] \cdot \nabla w \, dz \\ + \frac{C_\alpha}{h} \int_0^{t+h} \int_\Omega \left(|u_1|^{\alpha(x,t)-2} u_1 - |u_2|^{\alpha(x,t)-2} u_2 \right) w \, dz = 0. \end{aligned}$$

By the Lebesgue differentiation theorem, each term of this equality has the limit as $h \to 0$: for a.e. $t \in (0,T)$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|w\|_2^2 \right) + \int_\Omega \left[a(\|u_1\|_2^2) \nabla u_1 - a(\|u_2\|_2^2) \nabla u_2 \right] \cdot \nabla w \, dx \\ + C_\alpha \int_\Omega \left(|u_1|^{\alpha(x,t)-2} u_1 - |u_2|^{\alpha(x,t)-2} u_2 \right) w \, dx = 0. \end{aligned} \tag{4.3}$$

Notice that for $\alpha^- > 1$

$$\begin{aligned} \left(|u_1|^{\alpha(z)-2} u_1 - |u_2|^{\alpha(z)-2} u_2 \right) w &= w \int_0^1 \frac{d}{d\theta} \left| \theta u_1 + (1-\theta) u_2 \right|^{\alpha(z)-2} \left(\theta u_1 + (1-\theta) u_2 \right) d\theta \\ &= w^2 (\alpha(z) - 1) \int_0^1 \left| \theta u_1 + (1-\theta) u_2 \right|^{\alpha(z)-2} d\theta \geq 0. \end{aligned} \tag{4.4}$$

Thus from (4.4) and (4.3), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|_2^2\right)+\int_\Omega\left[a(\|u_1\|_2^2)\nabla u_1-a(\|u_2\|_2^2)\nabla u_2\right]\cdot\nabla w\,dx\le 0. \tag{4.5}$$

Since

$$\begin{aligned}&\int_\Omega\left[a(\|u_1\|_2^2)\nabla u_1-a(\|u_2\|_2^2)\nabla u_2\right]\cdot\nabla(u_1-u_2)\,dx\\&=a(\|u_1\|_2^2)\int_\Omega\nabla(u_1-u_2)\cdot\nabla(u_1-u_2)\,dx-\left[a(\|u_2\|_2^2)-a(\|u_1\|_2^2)\right]\int_\Omega\nabla u_2\cdot\nabla(u_1-u_2)\,dx.\end{aligned}$$

Substituting this into (4.5), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|_2^2\right)+a(\|u_1\|_2^2)\|\nabla w\|_2^2\le\left[a(\|u_2\|_2^2)-a(\|u_1\|_2^2)\right]\int_\Omega\nabla u_2\cdot\nabla(u_1-u_2)\,dx \tag{4.6}$$

So by the Cauchy-Schwarz inequality and using (4.2) and (3.1), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|_2^2\right)+m\|\nabla w\|_2^2\le L(\|u_1\|_2^2-\|u_2\|_2^2)\|\nabla u_2\|_2\|\nabla w\|_2. \tag{4.7}$$

By (4.7), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|_2^2\right)+m\|\nabla w\|_2^2\le L(\|u_1\|_2+\|u_2\|_2)\|w\|_2\|\nabla u_2\|_2\|\nabla w\|_2. \tag{4.8}$$

Then applying Young's inequality to the right hand side of (4.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|_2^2\right)+m\|\nabla w\|_2^2\le\frac{m}{2}\|\nabla w\|_2^2+\frac{1}{2m}(L(\|u_1\|_2+\|u_2\|_2)\|\nabla u_2\|)^2\|w\|_2^2. \tag{4.9}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|_2^2\right)\le C(t)\|w\|_2^2 \tag{4.10}$$

where $C(t)\in L^1(0,T)$. Since $\|w(0)\|_2=0$, the assertion follows after integration in t. □

Now, we show that if we take $f\equiv 0$ in problem (4.1) then the solutions are trivial.

Theorem 4.2 *Let* $f\equiv 0$ *and* **(H.1)** is satisfied then problem (4.1) has only trivial solution.

Proof Let us take the weak solution $u\in S_0$ of problem (4.1) for the test-function in the integral identity (3.4) then for all $t,t+h\in[0,T]$, $h>0$,

$$\frac{1}{2h}\left.\|u(t)\|_2^2\right|_t^{t+h}+\frac{1}{h}\int_t^{t+h}a(\|u(s)\|_p^p)\left(\sum_{i=1}^n\int_\Omega|u(s)|^{p-2}|D_iu(s)|^2\,dx\right)ds=-\frac{C_\alpha}{h}\int_t^{t+h}\int_\Omega|u|^{\alpha(x,s)}\,dxds.$$

Letting $h \to 0^+$ and using the Lebesgue differentiation theorem we conclude that for a.e. $t \in (0, T)$

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_2^2\right) + a(\|u(t)\|_p^p)\sum_{i=1}^{n}\int_\Omega |u(t)|^{p-2}|D_i u(t)|^2\,dx = -C_\alpha \int_\Omega |u(z)|^{\alpha(z)}\,dx. \tag{4.11}$$

It follows from (4.11) that every weak solution satisfies the following relation,

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_2^2\right) + a(\|u\|_p^p)\sum_{i=1}^{n}\int_\Omega |u|^{p-2}|D_i u|^2\,dx + C_\alpha \int_\Omega |u|^{\alpha(z)}\,dx = 0. \tag{4.12}$$

Dropping the nonnegative terms on the left-hand side of (4.12), we obtain the inequality

$$\frac{d}{dt}\left(\|u(t)\|_2^2\right) \le 0.$$

Since $\|u(\cdot,0)\|_2 = 0$, integration of the last inequality yields $\|u(t)\|_2^2 = 0$ for all $t \ge 0$. $\square$

By taking sufficiently smooth initial data $u(x,0) = u_0$ different from zero, we derive exponential and power decay of $\|u(t)\|_{L^2(\Omega)}$. We note that in this case, the solvability results can be obtained by the virtue of methods using in [19, 26, 33] and Theorem 2.6. Since our goal of studying the model problem (4.1) is to provide a more understandable and explicit way for the established results, in this article we skip this part for the sake of brevity.

Theorem 4.3 *Let* $f \equiv 0$ *and* **(H.1)** is fulfilled. Suppose u is a solution of problem (4.1) with initial function $u_0 \in W^{1,p}(\Omega) \cap L^{\alpha^+}(\Omega)$, $\|u_0\|_2 > 0$.

(a) If $p > 2$, then there exists a constant C' such that

$$\|u(t)\|_2^{p-2} \le \frac{2\|u_0\|_2^{p-2}}{2 + C'(p-2)t} \qquad \text{for all } t > 0.$$

(b) If $p = 2$, then there exists a constant $\tilde{C}$ such that

$$\|u(t)\|_2^2 \le \|u_0\|_2^2 \mathrm{e}^{-\tilde{C}t} \quad \text{for all } t > 0.$$

Proof From (4.12), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_2^2\right) + a(\|u\|_p^p)\sum_{i=1}^{n}\int_\Omega |u|^{p-2}|D_i u|^2\,dx + C_\alpha \int_\Omega |u|^{\alpha(z)}\,dx = 0. \tag{4.13}$$

By using (3.1) and Poincaré inequality we have the following estimate

$$\begin{aligned} a(\|u\|_p^p)\sum_{i=1}^{n}\int_\Omega |u|^{p-2}|D_i u|^2\,dx &= a(\|u\|_p^p)\sum_{i=1}^{n}\int_\Omega |D_i(|u|^{p/2})|^2\,dx \\ &\ge m\tilde{C}\int_\Omega |u|^p\,dx \ge C^*\|u\|_2^p. \end{aligned} \tag{4.14}$$

Substitution into (4.13) leads to the inequality

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_2^2\right)+C^*\|u\|_2^p\le 0. \tag{4.15}$$

By denoting $y(t)=\|u(t)\|_2^2$, last inequality can be written as:

$$y'(t)+C_0^* y^{\frac{p}{2}}(t)\le 0. \tag{4.16}$$

By (4.16) $y'(t)\le 0$, whence $y(t)\le\|u_0\|_2^2$ for all $t\in(0,T)$.

Let us consider the function $z(t)=\dfrac{y(t)}{R}\le 1$, $R=1+\|u_0\|_2^2$. From (4.16) $z(t)$ satisfies the differential inequality

$$z'(t)+C'z^{\frac{p}{2}}(t)\le 0 \quad \text{in } (0,T), \qquad z(0)<1, \tag{4.17}$$

with the coefficient $C'=C_0^* R^{\frac{p-2}{2}}$.

If $p>2$, the straightforward integration of inequality (4.17) over the interval $(0,t)$ gives

$$z^{1-\frac{p}{2}}(t)\ge (z(0))^{1-\frac{p}{2}}+\frac{C'}{2}(p-2)t,$$

which yields (a).

In the case $p=2$, the inequality (4.16) for $y(t)$ is linear and can be immediately integrated. □

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