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A multidimensional diffusion coefficient determination problem for the time-fractional equation

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Abstract: In this paper, we consider a multidimensional inverse problem for a fractional diffusion equation. The inverse problem is reduced to the equivalent integral equation. For solving this equation the Schauder principle is applied. The local existence and uniqueness results are obtained.

Key words: Diffusion equation, Gerasimov–Caputo fractional derivative, overdetermination integral condition, Hölder space, integral equation

1. Introduction: statement of the inverse problem

Consider the $n \geq 2-$dimensional fractional diffusion equation defined by

$$\left({}^C\mathcal{D}_t^\alpha u\right)(x,t) - \triangle u + q(x)u = f(x,t), \qquad \text{in} \quad \mathbb{R}_T^n, \tag{1.1}$$

where $\left({}^C\mathcal{D}_t^\alpha u\right)(x,t)$, $0 < \alpha < 1$ is the Gerasimov–Caputo fractional derivative, defined by [8], [21]:

$$\left({}^C\mathcal{D}_t^\alpha u\right)(x,t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha} d\tau,$$

$\triangle-$Laplacian respect to the variable $x = (x_1, x_2, ..., x_n)$, $\mathbb{R}_T^n = \{(x,t) : x \in \mathbb{R}^n, 0 < t \leq T\}$ and $f(x,t)$ is given function.

Equations with fractional derivatives are applied in studying of anomalous diffusion and various processes in physics, mechanics, chemistry and engineering. The diffusion equation is a mathematical model of important physical phenomena ranging from fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [6] and references therein). In normal diffusion (described by the heat equation or more general parabolic equations) the mean square displacement of a diffusive particle behaves like $\text{const} \cdot t$ for $t \to \infty$. A typical behavior for anomalous diffusion is $\text{const} \cdot t^\alpha$, and this was the

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reason to invoke Equation (1.1), where this anomalous behavior is an easy mathematical fact. For connections to statistical mechanics (see, [9, 19]).

It is natural from the physical point of view to consider a usual Cauchy problem, with the initial condition

$$u(x,0) = \Phi(x), \qquad \text{on} \quad \mathbb{R}^n, \tag{1.2}$$

where $\Phi(x)$ is given.

The mathematical theory of fractional diffusion equations has made only its first steps. An expression for the fundamental solution of the Cauchy problem (1.1), (1.2) was found independently by Schneider and Wyss [23] and Kochubei [13]. It was also shown in [23] that the fundamental solution is nonnegative, which led later [15, 24, 25, 29] to a probabilistic interpretation of Equation (1.1). In [23] only initial functions $\Phi \in \mathcal{S}(\mathbb{R}^n)$ were considered. A more general situation was studied in [13] where Φ was permitted to be unbounded, with minimal smoothness assumptions. There are also some results regarding initial-boundary value problems (see [23, 28]).

For general problems (1.1), (1.2) in [13] a uniqueness theorem for bounded solutions, and an exact uniqueness theorem (for the case $n = 1$) for solutions with a possible exponential growth were proved. There are also several papers devoted to the Cauchy problem for abstract evolution equations (1.1) (it was [2–4, 7, 14], and others).

Besides, [6] clarified an evolution equation with the regularized fractional derivative of an order $\alpha \in (0,1)$ with respect to the time variable, and a uniformly elliptic operator with variable coefficients acting in the spatial variables. Such equations described diffusion on inhomogeneous fractals. A fundamental solution of the Cauchy problem was constructed and investigated.

Many inverse problems to such equations arise in many branches of science and engineering. Some inverse problems to diffusion equation with different unknown functions or parameters were investigated, for example, in [22]. In particular, the article [1] was devoted to determination of a source term for a time fractional diffusion equation with an integral type overdetermination condition. The article [17] demonstrated the unique solvability for an inverse problem for semilinear fractional telegraph equation. [20] discussed an inverse problem of determining spatial coefficient $q(x), x \in \Omega$ and/or order α of the fractional derivative by data $u|_{\omega\times(0,T)}$, where $\omega \subset \Omega$ is a subdomain. Some inverse problem of recovering a spatially varying potential term in a one-dimensional time-fractional diffusion equation from the flux measurements taken at a single fixed time corresponding to a given set of input sources (see, [10, 31]). Besides, [30] considered a fractional diffusion equation (FDE) $\left({}^C\mathcal{D}_t^\alpha u\right)(x,t) = a(t)u_{xx}$ with an undetermined time-dependent diffusion coefficient $a(t)$. In [5], the uniqueness was obtained for a one-dimensional fractional diffusion equation: $\left({}^C\mathcal{D}_t^\alpha u\right)(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}(x,t)\right), 0 < x < l$, where $0 < \alpha < 1$. In our case, multidimensional inverse problem and [27] devoted to identifying a time-dependent source term in a multidimensional time-fractional diffusion equation from boundary Cauchy data.

Our main problem is formulated as follows:

Inverse problem. Find the function $q(x)$, $x \in \mathbb{R}^n$ in (1.1), if the solution to Cauchy problem (1.1), (1.2) satisfies

$$\int_0^T u(x,t)\chi(t)dt = g(x), \qquad x \in \mathbb{R}^n, \tag{1.3}$$

where $\chi(t)$, $g(x)$ are given.

In the present paper, we establish sufficient condition under which the solution of the inverse problem (1.1)–(1.3) exists and is unique. For the case $\alpha = 1$, closely related results were obtained in [11].

As in [6], we call a function $u(x,t)$ a classical solution to Cauchy problem (1.1) and (1.2), if:

(i) $u(x,t)$ is twice continuously differentiable in x for each $t>0$;

(ii) for each $x \in \mathbb{R}^n$ $u(x,t)$ is continuous in t on $[0,T]$, and its fractional integral

$$(I_{0+}^{1-\alpha}u)(x,t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-\tau)^{-\alpha} u(x,\tau)d\tau$$

is continuously differentiable in t for $t>0$.

(iii) $u(x,t)$ satisfies (1.1) and (1.2).

Let $u(x,t)$ be a classical solution to the Cauchy problem (1.1), (1.2) and f, Φ, χ, g be enough smooth functions. We carry out the next converting of the inverse problem (1.1)–(1.3). So that, denote $\left({}^C\mathcal{D}_t^\alpha u\right)(x,t)$ by $v = v(x,t)$, $(x,t) \in \mathbb{R}_T^n$. Making Gerasimov–Caputo fractional derivative for Equation (1.1) once, we get

$$\left({}^C\mathcal{D}_t^\alpha v\right)(x,t) - \triangle v + q(x)v = \left({}^C\mathcal{D}_t^\alpha f\right)(x,t), \;\; (x,t) \in \mathbb{R}_T^n. \tag{1.4}$$

Also, we can easily get the following initial condition by using (1.1) and (1.2):

$$v(x,0) = \triangle\Phi(x) - q(x)\Phi(x) + f(x,0), \;\; x \in \mathbb{R}^n. \tag{1.5}$$

Later, we will investigate the last direct problem.

In this paper, we use Hölder space with corresponding norm conventional sense (see, e.g., [12, 16]). Let $C(\mathbb{R}_T^n)$, $C(\overline{\mathbb{R}}_T^n)$ be spaces of continuous functions on $\mathbb{R}_T^n$, $\overline{\mathbb{R}}_T^n$, respectively. Everywhere in this paper we will denote by $C^l(\mathbb{R}^n)$ a space of bounded continuous functions on $\mathbb{R}^n$ satisfying locally Hölder continuity condition with exponent $l \in (0,1)$. By $C^l(\mathbb{R}_T^n)$ be space of bounded continuous functions $f(x,t)$ on $\mathbb{R}_T^n$ which for all $t \in (0,T]$, satisfies Hölder continuity condition with respect to space variables $x \in \mathbb{R}^n$, and $C^{2,\alpha}(\mathbb{R}_T^n) = \{f \in C(\mathbb{R}_T^n) |\, \triangle f, \left({}^C\mathcal{D}_t^\alpha f\right)(x,t) \in C(\mathbb{R}^n)\}$. For a fixed t, the norm of the function $\varphi(x,t)$ in $C^l(\mathbb{R}^n)$ will be denoted by $|\varphi|^l(t)$. The norm of a function $\varphi(x,t)$ in $C^l(\mathbb{R}_T^n)$ is defined by the equality

$$\|\varphi\|^l := \max_{t\in[0,T]} \left||\varphi|^l(t)\right|.$$

Let us introduce the set

$$B_d(0) := \{q(x) \in C^l(\mathbb{R}^n) : \;|q(x)|^l \le d\},$$

where $d = \text{const} > 0$. In all the subsequent arguments, we assume that the functions appearing in the input data of problem (1.1)–(1.3) satisfy the following conditions:

$$g(x) \in C^{l+2}(\mathbb{R}^n), \quad g(x) \ge g_0 > 0, \quad g_1 := |g|^{l+2}; \tag{1.6}$$

$$\Phi(x) \in C^{l+2}(\mathbb{R}^n), \quad |\Phi|^{l+2} \le \varphi_0, \;\; x \in \mathbb{R}^n, \quad \chi(t) \in C[0,T], \;\; \chi_0 := \max_{t\in[0,T]} |\chi(t)|; \tag{1.7}$$

$$F(x)=\int_0^T f(x,t)\chi(t)dt, \quad F_0:=\sup_{x\in\mathbb{R}^n}|F(x)|. \tag{1.8}$$

Besides, we will use the following useful inequality on Hölder spaces:

$$\text{If } f(x),g(x)\in C^l(\mathbb{R}^n), \text{ then for all } l\in(0,1) \text{ we have } \|fg\|_{C^l(\mathbb{R}^n)}\le\|f\|_{C^l(\mathbb{R}^n)}\|g\|_{C^l(\mathbb{R}^n)}. \tag{1.9}$$

2. Investigation of direct problem (1.1), (1.2)

In the paper [13] it was found the representation of the solution in terms of the fundamental solution to the following Cauchy problem

$$^C\mathcal{D}_t^\alpha u - Bu(x,t)=F(x,t), \quad x\in\mathbb{R}^n,\, t\in(0,T],$$

$$u\big|_{t=0}=u_0(x), \quad x\in\mathbb{R}^n,$$

where

$$B:=\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{j=1}^n b_j(x)\frac{\partial}{\partial x_j}+c(x)$$

is a uniformly second order elliptic differential operator with bounded continuous realvalued coefficients. In the case $B\equiv\triangle$, where $\triangle$ is $n-$dimensional laplacian, for any bounded continuous function $u_0(x)$ (locally Hölder continuous, if $n>1$) and any bounded continuous with respect to the both variables x, t and locally Hölder continuous in x function $F(x,t)$, it has the form

$$u(x,t)=\int_{\mathbb{R}^n} Z(x-\xi,t)u_0(\xi)d\xi+\int_0^t\int_{\mathbb{R}^n} Y(x-\xi,t-\tau)F(\xi,\tau)d\xi d\tau, \tag{2.1}$$

with

$$Z(x-y,t)=\pi^{-n/2}|x-y|^{-n}H_{1,2}^{2,0}\Big[\frac{1}{4}t^{-\alpha}|x-y|^2\Big|_{(n/2,1),(1,1)}^{(1,\alpha)}\Big],$$

$$Y(x-y,t-\lambda)=\pi^{-n/2}|x-y|^{-n}(t-\lambda)^{\alpha-1}H_{1,2}^{2,0}\Big[\frac{1}{4}(t-\lambda)^{-\alpha}|x-y|^2\Big|_{(n/2,1),(1,1)}^{(\alpha,\alpha)}\Big],$$

where H is Fox's $H-$function (see, [18]). Actually, $Y(x,t)$ is the Riemann-Liouville derivative of $Z(x,t)$ with respect to t of the order $1-\alpha$ (for $x\neq0$, $Z(x,t)\to0$ as $t\to0$), so that the Riemann-Liouville derivative coincides in this case with Grasimov-Caputo derivative, i.e. $Y(x,t)=\left(D_*^{1-\alpha}Z\right)(x,t)$, here $D_*^\alpha Z(x,t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t Z(x,\tau)(t-\tau)^{-\alpha}d\tau$. Fractional derivatives $D^\alpha Z(x,t)$ and $D_*^\alpha Z(x,t)$ are jointed by the formula

$$\left(^C\mathcal{D}_t^\alpha Z\right)(x,t)=D_*^\alpha Z(x,t)+\Gamma^{-1}(1-\alpha)t^{-\alpha}Z(x,+0).$$

Let us now look at the particular cases - solutions of classical equations. The fundamental solution of classical diffusion equation for $\alpha=1$

$$Z(x,t)=\pi^{-n/2}\,|x|^{-n}\,H_{1,2}^{2,0}\left[\frac{1}{4}t^{-1}|x|^2\Big|_{(n/2,1/2),(1,1)}^{(1,\,1)}\right]=\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}},$$

and so $D_*^0 Z(x,t) \equiv Z(x,t)$, then $Z(x,t) = Y(x,t)$ (this also follows from determination the functions $Z(x,t)$ and $Y(x,t)$). Thus, for $\alpha = 1$, formula (2.1) becomes the formula [16] for the solution of the Cauchy problem for the $n-$dimensional inhomogeneous classical heat conduction equations.

In (1.4), introducing the notation $F(x,t) := \left({}^C\mathcal{D}_t^\alpha f\right)(x,t) - q(x)v(x,t)$ and applying the formula (2.1) to direct problem (1.4), (1.5) for $n \geq 2$, we obtain the integral equation for determining $v(x,t)$:

$$v(x,t) = \bar{v}_0(x,t) - \int_{\mathbb{R}^n} Z(x-\xi,t)\Phi(\xi)q(\xi)d\xi - \int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)q(\xi)v(\xi,\tau)d\xi, \tag{2.2}$$

where

$$\bar{v}_0(x,t) := \int_{\mathbb{R}^n} Z(x-\xi,t)\Big[\triangle\Phi(\xi) + f(\xi,0)\Big]d\xi + \int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)\left({}^C\mathcal{D}_t^\alpha f\right)(\xi,\tau)d\xi. \tag{2.3}$$

The following assertion is hold:

Lemma 2.1 *Assume that conditions (1.6)–(1.8) hold. If* $q(x) \in C^l(\mathbb{R}^n)$, $f(x,t) \in C^{l,1}(\overline{\mathbb{R}}_T^n)$, $\Phi(x) \in C^{2+l}(\mathbb{R}^n)$, *then there exists a unique solution of the integral equation (2.2)* $v(x,t) \in C^{2,\alpha}(\overline{\mathbb{R}}_T^n)$, *where* $\overline{\mathbb{R}}_T^n := \{(x,t):\ x \in \mathbb{R}^n,\ 0 \leq t \leq T\}$, $\{l,\alpha\} \in (0,1)$.

Proof We use the method of successive approximations and consider the sequence of functions defined recursively by the formulas:

$$v_j(x,t) = -\int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)q(\xi)v_{j-1}(\xi,\tau)d\xi,\ j = 1,2,\dots, \tag{2.4}$$

where

$$v_0(x,t) := \bar{v}_0(x,t) - \int_{\mathbb{R}^n} Z(x-\xi,t)\Phi(\xi)q(\xi)d\xi.$$

Further, we need estimations for functions $Z(x,t)$, $Y(x,t)$ and their some derivatives. Assume $m = (m_1,\dots,m_n)$ is a multiindices with $|m| = m_1 + \dots + m_n$ and

$$D_x^m v = \frac{\partial^{|m|} v}{\partial x_1^{m_1} \partial x_2^{m_2} \dots \partial x_n^{m_n}},\ D_x^0 v = v.$$

Denote $R = t^{-\alpha}|x|^2$. Also, we will use the following estimates [6, 13] (here and below the letters C, σ will denote various positive constants):

(i) *if* $R \geq 1$, *then*

$$|D_x^m Z(x,t)| \leq Ct^{-\frac{\alpha(n+m)}{2}} e^{-\sigma t^{-\frac{\alpha}{2-\alpha}}|x|^{\frac{2}{2-\alpha}}},\ |m| \leq 3, \tag{2.5}$$

$$\left|{}^C\mathcal{D}_t^\alpha Z(x,t)\right| \leq Ct^{-\frac{\alpha(n+2)}{2}} e^{-\sigma t^{-\frac{\alpha}{2-\alpha}}|x|^{\frac{2}{2-\alpha}}}; \tag{2.6}$$

(ii) *if* $R \leq 1\, x \neq 0$, *then*

$$|D_x^m Z(x,t)| \leq Ct^\alpha |x|^{-n+2-|m|},\ |m| \leq 3, \tag{2.7}$$

if $n \geq 3$, *or* $n = 2$, $m \neq 0$;

$$|Z(t,x)| \leq Ct^{-\alpha}[|\log(t^{-\alpha}|x|^2)| + 1], \tag{2.8}$$

if $n = 2$;

$$\left|\frac{\partial^m Z(x,t)}{\partial x^m}\right| \leq Ct^{-\frac{(m+1)\alpha}{2}}, \tag{2.9}$$

if $n = 1$.

(iii) *If* $R \leq 1$, $x \neq 0$, *then*

$$\left|{}^C\mathcal{D}_t^{\alpha} Z(x,t)\right| \leq \begin{cases} Ct^{-2\alpha}|x|^{-n+2}, & \text{if } n \geq 3, \\ Ct^{-\alpha}[|\log(t^{-\alpha}|x|^2)| + 1], & \text{if } n = 2, \\ Ct^{-3\alpha/2}, & \text{if } n = 1. \end{cases} \tag{2.10}$$

Note that the orders of the singularities at $x = 0$ in (2.7), (2.8) and (2.10) are precise.

The next expressions contains estimates of the function Y and its derivatives:

(i) *If* $R \geq 1$, *then*

$$|D_x^m Y(x,t)| \leq Ct^{-\frac{\alpha(n+m)}{2} - 1 + \alpha} e^{-\sigma t^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{\alpha}{2-\alpha}})}, \ |m| \leq 3. \tag{2.11}$$

(ii) *If* $R \leq 1$, $x \neq 0$, $n > 4$, *then*

$$|D_x^m Y(x,t)| \leq Ct^{-\alpha-1}|x|^{-n+4-|m|}, \ |m| \leq 3. \tag{2.12}$$

(iii) *If* $R \leq 1$, $x \neq 0$, $n = 4$, *then*

$$|Y(x,t)| \leq Ct^{-\alpha-1}[|\log(t^{-\alpha}|x|^2)| + 1], \tag{2.13}$$

$$|D_x Y(x,t)| \leq Ct^{-\frac{3\alpha}{2}-1}, \tag{2.14}$$

$$|D_x^m Y(x,t)| \leq Ct^{-2\alpha-1}[|\log(t^{-\alpha}|x|^2)| + 1], \ |m| = 2, \tag{2.15}$$

$$|D_x^m Y(x,t)| \leq Ct^{-2\alpha-1}|x|^{-1}[|\log(t^{-\alpha}|x|^2)| + 1], \ |m| = 3. \tag{2.16}$$

(iv) *If* $R \leq 1$, $x \neq 0$, $n = 3$, *then*

$$|Y(x,t)| \leq Ct^{-\frac{\alpha}{2}-1}, \tag{2.17}$$

$$|D_x Y(x,t)| \leq Ct^{-\alpha-1}, \tag{2.18}$$

$$|D_x^m Y(x,t)| \leq Ct^{-\alpha-1}|x|^{-1}, \ |m| = 2, \tag{2.19}$$

$$|D_x^m Y(x,t)| \leq Ct^{-\alpha-1}|x|^{-2}, \ |m| = 3. \tag{2.20}$$

(v) *If* $R \leq 1,\ x \neq 0,\ n = 2$, *then*

$$|Y(x,t)| \leq Ct^{-1}, \tag{2.21}$$

$$|D_x Y(x,t)| \leq Ct^{-\frac{\alpha}{2}-1}, \tag{2.22}$$

$$|D_x^m Y(x,t)| \leq Ct^{-\alpha-1}[|\log(t^{-\alpha}|x|^2)|+1],\ \ |m|=2, \tag{2.23}$$

$$|D_x^m Y(x,t)| \leq Ct^{-\alpha-1}|x|^{-1}[|\log(t^{-\alpha}|x|^2)|+1],\ \ |m|=3. \tag{2.24}$$

(vi) *If* $R \leq 1,\ x \neq 0,\ n = 1$, *then*

$$|D_x^m Y(x,t)| \leq Ct^{-\frac{(m-1)\alpha}{2}-1},\ m=0,1,2,3. \tag{2.25}$$

Obtained estimations (2.5)–(2.25) are based on the asymptotic expansions of the $H-$functions and their derivatives under the small and greater values of the argument (see, [18]). We also note that it follows from the construction of the function $Z(x,t)$:

$$\int_{\mathbb{R}^n} Z(\xi,t)d\xi = 1, \tag{2.26}$$

and it is true the equality [13]

$$\int_{\mathbb{R}^n} Y(\xi,t)d\xi = C_0 t^{\alpha-1}, \quad t \in (0,T], \tag{2.27}$$

where C_0 depends only on n and α.

Set $q_0 := |q|^l$, $\varphi_0 := \|\Phi\|_{C^2(\mathbb{R}^n)}$ and $f_0 := \|f\|^l$. Using (2.4), (2.26), and (2.27), we estimate the modulus of $v_j(x,t)$ in the domain $\overline{\mathbb{R}}^n_T$. Then, we obtain

$$|v_0(x,t)| \leq \varphi_0 + \frac{f_0}{\Gamma(2-\alpha)}T^{1-\alpha} + \varphi_0 q_0 =: \varphi_{00},$$

$$|v_1(x,t)| \leq C_0 q_0 \varphi_{00}\int_0^t (t-\tau)^{\alpha-1}d\tau = C_0 q_0 \varphi_{00}\frac{t^\alpha}{\alpha} = C_0 q_0 \varphi_{00}\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)}t^\alpha,$$

$$|v_2(x,t)| \leq$$

$$\leq \varphi_{00}\left(C_0 q_0 \Gamma(\alpha)\right)^2 \frac{1}{\Gamma(1+\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\tau^\alpha d\tau}{(t-\tau)^{1-\alpha}} = \varphi_{00}\left(C_0 q_0 \Gamma(\alpha)\right)^2 \frac{1}{\Gamma(1+\alpha)} I^\alpha_{0+}t^\alpha,$$

where $I^\alpha_{0+}t^\alpha$ is the Riemann-Liouville fractional integral of the power function t^α and $\Gamma(\cdot)$ is the Euler's gamma function. It is not difficult to note (see, [12]) that the formula

$$I^\alpha_{0+}t^{j\alpha} = \frac{\Gamma(1+j\alpha)}{\Gamma(1+(j+1)\alpha)}t^{(1+j)\alpha},\ j=0,1,2,...$$

is valid. In accordance with this formula we continue to estimate $v_2(x,t)$:

$$|v_2(x,t)| \leq \varphi_{00}\frac{\left(C_0 q_0 \Gamma(\alpha)\right)^2}{\Gamma(1+\alpha)} I^\alpha_{0+}t^\alpha = \varphi_{00}\frac{\left(C_0 q_0 \Gamma(\alpha)\right)^2}{\Gamma(1+2\alpha)}t^{2\alpha}.$$

For arbitrary $j = 0, 1, 2, ...$ we have

$$|v_j(x,t)| \le \varphi_{00} \frac{\big(C_0 q_0 \Gamma(\alpha)\big)^j}{\Gamma(1+j\alpha)} t^{j\alpha}.$$

It follows from the above estimates that the series

$$v(x,t) = \sum_{j=0}^{\infty} v_j(x,t)$$

converges uniformly in $\overline{\mathbb{R}}^n_T$, since it can be majorized in $\overline{\mathbb{R}}^n_T$ by the convergent numerical series

$$\varphi_{00} \sum_{j=0}^{\infty} \frac{\big(C_0 q_0 \Gamma(\alpha) T^\alpha\big)^j}{\Gamma(1+j\alpha)}.$$

This means the following estimate for the solution of the integral equation (2.2) takes place:

$$|v(x,t)| \le \varphi_{00} \sum_{j=0}^{\infty} \frac{\big(C_0 q_0 \Gamma(\alpha) T^\alpha\big)^j}{\Gamma(1+j\alpha)} = \varphi_{00} E_\alpha\big(C_0 q_0 \Gamma(\alpha) T^\alpha\big) =: M_0, \quad (x,t) \in \overline{\mathbb{R}}^n_T, \tag{2.28}$$

where $E_\alpha(\cdot)$ is the Mittag–Leffler function of a nonnegative real argument (see, [12]).

Note that $v_0(x,t)$ is the solution to the problem (1.1), (1.2) for $q(x) \equiv 0$. Under the assumptions of 2.1 it is true that $v_0(x,t) \in C^{2,\alpha}(\overline{\mathbb{R}}^n_T)$. To prove this fact, let us write the integral in the left-hand side of (2.3) as the sum $v_0^1(x,t) + v_0^2(x,t)$ of two integrals corresponding to the decomposition $\mathbb{R}^n = \Omega_1 \cup \Omega_2$ where $\Omega_1 := \{y \in \mathbb{R}^n : |y - x_0| \ge t^\alpha\}$ and $\Omega_2 := \mathbb{R}^n \setminus \Omega_1$, fixing $x_0 \in \mathbb{R}^n$.

If the point x lies in a small neighborhood x^0, and $y \in \Omega_1$, then the value $|x - y|$ is separated from zero. Thus, to compute $\frac{\partial^2 v_0^1(x,t)}{\partial x_j^2}$ we can differentiate under the integral sign, so that

$$\frac{\partial^2 v_0^1(x,t)}{\partial x_j^2} = \int_{\Omega_1} \frac{\partial^2 Z(x-\xi,t)}{\partial x_j^2} \big[\triangle \Phi(\xi) + f(\xi,0) - \Phi(\xi) q(\xi)\big] d\xi +$$

$$+ \int_0^t \int_{\Omega_1} \frac{\partial^2 Y(x-\xi,t-\tau)}{\partial x_j^2} \left({}^C\mathcal{D}_t^\alpha f\right)(\xi,\tau) d\xi d\tau,$$

$j = 1, 2, ..., n,$ that is, $v_0^1(x,t) \in C^2(\Omega_1)$.

To calculate $\frac{\partial^2 v_0^2(x,t)}{\partial x_j^2}$, note that estimations (2.7), (2.8), (2.12), (2.3), (2.15), (2.16), (2.19), (2.20), (2.23), (2.24) for functions $Z(t, x-\xi)$, $Y(t-\tau, x-\xi)$ contain logarithmic singularity and it singularities of the type $|x-\xi|^{-k}$ with exponent $k > 0$. Consequently, such singularity will have integrals by Ω_2 in estimation functions $\frac{\partial v_0(x,t)}{\partial x_j}$ and $\frac{\partial^2 v_0^2(x,t)}{\partial x_j^2}$. From the theory of Newton potential it follows that improper integrals, having described above singularities converge uniformly on x and define continuous function in Ω_2, if only k less than number of the dimensions of the domain Ω_2, i.e. $k < n$, $n \ge 2$ [16]. It implies that the locally Hölder continuous of functions Φ, f in x, uniformly with respect to t, then it implies that the functions $\frac{\partial u_0^2(x,t)}{\partial x_j}$ and $\frac{\partial^2 v_0^2(x,t)}{\partial x_j^2}$ are continuous in Ω_2. Thus, $v_0(x,t) \in C^2\left(\overline{\mathbb{R}}_T^n\right)$.

Since the functions $Z(x-\xi,t)$, $Y(x-\xi,t-\tau)$ satisfy homogeneous equation, corresponding to (1.1), then ${}^C\mathcal{D}_t^\alpha v_0(x,t) \in C^2\left(\overline{\mathbb{R}}_T^n\right)$. Therefore, $v_0(x,t) \in C^{2,\alpha}\left(\overline{\mathbb{R}}_T^n\right)$. From (2.4) it follows $v_j(x,t) \in C^{2,\alpha}\left(\overline{\mathbb{R}}_T^n\right)$ for all $j = 1, 2, \ldots$. Then, according to the general theory of integral equations, this implies that the same property will be possessed the function $v(x,t)$. The function thus constructed is a classical solution to the problem (1.1), (1.2).

Let us derive an estimate for the norm of the difference between the solution of the original integral equation (2.2) and the solution of this equation with perturbed functions $\tilde{q}$, $\tilde{f}$ and $\tilde{\Phi}$. Let $\tilde{v}(x,t)$ be a solution of the integral equation (2.2) corresponding to the functions $\tilde{q}$, $\tilde{f}$ and $\tilde{\Phi}$; i.e.

$$\tilde{v}(x,t) = \tilde{\bar{v}}_0(x,t) - \int_{\mathbb{R}^n} Z(x-\xi,t)\tilde{\Phi}(\xi)\tilde{q}(\xi)d\xi - \int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)\tilde{q}(\xi)\tilde{v}(\xi,\tau)d\xi, \tag{2.29}$$

where in accordance with (2.3)

$$\tilde{\bar{v}}_0(x,t) := \int_{\mathbb{R}^n} Z(x-\xi,t)\left[\triangle\tilde{\Phi}(\xi) + \tilde{f}(\xi,0)\right]d\xi + \int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)\left({}^C\mathcal{D}_t^\alpha \tilde{f}\right)(\xi,\tau)d\xi. \tag{2.30}$$

Composing the difference $v-\tilde{v}$ with the help of Equations (2.2) and (2.29), for it we obtain the integral equation

$$v(x,t) - \tilde{v}(x,t) = \bar{v}_0(x,t) - \tilde{\bar{v}}_0(x,t) - \int_{\mathbb{R}^n} Z(x-\xi,t)\left[\Phi(\xi) - \tilde{\Phi}(\xi)\right]\tilde{q}(\xi)d\xi-$$

$$-\int_{\mathbb{R}^n} Z(x-\xi,t)\Phi(\xi)\left[q(\xi)-\tilde{q}(\xi)\right]d\xi - \int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)\left[q(\xi)-\tilde{q}(\xi)\right]v(\xi,\tau)d\xi-$$

$$-\int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)\left[v(\xi,\tau)-\tilde{v}(\xi,\tau)\right]\tilde{q}(\xi)d\xi,$$

from which, is derived the following linear integral inequality in $|v(x,t)-\tilde{v}(x,t)|$:

$$|v(x,t)-\tilde{v}(x,t)| \leq |\bar{v}_0(x,t) - \tilde{\bar{v}}_0(x,t)| + \tilde{q}_0|\Phi - \tilde{\Phi}|^l + \left[\varphi_0 + C_0 M_0 \frac{T^\alpha}{\alpha}\right]|q-\tilde{q}|^l +$$

$$+\tilde{q}_0 \int_0^t \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)\left|v(\xi,\tau)-\tilde{v}(\xi,\tau)\right| d\xi d\tau, \tag{2.31}$$

where $\tilde{q}_0 := |\tilde{q}|^l$. It follows from the equalities (2.3) and (2.30) the estimate

$$|v_0(x,t) - \tilde{v}_0(x,t)| \leq |\Phi - \tilde{\Phi}|^l + \left(1 + C_0\frac{T^\alpha}{\alpha}\right)\|f-\tilde{f}\|^l.$$

Let $\lambda = \lambda\left(\alpha, T, q_0, \varphi_0, f_0\right) = \max\left\{1+\tilde{q}_0,\, 1 + C_0\frac{T^\alpha}{\alpha},\, \varphi_0 + C_0 M_0 \frac{T^\alpha}{\alpha}\right\}$. Applying the successive approximation method to inequality (2.31) with the help of the scheme

$$\left|v(x,t)-\tilde{v}(x,t)\right|_0 \leq \lambda\left(|\Phi-\tilde{\Phi}|^l + \|f-\tilde{f}\|^l + |q-\tilde{q}|^l\right),$$

$$|v(x,t)-\tilde{v}(x,t)|_j \leq \tilde{q}_0 \int_0^t \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)\,|v(\xi,\tau)-\tilde{v}(\xi,\tau)|_{j-1}\,d\xi d\tau,\ j=1,2,...,$$

we arrive at the estimate

$$|v(x,t)-\tilde{v}(x,t)| \leq \lambda M_0 \left(|\Phi-\tilde{\Phi}|^l + \|f-\tilde{f}\|^l + |q-\tilde{q}|^l\right), \tag{2.32}$$

which will be used in the next section of the paper. Indeed the expression (2.32) is the stability estimate for the solution to the Cauchy problem (1.1) and (1.2). The uniqueness for this solution follows from (2.32). □

3. Existence and uniqueness of the solution of the inverse problem (1.1)-(1.3)

Let us consider the inverse problem (1.1)–(1.3) and obtain an operator equation for the coefficient $q(x)$. Thus, let $q(x)$ be an arbitrary function from $C^l(\mathbb{R}^n)$. Let us multiply Eq. (1.1) by $\chi(t)$ and integrate over the closed interval $[0,T]$. Taking into account conditions (1.2), (1.3), and (2.2) integral equation, and assumptions (1.6)–(1.8), we obtain the next relation

$$q(x) = q_0(x) + \frac{1}{g(x)} \int_0^T \chi(t)dt \int_{\mathbb{R}^n} Z(x-\xi,t)\Phi(\xi)q(\xi)d\xi +$$

$$+\frac{1}{g(x)} \int_0^T \chi(t)dt \int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)q(\xi)v(\xi,\tau)d\xi, \tag{3.1}$$

where

$$q_0(x) := \frac{1}{g(x)}\Big[F(x) + \triangle g(x) - \int_0^T \chi(t)\bar{v}_0(x,t)dt\Big].$$

Let us introduce the nonlinear operator $\mathcal{A}: C^l(\mathbb{R}^n) \to C^l(\mathbb{R}^n)$, $l \in (0,1)$ by the formula

$$\mathcal{A}(q) = q_0(x) + \frac{1}{g(x)} \int_0^T \chi(t)dt \int_{\mathbb{R}^n} Z(x-\xi,t)\Phi(\xi)q(\xi)d\xi +$$

$$+\frac{1}{g(x)} \int_0^T \chi(t)dt \int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)q(\xi)v(\xi,\tau)d\xi, \tag{3.2}$$

where $v(x,t)$ is defined by (2.2), that is solution of the direct problem (1.4), (1.5) with coefficient $q(x) \in C^l(\mathbb{R}^n)$ in Eq. (1.4).

In view of conditions (1.6)–(1.8), the operator $\mathcal{A}$ acts from $C^l(\mathbb{R}^n)$ to $C^l(\mathbb{R}^n)$, and relation (3.2) can be rewritten as

$$q = \mathcal{A}(q). \tag{3.3}$$

Lemma 3.1 *Assume that conditions (1.6)–(1.8) hold. Then, for all $q(x) \in C^l(\mathbb{R}^n)$, the following estimate holds:*

$$\mathcal{A}(q) \leq d_0, \tag{3.4}$$

where

$$d_0 := \frac{1}{g_0}\Big[F_0 + g_1 + \chi_0 M_0 T + q_0\chi_0 T\Big(\varphi_0 + C(\alpha)M_0T^\alpha\Big)\Big]. \tag{3.5}$$

Proof Estimate (3.4) is a direct consequence of the definition of the operator $\mathcal{A}$, conditions (1.6)–(1.8), and estimate (2.28). □

Lemma 3.2 *Suppose that conditions (1.6)-(1.8) hold. Then the operator $\mathcal{A}$ is continuous on the set B_d for all $d > 0$.*

Proof Let $q^{(1)}(x)$, $q^{(2)}(x) \in B_d$, and let $v^{(1)}(x,t)$ and $v^{(2)}(x,t)$ be the corresponding solutions of the direct problem (1.4), (1.5). Set

$$\hat{v}(x,t) = v^{(1)}(x,t) - v^{(2)}(x,t), \quad \hat{q}(x) = q^{(1)}(x) - q^{(2)}(x).$$

For these functions, the following relations hold:

$$\left({}^C\mathcal{D}_t^\alpha \hat{v}\right)(x,t) - \triangle \hat{v} + q^{(1)}(x)\hat{v} = -v^{(2)}(x,t)\hat{q}(x), \ \ (x,t) \in \mathbb{R}^n_T,$$

$$\hat{v}(x,0) = -\hat{q}(x)\Phi(x), \ \ x \in \mathbb{R}^n.$$

According to the (2.2), we have

$$|\hat{v}(x,t)| \leq \Big(\varphi_0 + C(\alpha)M_1T^\alpha\Big)e^{C(\alpha)T^\alpha d}|\hat{q}|^l, \quad (x,t) \in \bar{\mathbb{R}}^n_T. \tag{3.6}$$

On the other hand, taking into account the definition of the operator $\mathcal{A}$ in (3.2), we obtain

$$|\mathcal{A}(q^{(1)}) - \mathcal{A}(q^{(2)})|^l \leq \frac{\chi_0}{g_0}T\Big(\varphi_0 + C_1(\alpha)M_1T^\alpha\Big)|\hat{q}|^l + \frac{\chi_0}{g_0}C_1(\alpha)q_0T^{\alpha+1}\|\hat{v}\|^\alpha. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$|\mathcal{A}(q^{(1)}) - \mathcal{A}(q^{(2)})|^l \to 0 \qquad \text{for} \qquad |q^{(1)} - q^{(2)}|^l \to 0.$$

The lemma is proved. □

Lemma 3.3 *Let conditions (1.6)-(1.8) hold. Let $d > 0$ be arbitrary. Then the operator $\mathcal{A}$ is a completely continuous operator on the set B_d.*

Proof The assertion of the lemma is a direct consequence of estimate (2.28), the compactness of the embedding of the space $C^l(\bar{\mathbb{R}}^n_T))$ in $C(\mathbb{R}^n_T)$, the definition of the operator $\mathcal{A}$, and 3.2. □

Theorem 3.4 *Let conditions (1.6)–(1.8) and inequality (3.4) hold. Then there exists a solution $q(x)$ of the inverse problem (1.1)-(1.3), with $q(x)$ satisfying the estimate*

$$q(x) \leq d_0, \tag{3.8}$$

where d_0 is from (3.5).

Proof By 3.1-3.3, the operator $\mathcal{A}$ is a completely continuous operator taking the set B_{d_0} into itself. Therefore, by Schauder's fixed point theorem (see [26], Chap. 8, Sec. 35), there exists a solution $q(x)$ of Eq. (3.3) belonging to the set B_{d_0} such that $q(x)$ satisfies estimate (3.8). □

Theorem 3.5 *Let conditions (1.6)–(1.8) hold. Let M_0 be the constant from estimate (2.28). Suppose that*

$$M^* < g_0, \tag{3.9}$$

where $M^* := T\chi_0\varphi_0 + C_1(\alpha)q_0M_0T^{\alpha+1} + C_0(\alpha)M_1T^{\alpha}$. Then the inverse problem (1.1)–(1.3) cannot have more than one solution.

Proof Suppose that there exist two different solutions of the inverse problem (1.1)–(1.3)), namely, the pairs $\{u^{(1)}(x,t), q^{(1)}(x)\}$ and $\{u^{(2)}(x,t), q^{(2)}(x)\}$. Then, necessarily,

$$q^{(1)}(x) \not\equiv q^{(2)}(x). \tag{3.10}$$

Note that, by the assumptions of 3.5, the functions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ satisfy estimate (2.28) with constant M_0 satisfying condition (3.10). Set

$$U(x,t) := u^{(1)}(x,t) - u^{(2)}(x,t), \qquad Q(x) := q^{(1)}(x) - q^{(2)}(x).$$

Then this pair of functions satisfies the relations

$$\left({}^C\mathcal{D}_t^{\alpha}U\right)(x,t) - \triangle U + q^{(1)}(x)U = -Q(x)u^{(2)}(x,t), \qquad \text{in} \quad \mathbb{R}^n_T, \tag{3.11}$$

$$U(x,0) = 0, \qquad \text{on} \quad \mathbb{R}^n. \tag{3.12}$$

$$\int_0^T U(x,t)\chi(t)dt = 0, \quad x \in \mathbb{R}^n. \tag{3.13}$$

Multiplying (3.12) by $\chi(t)$ and integrating over t between 0 and T, taking into account proposition, we obtain the relation

$$Q(x) = \frac{1}{g(x)}\int_0^T \left[\triangle U(x,t) - \left({}^C\mathcal{D}_t^{\alpha}U\right)(x,t)\right]dt. \tag{3.14}$$

Then, using results from (2.28), (2.32) applied to problem (3.6), (3.12), (3.13), and relation (3.14) implies the inequality

$$|Q(x)|^l \leq \frac{M^*}{g_0}|Q(x)|^l. \tag{3.15}$$

In view of assumption (3.10), it follows (3.15) that $Q(x) \equiv 0$ in $\mathbb{R}^n$, which contradicts (3.11).

The theorem is proved. □

4. Conclusion

We showed the existence and uniqueness of the solution of the direct problem using the method of successive approximations. Besides, the inverse problem has been shown to be of a similar character such a direct problem using by Schauder principle.

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