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On the extended zero-divisor graph of strictly partial transformation semigroup

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Abstract: Given a commutative ring R, the zero-divisor graph of R is an undirected simple graph with vertices the nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. In [8], Redmond presented different versions of zero-divisor graphs of noncommutative rings. The main aim of this paper is to analyse these graphs for the semigroup $\mathcal{SP}_n$ of all strictly partial transformations on the set $X_n = \{1, 2, \ldots, n\}$.

Key words: Strictly partial transformation, zero-divisor graph, clique number, chromatic number

1. Introduction

The concept of zero-divisor graphs was introduced by Beck in 1988 [2]. In this pioneering work, the author predominantly focused on the coloring of rings and let all elements of R be vertices of the graphs. Several decades later in [1], Anderson and Livingston presented the standard definition of zero-divisor graphs on commutative rings with identity. In this study, the authors let R be a commutative ring and let $Z(R)$ be its set of zero-divisors and considered the zero-divisor of R as a simple undirected graph having vertex set $Z(R)\backslash\{0\}$ in which two distinct vertices x, y are adjacent if and only if $xy = 0$. DeMeyer et al. utilized this standard definition on commutative semigroups with zero in [6, 7]. Further, several other studies (e.g., [4, 5, 7]) have extensively explored the zero-divisor graph of commutative semigroups.

For any set X contained in a ring R, let $X^\star = X\backslash\{0\}$. For a ring R, let $Z(R) = \{x \in R : xy = 0 \text{ or } zx = 0 \text{ for some } y, z \in R^\star\}$ is the set of zero-divisors of R and $T(R) = \{x \in R : xy = 0 = zx \text{ for some } y, z \in R^\star\}$ is the set of two-sided zero-divisors of R. Afterwards, Redmond generalized the zero-divisor graph to noncommutative rings in numerous ways in [8] and presented the following:

- Let R be a noncommutative ring. We define a directed graph $G(R)$ with vertices $Z(R)^\star = Z(R) \setminus \{0\}$, where $x \to y$ is an edge between distinct vertices x and y if and only if $xy = 0$.
- Let R be a noncommutative ring. We define an undirected graph $\overline{G}(R)$ with vertices $Z(R)^\star = Z(R)\backslash\{0\}$, where distinct vertices x and y are adjacent if and only if either $xy = 0$ or $yx = 0$.
- Let R be a noncommutative ring. We define an undirected graph $\overline{G}(R)$ with vertices $T(R)^\star = T(R)\backslash\{0\}$, where distinct vertices x and y are adjacent if and only if $xy = yx = 0$.

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- Let R be a noncommutative ring. We define an undirected graph $\overline{G}(R)$ with vertices $Z(R)^{\star} = Z(R)\setminus\{0\}$, where distinct vertices x and y are adjacent if and only if $xy = yx = 0$.

The third definition coincides with the usual definition of the zero-divisor graph when R is a commutative ring. These concepts carry over to noncommutative semigroups by taking the semigroup S with zero instead of R. There are several papers (e.g., [12–14]) presenting the study of these definitions for noncommutative semigroups.

Notation 1.1 *For $\alpha \in \mathcal{SP}_n$ we denote the domain of α by $dom(\alpha)$, the image of α by $im(\alpha)$ and the codomain of α by $codom(\alpha)$. A partial transformation $\alpha : dom(\alpha) \subseteq X_n \to im(\alpha) \subseteq X_n$ is called strictly partial provided that $dom(\alpha) \neq X_n$, that is, $codom(\alpha) \neq \emptyset$. For $\alpha \in \mathcal{SP}_n$ and $i \in X_n$, we will write $i\alpha = -$ if and only if $i \in codom(\alpha)$. The strictly partial transformation on X_n will be denoted by 0 if $codom(\alpha) = X_n$. For any $\alpha, \beta \in \mathcal{P}_n$, we shall use the notation $x\alpha$ instead of $\alpha(x)$, so that the composition $(x\alpha)\beta$ is written as $x(\alpha\beta)$.*

Let $\mathcal{SP}_n^{\star} = \mathcal{SP}_n \setminus \{0\}$. For $n \geq 2$, we define the following sets:

$$\begin{aligned}
L &= L(\mathcal{SP}_n) &&= \{\alpha \in \mathcal{SP}_n : \alpha\beta = 0 \text{ for some } \beta \in \mathcal{SP}_n^{\star}\},\\
R &= R(\mathcal{SP}_n) &&= \{\alpha \in \mathcal{SP}_n : \gamma\alpha = 0 \text{ for some } \gamma \in \mathcal{SP}_n^{\star}\},\\
T &= T(\mathcal{SP}_n) &&= \{\alpha \in \mathcal{SP}_n : \alpha\beta = 0 = \gamma\alpha \text{ for some } \beta, \gamma \in \mathcal{SP}_n^{\star}\},\\
Z &= Z(\mathcal{SP}_n) &&= \{\alpha \in \mathcal{SP}_n : \alpha\beta = 0 \text{ or } \gamma\alpha = 0 \text{ for some } \beta, \gamma \in \mathcal{SP}_n^{\star}\}
\end{aligned}$$

which are called the set of left zero-divisors, right zero-divisors, two-sided zero-divisors, and the set of zero-divisors of $\mathcal{SP}_n$. Let $\mathcal{S}_n$, $\mathcal{T}_n$, $\mathcal{P}_n$, and $\mathcal{SP}_n = \mathcal{P}_n \setminus \mathcal{T}_n$ be the symmetric group, (full) transformations semigroup, partial transformations semigroup and strictly partial transformations semigroup on the set $X_n = \{1, \ldots, n\}$, respectively. It is known that $|\mathcal{P}_n| = (n+1)^n$, $|\mathcal{T}_n| = (n)^n$ and $|\mathcal{SP}_n| = (n+1)^n - n^n$. The reader is referred to [9, 10] and [11] for more details in semigroup theory and graph theory, respectively.

The paper is organized as follows: In Section 1, we summarize relevant definitions and notations. In Section 2, we show that $L = R = T = Z = \mathcal{SP}_n$. Therefore, if we consider Redmond's generalizations for $\mathcal{SP}_n$, then the last two definitions of Redmond's coincide. In [14], Toker showed that $\mathcal{SP}_n$ is the set of two-sided zero-divisors of $\mathcal{P}_n$. He has also analysed the zero-divisor graph of $\mathcal{P}_n$. For this reason, we consider the simple directed graph, denoted by $\Gamma(\mathcal{SP}_n)$, (that is, with no multiple edges or loops) with vertices $\mathcal{SP}_n^{\star} = \mathcal{SP}_n \setminus \{0\}$, where $\alpha \to \beta$ is an edge between distinct vertices α and β if and only if $\alpha\beta = 0$. Then, we analyse of this graph in terms of its properties. In Section 3, we consider the simple undirected graph, denoted by $\overline{\Gamma}(\mathcal{SP}_n)$, (that is with no multiple edges or loops) with vertices $\mathcal{SP}_n^{\star} = \mathcal{SP}_n \setminus \{0\}$, where $\alpha - \beta$ is an edge between distinct vertices α and β if and only if either $\alpha\beta = 0$ or $\beta\alpha = 0$. Then, we analyse of this graph in terms of its properties.

2. A directed zero-divisor graph of $\mathcal{SP}_n$

In this section, we consider the simple directed graph $\Gamma = \Gamma(\mathcal{SP}_n) = (V(\Gamma), E(\Gamma))$ with vertices $\mathcal{SP}_n^{\star}$, where $\alpha \to \beta$ is an edge between distinct vertices α and β if and only if $\alpha\beta = 0$. For $\alpha, \beta \in V(\Gamma)$, we will write $\alpha \leftrightarrow \beta$ to mean that $\alpha \to \beta$ is an edge and $\beta \to \alpha$ is an edge. We begin with some results about (left/right) zero-divisors of $\mathcal{SP}_n$ that will be needed to analyse Γ. The following simple lemma is stated without proof.

Lemma 2.1 *Given $\alpha, \beta \in \mathcal{SP}_n$, the following statements are satisfied:*

(i) $\alpha\beta = 0$ *if and only if* $im(\alpha) \subseteq codom(\beta)$.

(ii) $\alpha^2 = 0$ *if and only if* $im(\alpha) \subseteq codom(\alpha)$.

Lemma 2.2 *For $n \geq 2$,*

$$L = R = T = Z = \mathcal{SP}_n.$$

Proof It is clear that $0 \in L,\ R,\ \mathcal{SP}_n$. For any $0 \neq \alpha \in \mathcal{SP}_n$, let

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r & A_{r+1} \\ a_1 & \cdots & a_r & - \end{pmatrix},$$

where $1 \leq r \leq n-1$. For $1 \leq i \leq n$, if we take

$$i\beta = \begin{cases} i & i \in codom(\alpha) \\ - & i \in X_n \setminus codom(\alpha) \end{cases} \quad \text{and} \quad i\gamma = \begin{cases} - & i \in im(\alpha) \\ i & i \in X_n \setminus im(\alpha) \end{cases}$$

then it is clear that $\beta, \gamma \in \mathcal{SP}_n^{\star}$. Since $im(\beta) \subseteq codom(\alpha)$, it follows quickly from Lemma 2.1 that $\beta\alpha = 0$, and so $\mathcal{SP}_n \subseteq R$. Similarly, since $im(\alpha) \subseteq codom(\gamma)$, it follows quickly from Lemma 2.1 that $\alpha\gamma = 0$, and so $\mathcal{SP}_n \subseteq L$. The proof of the lemma follows since $T = L \cap R$ and $Z = L \cup R$.

Let $G = (V, E)$ is a *directed graph*, where V is a finite nonempty set of vertices and $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$ is the set of edges. We will write $u \to v$ to mean that $\{u, v\} \in E$. For $u, v \in V$ if there exist distinct vertices $v_0, v_1, \ldots, v_n \in V(G)$ such that $v_0 = u, v_n = v$ and $v_{i-1} \to v_i$ is an edge in E for each in $1 \leq i \leq n$, then $u \to v_1 \to \cdots \to v_{n-1} \to v$ is called a *path* from u to v of length n in G. If there is a path between any two vertices in G, then G is called *strongly connected graph*. The length of the shortest path between u and v in G denoted by $d_G(u, v)$. The *diameter* of the graph G is defined by $\operatorname{diam}(G) = \max\{d_G(u, v) : u, v \in V\}$.

Lemma 2.3 Γ *is strongly connected and* $\operatorname{diam}(\Gamma) = 2$ *for* $n \geq 2$.

Proof We define $\theta_{ij} \in \mathcal{SP}_n$ such that $dom(\theta_{ij}) = \{i\}$ and $i\theta_{ij} = j$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. It is clear that $\theta_{ij} \in V(\Gamma)$. Now let $\Omega = \{\theta_{ij} \in \mathcal{SP}_n : 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}$ and $\lambda \in V(\Gamma) \setminus \Omega$. Since there exist $i, j \in X_n$ such that $i \notin im(\lambda)$ and $j \in codom(\lambda)$, then it follows quickly from Lemma 2.1 that $\lambda \leftrightarrow \theta_{ij}$ is a path. Moreover, for any $i, j, k, l \in X_n$ such that $i \neq k \neq j$, if we consider the transformations $\theta_{ij}, \theta_{kl} \in \Omega$, then $\theta_{ij} \leftrightarrow \theta_{kl}$ is a path. Hence, we prove that Γ is strongly connected and $\operatorname{diam}(\Gamma) \leq 2$. Now let α and β be distinct nonadjacent vertices of Γ. Then, it is clear that there are $i, j \in X_n$ such that $i \notin im(\alpha)$ and $j \in codom(\beta)$. Therefore, $\alpha \to \theta_{ij} \to \beta$ is path, and so $\operatorname{diam}(\Gamma) = 2$, as required.

The *girth* of the G, denoted by $\operatorname{gr}(G)$, is the length of the shortest cycle in G. The girth is infinite if G does not contain any cycles.

Lemma 2.4 $\operatorname{gr}(\Gamma) = 2$ *for* $n \geq 2$.

Proof For any two distinct $i, j \in X_n$, if we take $\theta_{ii}, \theta_{jj} \in \Omega$ which is defined as in Lemma 2.3, then it is clear that $\theta_{ii} \to \theta_{jj} \to \theta_{ii}$ is a cycle of length 2 in Γ, and so $\operatorname{gr}(\Gamma) = 2$.

Let v be a vertex of G. The *open out-neighbourhood* of v is $\mathcal{N}^+(v) = \{u \in V : \{v,u\} \in E\}$ and the *open in-neighbourhood* of v is $\mathcal{N}^-(v) = \{u \in V : \{u,v\} \in E\}$. The *closed out-neighbourhood* of v is $\mathcal{N}^+[v] = \mathcal{N}^+(v) \cup \{v\}$ and the *closed in-neighbourhood* of v is $\mathcal{N}^-[v] = \mathcal{N}^-(v) \cup \{v\}$. The *indegree* $\deg_G(v^+)$ of a vertex v is $\deg_G(v^+) = |\mathcal{N}^+(v)|$ and the *outdegree* $\deg_G(v^+)$ of a vertex v is $\deg_G(v^-) = |\mathcal{N}^-(v)|$. Furthermore, the *maximum indegree* and the *maximum outdegree* are denoted by $\triangle^-(G)$ and $\triangle^+(G)$, respectively. $\delta^-(G)$ is the *minimum indegree* and $\delta^+(G)$ is the *minimum outdegree*.

Lemma 2.5 *For $n \geq 2$, let $\alpha \in V(\Gamma)$ with $|im(\alpha)| = r$ and $|codom(\alpha)| = k$. Then,*

$$\deg_\Gamma(\alpha^+) = \begin{cases} (n+1)^{n-r} - 1 & im(\alpha) \nsubseteq codom(\alpha) \\ (n+1)^{n-r} - 2 & im(\alpha) \subseteq codom(\alpha) \end{cases} \quad and$$

$$\deg_\Gamma(\alpha^-) = \begin{cases} (k+1)^{n} - k^n - 1 & im(\alpha) \nsubseteq codom(\alpha) \\ (k+1)^{n} - k^n - 2 & im(\alpha) \subseteq codom(\alpha). \end{cases}$$

Proof Let $\alpha \in V(\Gamma)$ such that $|im(\alpha)| = r$ and $|codom(\alpha)| = k$ (for $1 \leq r \leq n-1$ and $1 \leq k \leq n-1$).
Case 1: Let $im(\alpha) \nsubseteq codom(\alpha)$. If $\beta \in \mathcal{N}^+(\alpha)$, then $\alpha\beta = 0$, and so $im(\alpha) \subseteq codom(\beta)$. This means that if $i \in im(\alpha)$, then $i\beta = -$ and if $i \in X_n \setminus im(\alpha)$ then $i\beta \in X_n$ or $i\beta = -$. However, there are $(n+1)^{n-r}$ elements in this way including 0, and so $\deg_\Gamma(\alpha^+) = (n+1)^{n+r} - 1$. If $\beta \in \mathcal{N}^-(\alpha)$, then $\beta\alpha = 0$, and so $im(\beta) \subseteq codom(\alpha)$. This means that for $j \in X_n$, we have $j\beta \in codom(\alpha)$ or $j\beta = -$. It is clear that there are $(k+1)^n$ elements in this way including 0. But, we must exclude those elements which are chosen from $codom(\alpha)$ such that $codom(\beta) = \emptyset$, as those elements do not belong to $V(\Gamma)$. Since there are k^n elements in this way, we have $\deg_\Gamma(\alpha^-) = (k+1)^n - k^n - 1$.
Case 2: Let $im(\alpha) \subseteq codom(\alpha)$. By considering $\alpha^2 = 0$, the proof is similar to above case.

Lemma 2.5 gives us immediately the proof of the next corollary.

Corollary 2.6 $\Delta^+(\Gamma) = (n+1)^{n-1} - 1$, $\Delta^-(\Gamma) = n^n - 1$, $\delta^+(\Gamma) = n$, *and* $\delta^-(\Gamma) = 2^n - 1$ *for* $n \geq 2$.

A nonempty subset D of $V(G)$ is called a *dominating set* of G if $\bigcup_{v \in D} \mathcal{N}^+[v] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Theorem 2.7 $\gamma(\Gamma) = n$ *for* $n \geq 2$.

Proof For $1 \leq i \leq n$, let $\mathcal{D} = \{\alpha_{ii} \in V(\Gamma) : dom(\alpha_{ii}) = im(\alpha_{ii}) = i\}$ and $\gamma \in V(\Gamma) \setminus \mathcal{D}$. Since there exists $j \in codom(\gamma)$, it follows quickly that $\alpha_{jj}\gamma = 0$. Furthermore, for any two distinct $i, j \in X_n$, it is clear that $\alpha_{ii}\alpha_{jj} = 0$. This yields, $\bigcup_{\lambda \in \mathcal{D}} \mathcal{N}^+[\lambda] = V(\Gamma)$, that is, $\mathcal{D}$ is a dominating set of Γ. For $1 \leq i \leq n$, let $\beta_{ii} \in V(\Gamma)$ such that $codom(\beta_{ii}) = i$ and $j\beta_{ii} = j$ for all $i \neq j \in X_n$. Then, we consider the set $\mathcal{A} = \{\beta_{ii} \in V(\Gamma) : 1 \leq i \leq n\}$. Now suppose that $\alpha \in V(\Gamma)$ is an adjacent β_{ii} and β_{jj} for any two distinct $i, j \in X_n$. This yields, $\alpha\beta_{ii} = 0$ and $\alpha\beta_{jj} = 0$ if and only if $\alpha = 0$. Similarly, $\beta_{ii}\alpha = 0$ and $\beta_{jj}\alpha = 0$ if and only if $\alpha = 0$. That is contradiction. Moreover, for any two distinct $i, j \in X_n$, it is clear that β_{ii} and β_{jj} are nonadjacent vertices. Thus, we show that if $\mathcal{B}$ is a minimum dominating set of Γ, then $|\mathcal{B}| \geq |\mathcal{A}| = n$. Since $\mathcal{D}$ is a dominating set of Γ and $|\mathcal{D}| = n$, this proves the assertion.

The *chromatic number* of G is the minimum number of colours required to colour all vertices in G so that no two adjacent vertices receive the same colour and it is denoted by $\chi(G)$.

A *complete graph* G is a simple graph such that every vertex is adjacent to every other vertex. A complete graph on n vertices is denoted by K^n. A subset C of V is called a *clique* in G if $u \to v$ for all distinct $u, v \in C$. The *clique number* of G, denoted by $\omega(G)$, is the greatest integer r such that G has a clique K^r.
For $n \geq 2$, we give a lower bound for the clique number of Γ in the following theorem.

Theorem 2.8 *If $n \geq 2$, then $\omega(\Gamma) \geq (r+1)^{n-r} - 1$ for $1 \leq r \leq n-1$.*

Proof The proof is the same in the given [14, Theorem 3.7].

For any graph G, it is proved in [3, Corollary, 6.2] that $\chi(G) \geq \omega(G)$. Thus, we have the next corollary.

Corollary 2.9 *If $n \geq 2$, then $\chi(\Gamma) \geq (r+1)^{n-r} - 1$ for $1 \leq r \leq n-1$.*

3. Undirected zero-divisor graph of $\mathcal{SP}_n$

In this section, we define the simple undirected graph $\overline{\Gamma} = \overline{\Gamma}(\mathcal{SP}_n) = (V(\overline{\Gamma}), E(\overline{\Gamma}))$ with vertices $\mathcal{SP}_n^{\star}$, where $\alpha - \beta$ is an edge between distinct vertices α and β if and only if either $\alpha\beta = 0$ or $\beta\alpha = 0$.

Let $\overline{G} = (\overline{V}, \overline{E})$ is an *undirected graph*, where $\overline{V}$ is a finite nonempty set of vertices and $\overline{E} \subsetneq \{(u,v) : u, v \in \overline{V}, u \neq v\}$ is the set of edges. We will write $u - v$ to mean that $(u,v) \in \overline{E}$. For $u, v \in \overline{V}$ if there exist distinct vertices $v_0, v_1, \ldots, v_n \in \overline{V}$ such that $v_0 = u, v_n = v$ and $v_{i-1} - v_i$ is an edge in $\overline{E}$ for each in $1 \leq i \leq n$, then $u - v_1 - \cdots - v_{n-1} - v$ is called a *path* from u to v of length n in $\overline{G}$. If there is a path between any two vertices in $\overline{G}$, then $\overline{G}$ is called *connected graph*. The length of the shortest path between u and v in $\overline{G}$ is denoted by $d_{\overline{G}}(u,v)$. The *diameter* of the graph $\overline{G}$ is defined by $\operatorname{diam}(\overline{G}) = \max\{d_{\overline{G}}(u,v) : u, v \in \overline{V}\}$.

Lemma 3.1 *$\overline{\Gamma}$ is connected and $\operatorname{diam}(\overline{\Gamma}) = 2$ for $n \geq 2$.*

Proof As defined in Lemma 2.3, let $\theta_{ij} \in \mathcal{SP}_n$ such that $dom(\theta_{ij}) = \{i\}$ and $i\theta_{ij} = j$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. Then, we consider the set $\Omega = \{\theta_{ij} \in \mathcal{SP}_n : 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}$. It is clear that $\Omega \subsetneq V(\overline{\Gamma})$.
Now assume that α and β be distinct vertices of $\overline{\Gamma}$.
<u>Case 1:</u> $\alpha\beta = 0$ or $\beta\alpha = 0$. Then, $\alpha - \beta$ is a path.
Now suppose that $\alpha\beta \neq 0$ and $\beta\alpha \neq 0$.
<u>Case 2:</u> Let $im(\alpha) \leq n-1$ and $im(\beta) \leq n-2$. Then, it is clear that there exist $i, j \in X_n$ such that $i \notin im(\alpha)$ and $j \in codom(\beta)$. Now if we consider the transformation $\theta_{ij} \in \Omega$, then $\alpha\theta_{ij} = 0$ and $\theta_{ij}\beta = 0$. This yields, $\alpha - \theta_{ij} - \beta$ is a path.
<u>Case 3:</u> Let $im(\alpha) \leq n-2$ and $im(\beta) \leq n-1$. The result follows by using an argument similar to the aforementioned case.
<u>Case 4:</u> Let $im(\alpha) = n-1$ and $im(\beta) = n-1$.
<u>Subcase 1:</u> For, $i, j \in X_n$, let $i \notin im(\alpha)$ and $j \notin im(\beta)$ such that $i \neq j$. It is clear that there exists $k \in X_n$ such that $k \in codom(\beta)$. Now if we consider the transformation $\theta_{ik} \in \Omega$, then $\alpha\theta_{ik} = 0$ and $\theta_{ik}\beta = 0$. This yields, $\alpha - \theta_{ik} - \beta$ is a path.

<u>Subcase 2:</u> For $i \in X_n$, let $i \notin im(\alpha)$ and $i \notin im(\beta)$. If we consider the transformation $\theta_{ii} \in \Omega$, then $\alpha\theta_{ii} = 0$ and $\beta\theta_{ii} = 0$. This yields, $\alpha - \theta_{ii} - \beta$ is a path.

Thus, we prove that $\overline{\Gamma}$ is connected and $\mathrm{diam}(\overline{\Gamma}) \leq 2$. If we take θ_{ij} and θ_{ji} in Ω, then it is clear that θ_{ij} and θ_{ji} are nonadjacent vertices in $\overline{\Gamma}$, and so $\mathrm{diam}(\overline{\Gamma}) = 2$.

The *girth* of the $\overline{G}$, denoted by $\mathrm{gr}(\overline{G})$, is the length of the shortest cycle in $\overline{G}$. The girth is infinite if $\overline{G}$ does not contain any cycles.

Lemma 3.2 $\mathrm{gr}(\overline{\Gamma}) = 3$ *for* $n \geq 2$.

Proof Since $\overline{\Gamma}$ is a simple undirected graph, it is clear that $\mathrm{gr}(\overline{\Gamma}) \geq 3$. Let i, j, and k be three distinct elements of X_n. If we consider the set $\Omega \subsetneq V(\overline{\Gamma})$ as defined in Lemma 3.1, then we have a cycle $\theta_{ii} - \theta_{jj} - \theta_{kk} - \theta_{ii}$, and so $\mathrm{gr}(\overline{\Gamma}) = 3$.

The *degree* of a vertex v of $\overline{V}$, denoted $\deg_{\overline{G}}(v)$, is the number of adjacent vertices to v in $\overline{G}$. The *maximum vertex degree* and *minimum vertex degree* in $\overline{G}$ are denoted by $\triangle(\overline{G})$ and $\delta(\overline{G})$, respectively.

Lemma 3.3 *For* $n \geq 2$*, let* $\alpha \in V(\overline{\Gamma})$ *with* $|im(\alpha)| = r$ *and* $|codom(\alpha)| = k$*. Then,*

$$\deg_{\overline{\Gamma}}(\alpha) = \begin{cases} (n+1)^{n-r} + (k+1)^n - k^n - (k+1)^{n-r} - 1 & im(\alpha) \nsubseteq codom(\alpha) \\ (n+1)^{n-r} + (k+1)^n - k^n - (k+1)^{n-r} - 2 & im(\alpha) \subseteq codom(\alpha). \end{cases}$$

Proof Let $\alpha \in V(\overline{\Gamma})$ such that $|im(\alpha)| = r$ and $|codom(\alpha)| = k$ (for $1 \leq r \leq n-1$ and $1 \leq k \leq n-1$).

<u>Case 1:</u> Let $im(\alpha) \nsubseteq codom(\alpha)$. For $\beta \in \mathcal{SP}_n$, if $\alpha\beta = 0$, then $im(\alpha) \subseteq codom(\beta)$. For $\gamma \in \mathcal{SP}_n$, if $\gamma\alpha = 0$, then $im(\gamma) \subseteq codom(\alpha)$. Using a similar method as in the proof of Lemma 2.5, it can be easily obtained that there are $(n+1)^{n-r} - (k+1)^n - k^n$ adjacent vertices of α in $V(\overline{\Gamma})$. Now we must exclude those elements which are counted twice. For $\lambda \in \mathcal{SP}_n$, if $\alpha\lambda = 0 = \lambda\alpha$, then $im(\alpha) \subseteq codom(\lambda)$ and $im(\lambda) \subseteq codom(\alpha)$. This means that if $i \in im(\alpha)$, then $i\lambda = -$ and if $i \in X_n \setminus im(\alpha)$, then $i\lambda \in codom(\alpha)$ or $i\lambda = -$. However, there are $(k+1)^{n-r}$ elements in this way including 0. Thus, $\deg_{\overline{\Gamma}}(\alpha) = (n+1)^{n-r} + (k+1)^n - k^n - (k+1)^{n-r} - 1$, as required.

<u>Case 2:</u> Let $im(\alpha) \subseteq codom(\alpha)$.

The proof can be obtained by using an argument similar to the aforementioned case in view of the fact that we consider α is adjacent itself.

Lemma 3.3 gives us immediately the proof of the next corollary.

Corollary 3.4 $\Delta(\overline{\Gamma}) = (n+1)^{n-1} + n^n - (n-1)^n - n^{n-1} - 1$ *and* $\delta(\overline{\Gamma}) = n + 2^n - 3$ *for* $n \geq 2$.

A nonempty subset D of $\overline{V}$ is called a *dominating set* of $\overline{G}$ if every vertex $v \in \overline{V}$ is either in D or is adjacent to a vertex in D. The *dominating number* $\gamma(\overline{G})$ of a graph $\overline{G}$ is the minimum cardinality of a dominating set in $\overline{G}$.

Theorem 3.5 $\gamma(\overline{\Gamma}) = n$ *for* $n \geq 2$.

Proof For $1 \leq i \leq n$, let $\mathcal{D} = \{\alpha_{ii} \in V(\overline{\Gamma}) : dom(\alpha_{ii}) = im(\alpha_{ii}) = i\}$ and $\lambda \in V(\overline{\Gamma}) \setminus D$. Since there exists $j \in X_n$ such that $j \in codom(\lambda)$, it follows quickly that $\alpha_{jj}\lambda = 0$. Furthermore, for any two distinct $i, j \in X_n$, it is clear that $\alpha_{ii}\alpha_{jj} = 0$. This yields, D is a dominating set of $\overline{\Gamma}$. For $1 \leq i \leq n$, let $\beta_{ii} \in V(\overline{\Gamma})$ such that $codom(\beta_{ii}) = i$ and $j\beta_{ii} = j$ for all $i \neq j \in X_n$. Then, we consider the set $\mathcal{A} = \{\beta_{ii} \in V(\overline{\Gamma}) : 1 \leq i \leq n\}$. Now suppose that $\alpha \in V(\overline{\Gamma})$ is an adjacent β_{ii} and β_{jj} for any two distinct $i, j \in X_n$. This yields, $\alpha\beta_{ii} = 0$ and $\alpha\beta_{jj} = 0$ if and only if $\alpha = 0$. Similarly, $\beta_{ii}\alpha = 0$ and $\beta_{jj}\alpha = 0$ if and only if $\alpha = 0$. That is a contradiction. Moreover, for any two distinct $i, j \in X_n$, it is clear that β_{ii} and β_{jj} are nonadjacent. Thus, we show that if $\mathcal{B}$ is a minimum dominating set of $\overline{\Gamma}$, then $|\mathcal{B}| \geq |\mathcal{A}| = n$. Since $\mathcal{D}$ is a dominating set of $\overline{\Gamma}$ and $|\mathcal{D}| = n$, this proves the assertion.

Recall that we use $u - v$ to mean that $(u, v) \in E(\overline{G})$. By taking into account of this, the *chromatic number*, *clique*, and *clique number* for $\overline{G}$ are defined as in Section 2.

For $n \geq 2$, we give a better lower bound for the clique number of $\overline{\Gamma}$ in the following theorem.

Theorem 3.6 *If* $n \geq 2$, *then* $\omega(\overline{\Gamma}) \geq (r+1)^{n-r} + \frac{r(r+1)}{2} - 1$ *for* $1 \leq r \leq n-1$.

Proof Let $A = \{k_1, k_2, \ldots, k_r\} \subseteq X_n$ for $1 \leq r \leq n-1$. Let $\beta_{ji} \in \mathcal{SP}_n$ such that $dom(\beta_{ji}) = j$ and $j\beta_{ji} = i$ for all $i, j \in A$ with $i \leq j$. Now we consider the sets:

$$\begin{aligned} B &= \{\alpha \in \mathcal{SP}_n : A \subseteq codom(\alpha) \text{ and } \emptyset \neq im(\alpha) \subseteq A\}, \\ C &= \{\beta_{ji} \in \mathcal{SP}_n : 1 \leq i \leq j \leq r\}. \end{aligned}$$

It is clear that $B \neq \emptyset \neq C$ and if $\alpha \in B \cup C$, then $\alpha \in V(\overline{\Gamma})$. For any two distinct $\beta, \lambda \in B \cup C$, it is easy to see that $im(\beta) \subseteq codom(\lambda)$ and $im(\lambda) \subseteq codom(\beta)$. This yields, β and λ are adjacent vertices in $V(\overline{\Gamma})$ from Lemma 2.1. Thus, $\overline{\Gamma}$ has a clique $K^{|B \cup C|}$. Since $|B \cup C| = |B| \cup |C| = (r+1)^{n-r} - 1 + \frac{r(r+1)}{2}$, it follows quickly that $\omega(\overline{\Gamma}) \geq (r+1)^{n-r} + \frac{r(r+1)}{2} - 1$ for $1 \leq r \leq n-1$, as required.

For any graph G, it is proved in [3, Corollary, 6.2] that $\chi(G) \geq \omega(G)$. Hence, we have the following corollary.

Corollary 3.7 *If* $n \geq 2$, *then* $\chi(\overline{\Gamma}) \geq (r+1)^{n-r} + \frac{r(r+1)}{2} - 1$ *for* $1 \leq r \leq n-1$.

Together, our study contributes to the research conducted on zero divisor graphs by revealing the properties of the extended zero-divisor graphs of $\mathcal{SP}_n$.

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