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Characterization of exponential polynomial as solution of certain type of nonlinear delay-differential equation

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Abstract: In this paper, we have characterized the nature and form of solutions of the following nonlinear delay-differential equation:

$$f^n(z)+\sum_{i=1}^{n-1} b_i f^i(z)+q(z)e^{Q(z)}L(z,f)=P(z),$$

where $b_i \in \mathbb{C}$, $L(z,f)$ are a linear delay-differential polynomial of f; n is positive integers; q, Q and P respectively are nonzero, nonconstant and any polynomials. Different special cases of our result will accommodate all the results of [J. Math. Anal. Appl., 452(2017), 1128-1144; Mediterr. J. Math., 13(2016), 3015-3027; Open Math., 18(2020), 1292-1301]. Thus our result can be considered as an improvement of all of them. We have also illustrated a handful number of examples to show that all the cases as demonstrated in our theorem actually occur and consequently the same are automatically applicable to the previous results.

Key words: Exponential polynomial, differential-difference equation, convex hull, Nevanlinna theory

1. Introduction, results and examples

Throughout the paper, we denote by f a meromorphic function in the complex plane $\mathbb{C}$ and related to the function, we assume that the readers are familiar with the basic terms like $T(r,f)$, $N(r,f)$, $m(r,f)$, of Nevanlinna value distribution theory of meromorphic functions (see [6, 8]). The notation $S(r,f)$ will be used to define any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside a set E of r of finite logarithmic measure. In addition, we will respectively use the symbols $\rho(f)$, $\lambda(f)$ and $\tau(f)$ to denote the order, exponent of convergent and type of f. The symbol $L(f)$ will be used to represent a linear differential polynomial in f with polynomial coefficients. Also, throughout this paper, by $card(S)$, we mean the cardinality of a set S, i.e. the number of elements in S.

Considering the nonlinear differential equation

$$L(f)-p(z)f^n(z)=h(z),$$

in 2001, Yang [17] investigated about the transcendental finite order entire solutions f of the equation, where $p(z)$ is a nonvanishing polynomial, $h(z)$ is entire and $n\geq 4$ is an integer.

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In 2010, Yang-Laine [18] showed that the equation

$$f(z)^2 + q(z)f(z+1) = p(z),$$

where $p(z)$, $q(z)$ are polynomials, admits no transcendental entire solutions of finite order.

In the last two decades researchers mainly studied (see [1, 9, 11, 17, 20], etc.) about the following three distinct features of solutions of shift or delay-differential or differential equations:
i) existence and nonexistence conditions,
ii) order of growth, and
iii) different types of forms of solutions.

Next, let us consider the exponential polynomial $f(z)$, defined by the form

$$f(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)}, \tag{1.1}$$

where P_j's and Q_j's are polynomials in z. Steinmetz [14] showed that (1.1) can be written in the normalized form

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z^t} + \cdots + H_m(z)e^{\omega_m z^t}, \tag{1.2}$$

where H_j are either exponential polynomials of order $< t$ or ordinary polynomials in z, the leading coefficients ω_j are pairwise distinct and $m \leq k$.

Let $co(\mathcal{W})$ be the convex hull of a set $\mathcal{W} \subset \mathbb{C}$ which is the intersection of all convex sets containing $\mathcal{W}$. If $\mathcal{W}$ contains finitely many elements then $co(\mathcal{W})$ is obtained as an intersection of finitely many half-planes, then $co(\mathcal{W})$ is either a compact polygon with a nonempty interior or a line segment. We denote by $C(co(\mathcal{W}))$, the circumference of $co(\mathcal{W})$. If $co(\mathcal{W})$ is a line-segment, then $C(co(\mathcal{W}))$ is equals to twice the length of this line segment. Throughout the paper, we denote $W = \{\bar{\omega}_1, \bar{\omega}_2, \ldots, \bar{\omega}_m\}$ and $W_0 = W \cup \{0\}$.

Nowadays, to find the form of exponential polynomials as solution of certain nonlinear differential-difference equation has become an interesting topic among researchers (see [3, 4, 16]). Most probably, in this regard, the first attempt was made by Wen–Heittokangas–Laine [15]. In 2012, they considered the equation

$$f(z)^n + q(z)e^{Q(z)}f(z+c) = P(z), \tag{1.3}$$

where $q(z)$, $Q(z)$, $P(z)$ are polynomials, $n \geq 2$ is an integer, $c \in \mathbb{C}\backslash\{0\}$. Wen–Heittokangas–Laine [15] also pointed out that for a nonconstant polynomial $\alpha(z)$ and $d \in \mathbb{C}$, every solution f of the form (1.2) reduces to a function which belongs to one of the following classes:

$$\begin{aligned} \Gamma_1 &= \{e^{\alpha(z)} + d\}, \\ \Gamma_0 &= \{e^{\alpha(z)}\} \end{aligned}$$

and classified finite order entire(meromorphic) solutions of (1.3) as follows:

Theorem 1.1 [15] *Let $n \geq 2$ be an integer, let $c \in \mathbb{C}\backslash\{0\}$, $q(z)$, $Q(z)$, $P(z)$ be polynomials such that $Q(z)$ is not a constant and $q(z) \not\equiv 0$. Then the finite order entire solutions f of Equation (1.3) satisfies the following conclusions:*

(a) Every solution f satisfies $\rho(f) = \deg Q$ and is of mean type.

(b) Every solution f satisfies $\lambda(f) = \rho(f)$ if and only if $P(z) \not\equiv 0$.

(c) A solution f belongs to Γ_0 if and only if $P(z) \equiv 0$. In particular, this is the case if $n \geq 3$.

(d) If a solution f belongs to Γ_0 and if g is any other finite order entire solution of (1.3), then $f = \eta g$, where $\eta^{n-1} = 1$.

(e) If f is an exponential polynomial solution of the form (1.2), then $f \in \Gamma_1$. Moreover, if $f \in \Gamma_1 \backslash \Gamma_0$, then $\rho(f) = 1$.

Inspired by *Theorem 1.1*, in 2016, Liu [12] replaced $f(z+c)$ by $f^{(k)}(z+c)$ in (1.3) and for two polynomials $p_1(z)$, $p_2(z)$ and a nonconstant polynomial $\alpha(z)$, introduced two new classes of solutions:

$$\begin{aligned} \Gamma_1' &= \{p_1(z)e^{\alpha(z)} + p_2(z)\}, \\ \Gamma_0' &= \{p_1(z)e^{\alpha(z)}\} \end{aligned}$$

to obtain the following theorem.

Theorem 1.2 [12] *Under the same situation as in* Theorem 1.1 *with $k \geq 1$, the finite-order transcendental entire solution f of*

$$f(z)^n + q(z)e^{Q(z)} f^{(k)}(z+c) = P(z) \tag{1.4}$$

should satisfy the results (a), (b), (d) *and*

(1) a solution f belongs to Γ_0' if and only if $P(z) \equiv 0$. In particular, this is the case if $n \geq 3$,

(2) if f is an exponential polynomial solution of (1.4) of the form (1.2), then $f \in \Gamma_1'$.

Recently, Liu–Mao–Zheng [13] considered $\Delta_c f(z)$ instead of $f(z+c)$ in (1.3) and proved the following theorem.

Theorem 1.3 [13] *Under the same situation as in* Theorem 1.1*, the finite order entire solutions f of the equation*

$$f(z)^n + q(z)e^{Q(z)} \Delta_c f(z) = P(z), \tag{1.5}$$

satisfies the results (a), (b) *and*

(1) $\lambda(f) = \rho(f) - 1$ if and only if $P(z) \equiv 0$. In particular, this is the case if $n \geq 3$,

(2) if $n \geq 3$ or $P(z) \equiv 0$, f is of the form $f(z) = A(z)e^{\omega z^s}$, where $s = \deg Q$, ω is a nonzero constant and $A(z)(\not\equiv 0)$ is an entire function satisfying $\lambda(A) = \rho(A) = \deg Q - 1$. In particular, if $\deg Q = 1$, then $A(z)$ reduces to a polynomial,

(3) if f is an exponential polynomial solution of (1.5) of the form (1.2), then f is of the form

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z},$$

where $H_1(z)$, $H_2(z)$ are nonconstant polynomials and ω_1 is a nonzero constant satisfying $e^{\omega_1 c} = 1$.

In 2017, Li–Yang [10] considered the following form of equation

$$f^n(z) + a_{n-1}f^{n-1}(z) + \cdots + a_1 f(z) + q(z)e^{Q(z)}f(z+c) = P(z), \tag{1.6}$$

where $a_i \in \mathbb{C}$ and proved the following results.

Theorem 1.4 [10] *Under the same situation as in* Theorem 1.1*, the finite order entire solutions f of Equation (1.6) satisfies the results* (a), (d) *and*

(1) if zero is a Borel exceptional value of $f(z)$, then we have $a_{n-1} = \cdots = a_1 = P(z) \equiv 0$,

(2) if $P(z) \equiv 0$, then we have $z^{n-1} + a_{n-1}z^{n-2} + \cdots + a_1 = (z + a_{n-1}/n)^{n-1}$. Furthermore, if there exists $i_0 \in \{1, \ldots, n-1\}$ such that $a_{i_0} = 0$, then all of the $a_j (j = 1, \ldots, n-1)$ must be zero as well and we have $\lambda(f) < \rho(f)$; otherwise we have $\lambda(f) = \rho(f)$,

(3) a solution f belongs to Γ_0 if and only if $P(z) \equiv 0$ and there exists an $i_0 \in \{1, \ldots, n-1\}$ such that $a_{i_0} = 0$,

(4) when $n \geq 3$, if there exists an $i_0 \in \{1, \ldots, n-1\}$ such that $a_{i_0} = 0$ and card$\{z : p(z) = p'(z) = p''(z) = 0\} \geq 1$ or card$\{z : p(z) = p'(z) = 0\} \geq 2$, where $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z$, then f belongs to Γ_0 and $a_{n-1} = \cdots = a_1 = 0 \equiv P(z)$.

In the same paper, Li–Yang [10] also proved the following result.

Theorem 1.5 [10] *If f is an exponential polynomial solution of the form (1.2) of Equation (1.6) for $n = 2$ and $a_1 \neq 0$, then the following conclusions hold.*

(1) When $m \geq 2$, there exists $i, j \in \{1, 2, \ldots, m\}$ such that $\omega_i = 2\omega_j$.

(2) When $m = 1$, then $f \in \Gamma_1$. Moreover, if $f \in \Gamma_1 \backslash \Gamma_0$, then $\rho(f) = 1$, $f(z) = Ke^{\frac{1}{c}(2k\pi i - \log\frac{2d+a_1}{d})z}$, $Q(z) = \frac{1}{c}(2k\pi i - \log\frac{2d+a_1}{d})z$, $q(z) = -\frac{2d+a_1}{d}$ and $d^2 + a_1 d = P(z)$, where $K, d \in \mathbb{C}\backslash\{0\}$ and $k \in \mathbb{Z}$.

We now introduce the generalized linear delay-differential operator of $f(z)$,

$$L(z, f) = \sum_{i=0}^{k} b_i f^{(r_i)}(z + c_i) \; (\not\equiv 0), \tag{1.7}$$

where $b_i, c_i \in \mathbb{C}$, r_i are nonnegative integers, $c_0 = 0$, $r_0 = 0$. In view of the above theorems it is quiet natural to characterize the nature of exponential polynomial as solution of certain nonlinear complex equation involving generalized linear delay-differential operator. In this regard, we consider the following nonlinear delay-differential equation

$$f^n(z) + \sum_{i=1}^{n-1} a_i f^i(z) + q(z)e^{Q(z)}L(z, f) = P(z), \tag{1.8}$$

where $a_i \in \mathbb{C}$, n are nonnegative integers; q, Q, P respectively are nonzero, nonconstant, any polynomials. We also introduce, for any polynomials $p_i(z)$ and nonconstant polynomials $\alpha_i(z)$, a new class of solution as follows:

$$\Gamma_2' \quad = \quad \{p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)} + p_3(z)\}.$$

Now, we are at a state to present our main result which improves all the above mentioned results as follows:

Theorem 1.6 *Under the same situation as in* Theorem 1.1, *the finite order entire solutions f of equation (1.8) satisfies the following conclusions.*

(i) Every solution f satisfies $\rho(f) = \deg Q$ and is of mean type.

(ii) If zero is a Borel exceptional value of $f(z)$, then we have $a_{n-1} = \cdots = a_1 = P(z) \equiv 0$. Conversely, if $P(z) \equiv 0$ and there exists an $i_0 \in \{1, \ldots, n-1\}$ such that $a_{i_0} = 0$, then all of a_j's $(j = 1, \ldots, n-1)$ must be zero and we have $\lambda(f) < \rho(f)$; otherwise we have $\lambda(f) = \rho(f)$.

(iii) If a solution f belongs to Γ_0', then $a_{n-1} = \cdots = a_1 = P(z) \equiv 0$. Conversely, let $P(z) \equiv 0$ and there exists an $i_0 \in \{1, \ldots, n-1\}$ such that $a_{i_0} = 0$, then either $\lambda(f) = \rho(f) - 1$ for $c_i = c_j$, $1 \le i, j \le k$ or f belongs to Γ_0'.

(iv) Let $n \ge 3$. If at least one $a_{i_0} = 0$ $(i_0 = 1, 2, \ldots, n-1)$ and $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z$ such that card$\{z : p(z) = p'(z) = p''(z) = 0\} \ge 1$ or card$\{z : p(z) = p'(z) = 0\} \ge 2$, then $P(z) \equiv 0 = a_{n-1} = \cdots = a_1 = 0$ and $f \in \Gamma_0'$. Moreover, card$\{z : p(z) = p'(z) = 0\} \ge 2$ is not possible.

(v) Let f be given by (1.2), which is a solution of (1.8) for $n = 2$ and $a_1 \ne 0$. Then the following conclusions hold:

(a) when $m \ge 2$, there exists $i, j \in \{1, 2, \ldots, m\}$ such that $\omega_i = 2\omega_j$. In this case, $f \in \Gamma_2'$.

(b) when $m = 1$, then f takes the form $f(z) = H_0(z) + H_1(z)e^{\omega_1 z^t}$, i.e. $f \in \Gamma_1'$. In this case,

(I) either $t = 1$, $\rho(f) = 1$ and $H_0(z)$, $H_1(z)$ are polynomials and $Q(z)$ is a polynomial of degree 1

(II) or $H_0(z) = -\frac{a_1}{2}$, $P(z) = -\frac{a_1^2}{4}$, $H_1^2(z) = \frac{b_0 a_1}{2} q(z) e^{Q_{t-1}(z)}$ and $L(z, f) = b_0 H_0(z)$

(III) or $H_0(z) = -\frac{a_1}{2}$, $P(z) = -\frac{a_1^2}{4}$, $H_1^2(z) = -q(z)e^{Q_{t-1}(z)}\mathcal{A}_1(z)$ and $L(z, f) = \mathcal{A}_1(z)e^{\omega_1 z^t}$, where $\mathcal{A}_1(z) = \sum\limits_{i=0}^{k} b_i \tilde{H}_1(z + c_i)e^{\omega_1(z+c_i)^t - \omega_1 z^t}$ such that $\tilde{H}_1(z + c_i)$ are the delay-differential polynomial of $H_1(z)$.

Remark 1.1 *Note that* Cases (i)–(iv) *and* (v) *of* Theorem 1.6 *improve* Theorems 1.4 *and* 1.5, *respectively. Also, since $L(z, f)$ includes $f^{(k)}(z + c)$ and $\Delta_c f(z)$,* Theorem 1.6 *improves* Theorems 1.2-1.3 *as follows:*

(I) Cases (i), (ii) *and* (iii)–(iv) *of* Theorem 1.6 *improve* Case (a), (b) *and* (1) *of each of* Theorems 1.2-1.3, *respectively.*

(II) Case (v)–(b) *of* Theorem 1.6 *improves, respectively,* Case (2) *and* Case (3) *of* Theorem 1.2 *and* Theorem 1.3.

This following three examples clarify *Cases (ii)-(iii)*.

Example 1.1 *Take* $L(z,f)=f''(z+c)$. *Then the function* $f=e^{2z}$ *satisfies the equation* $f^2-\frac{1}{4}e^{2z}L(z,f)=0$ *such that* $e^{2c}=1$. *Clearly,* $0=\lambda(f)=\rho(f)-1$. *This example clarifies* Theorem 1.2 *as well.*

Example 1.2 *Let* $L(z,f)=\Delta_c f(z)$. *Then the function* $f=e^{\alpha z}$ *satisfies the equation* $f^2-\frac{1}{2}e^{\alpha z}L(z,f)=0$ *such that* $e^{\alpha c}=3$. *Clearly,* $0=\lambda(f)=\rho(f)-1$. *This example also satisfies* Theorem 1.3.

Example 1.3 *Let* $L(z,f)=f(z+1)+f'(z+1)-f''(z+1)$. *Then the function* $f=(z+1)e^z$ *satisfies the equation* $f^2-(z+1)e^{z-1}L(z,f)=0$. *Note that, here* $c_1=c_2=c_3=1$ *and* $f\in\Gamma'_0$.

The next example satisfies *Case (iv)*.

Example 1.4 *Let* $L(z,f)=f(z+\log 2)+f''(z+\pi i)$. *Then the function* $f=e^{iz}$ *satisfies the equation* $f^3+qe^{2iz}L(z,f)=0$, *where* $q=\frac{1}{e^{-\pi}-2^i}$. *Note that, here* $p(z)=z^3$ *and* $card\{z:p(z)=p'(z)=p''(z)=0\}=1$. *Also,* $a_2=a_1=0\equiv P(z)$ *and* $f\in\Gamma'_0$.

By the following example, it is clear that the *Case (v)–(a)* occurs significantly.

Example 1.5 *Take* $L(z,f)=f'(z+\log 4)-4f(z+\log 3)$ *and* $m=2$. *Then the function* $f=e^{2z}-e^z+1$ *satisfies the equation* $f^2-2f+\frac{1}{4}e^{2z}L(z,f)=-1$. *Note that here* $f\in\Gamma'_2$.

The following two examples show that the *Case (v)–(b)–(I)* actually holds.

Example 1.6 *We take* $L(z,f)=f(z+c)$. *Then the function* $f=d+e^{\alpha z}$ *satisfies the equation* $f^2-df-e^{\alpha z}L(z,f)=0$ *such that* $e^{\alpha c}=1$. *Here,* $P(z)\equiv 0$.
Also, the same function satisfies $f^2-3df+e^{\alpha z}L(z,f)=-2d^2$ *such that* $e^{\alpha c}=-1$. *Here,* $f\in\Gamma'_1$. *Here,* $P(z)\not\equiv 0$. *This example is true for* Theorem 1.5 *as well.*

Example 1.7 *Put* $L(z,f)=f(z+\log 2)+f'(z+\pi i)+f''(z+2\pi i)$ *and* $m=1$. *Then the function* $f=2+3e^z$ *satisfies the equation* $f^2-3f-\frac{3}{2}e^zL(z,f)=-2$. *Here,* $P(z)\not\equiv 0$.
Also, let $L(z,f)=f(z+\log 3)-f'(z+\log 4)+f''(z+\log 2)$ *and* $m=1$. *Then the function* $f=3+e^z$ *satisfies the equation* $f^2-3f-e^zL(z,f)=0$. *Here,* $P(z)\equiv 0$.

Next example shows that the *Case (v)–(b)–(II)* actually occurs.

Example 1.8 *Let* $L(z,f)=3f(z)+f'(z+\log 2)-3f''(z+2\pi i)$ *and* $m=1$. *Then the function* $f=-\frac{a_1}{2}+2e^{3z}$ *satisfies the equation* $f^2+a_1f+\frac{8}{3a_1}e^{6z}L(z,f)=-\frac{a_1^2}{4}$. *Note that here* $b_0=3$, $H_0=-\frac{a_1}{2}$ *and so,* $L(z,f)=-\frac{3a_1}{2}=b_0H_0$.

Next example shows that the *Case (v)–(b)–(III)* actually occurs.

Example 1.9 *Let* $L(z,f) = f(z) - f(z+\log 2) + \frac{1}{2}f'(z+\log 2) + \frac{2}{9}f'(z+\log 3) - \frac{1}{9}f''(z+\log 3)$ *and* $m = 1$. *Then the function* $f = -\frac{a_1}{2} + ze^{2z}$ *satisfies the equation* $f^2 + a_1 f - ze^{2z}L(z,f) = -\frac{a_1^2}{4}$. *Note that here,* $q(z) = -z$, $Q(z) = 2z$, $Q_{t-1}(z) = 0$. *Also,* $\mathcal{A}_1(z) = \sum_{i=0}^{5}\left(b_i\tilde{H}_1(z+c_i)e^{2c_i}\right) = z$. *So,* $H_1^2(z) = z^2 = -q(z)e^{Q_{t-1}(z)}\mathcal{A}_1(z)$ *and* $L(z,f) = \mathcal{A}_1(z)e^{2z} = \sum_{i=0}^{5}\left(b_i\tilde{H}_1(z+c_i)e^{2c_i}\right)e^{2z} = ze^{2z}$.

2. Lemmas

We give the following well-known results which are important to prove our theorems.

Lemma 2.1 [5] *Let* f *be a nonconstant meromorphic function and* c_1, c_2 *be two complex numbers such that* $c_1 \neq c_2$. *Let* $f(z)$ *be a meromorphic function with finite order* ρ, *then for each* $\epsilon > 0$,

$$m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = S(r, f(z)).$$

Lemma 2.2 [8, Corollary 2.3.4] *Let* f *be a transcendental meromorphic function and* $k \geq 1$ *be an integer. Then* $m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = S(r, f(z))$.

Combining Lemmas 2.1 and 2.2 we have the following lemma:

Lemma 2.3 *Let* $f(z)$ *be a transcendental meromorphic function of finite order and let* $c \in \mathbb{C}$, $k \geq 1$ *be an integer. Then* $m\left(r, \frac{f^{(k)}(z+c)}{f(z)}\right) = S(r, f(z))$.

Proof

$$\begin{aligned} m\left(r, \frac{f^{(k)}(z+c)}{f(z)}\right) &= m\left(r, \frac{f^{(k)}(z+c)}{f(z+c)}.\frac{f(z+c)}{f(z)}\right) \\ &\leq m\left(r, \frac{f^{(k)}(z+c)}{f(z+c)}\right) + m\left(r, \frac{f(z+c)}{f(z)}\right) \\ &= S(r, f(z+c)) + S(r, f(z)) = S(r, f(z)). \end{aligned}$$

□

Lemma 2.4 [10] *Let* f *be a nonconstant meromorphic function of hyper order less than* 1 *and* $c \in \mathbb{C}$. *Then*

$$N(r, 1/f(z+c)) = N(r, 0; f(z)) + S(r, f).$$

Lemma 2.5 [19] *Suppose* $f_j(z)$ $(j = 1, 2, ..., n+1)$ *and* $g_k(z)$ $(k = 1, 2, ..., n)$ $(n \geq 1)$ *are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}(z)$,

(ii) The order of $f_j(z)$ *is less than the order of* $e^{g_k(z)}$ *for* $1 \leq j \leq n+1$, $1 \leq k \leq n$ *and furthermore, the order of* $f_j(z)$ *is less than the order of* $e^{g_h(z)-g_k(z)}$ *for* $n \geq 2$ *and* $1 \leq j \leq n+1$, $1 \leq h < k \leq n$. *Then* $f_j(z) \equiv 0$, $(j = 1, 2, ..., n+1)$.

Lemma 2.6 [2, 6] *Let f be a meromorphic function and suppose that*

$$R(z) = a_n f(z)^n + \cdots + a_0(z)$$

has small meromorphic coefficients $a_j(z)$, $a_n \not\equiv 0$ in the sense of $T(r, a_j) = S(r, f)$. Moreover, assume that

$$\overline{N}\left(r, \frac{1}{R}\right) + \overline{N}(r, f) = S(r, f).$$

Then

$$R(z) = a_n \left(f + \frac{a_{n-1}}{n a_n}\right).$$

The following lemma gives the Nevanlinna characteristic and counting functions of an exponential polynomial.

Lemma 2.7 [14] *Let $f(z)$ be given by (1.2). Then*

$$T(r, f) = C(co(W_0))\frac{r^t}{2\pi} + o(r^t).$$

If $H_0(z) \not\equiv 0$, then

$$m(r, 1/f) = o(r^t),$$

while if $H_0(z) \equiv 0$, then

$$N(r, 1/f) = C(co(W))\frac{r^t}{2\pi} + o(r^t).$$

Next, we proof the following lemmas which are the core parts of our paper.

Lemma 2.8 *Let f be given by (1.2) which is a solution of (1.8) for $n = 2$ and $\omega_i \neq 2\omega_j$. If the points $0, \omega_1, \omega_2, \ldots, \omega_m$ are collinear, then $m = 1$.*

Proof Assume on the contrary to the assertion that $m \geq 2$. Take $\omega_i = \xi_i \omega$, for each $i \in \{1, 2, \ldots, m\}$, where the constants $\xi_i \in \mathbb{R}\backslash\{0\}$ are distinct, $\xi_0 = 0$ and $\omega \in \mathbb{C}\backslash\{0\}$. Moreover, we may suppose that $\xi_i > \xi_j$ for $i > j$. Equation (1.8) can be written as

$$\begin{aligned}
&\sum_{i,j=0}^{m} H_i(z)H_j(z)e^{(\xi_i+\xi_j)\omega z^t} + a_1 \sum_{l=0}^{m} H_l(z)e^{\xi_l \omega z^t} \\
&\quad + q(z)e^{Q_{t-1}(z)}\left[\mathcal{A}_0(z)e^{v_t z^t} + \sum_{h=1}^{m} \mathcal{A}_h(z)e^{(v_t+\xi_h\omega)z^t}\right] = P(z), \qquad (2.1)
\end{aligned}$$

where $Q_{t-1}(z) = Q(z) - v_t z^t$ with $\deg Q_{t-1}(z) \leq t-1$ and $\mathcal{A}_0(z) = \sum_{i=0}^{k} b_i H_0^{(r_i)}(z + c_i)$, $\mathcal{A}_h(z) = \sum_{i=0}^{k} b_i \tilde{H}_h(z + c_i)e^{\omega_h(z+c_i)^t - \omega_h z^t}$, $h = 1, 2, \ldots, m$ such that $\tilde{H}_h(z + c_i)$ are the delay-differential polynomial of $H_h(z)$.
Now we consider following two cases to derive contradiction.

Case 1. Let $\xi_m > 0$. Note that $\max\{\xi_i + \xi_j : i, j = 0, 1, \ldots, m\} = 2\xi_m$. Since $L(z, f) \not\equiv 0$, then at least one of $\mathcal{A}_h(z)$, $h = 0, 1, \ldots, m$ is not vanishing.

Case 1.1. Let all $\mathcal{A}_h(z) = 0$, $h = 1, 2, \ldots, m$. Then $\mathcal{A}_0(z) \not\equiv 0$., i.e. $H_0(z) \not\equiv 0$. If $2\xi_m\omega \neq v_t$, applying *Lemma 2.5* on (2.1), we obtain $H_m^2(z) \equiv 0$, a contradiction. Next, let $2\xi_m\omega = v_t$. Since, $\omega_i \neq 2\omega_j$, applying *Lemma 2.5* on (2.1), we obtain $H_1^2(z) \equiv 0$, a contradiction.

Case 1.2. Let at least one of $\mathcal{A}_h(z) \neq 0$, for $h = 1, 2, \ldots, m$.

Case 1.2.1. Let $\mathcal{A}_0(z) \neq 0$. Then by *Lemma 2.5*, from (2.1), there exists one $h_0 \in \{0, 1, \ldots, m\}$ such that $2\xi_m\omega = v_t + \xi_{h_0}\omega$. Otherwise, we have $H_m^2(z) \equiv 0$, a contradiction.

Case 1.2.1.1. If $h_0 = m$, then we have $v_t = \xi_m\omega$. Since $2\xi_i \neq \xi_j$, $j = 0, 1, \ldots, m$ and $2\xi_1 \notin \{\xi_i + \xi_j : 0 \leq i, j \leq m, (i, j) \neq (1, 1)\}$ and $2\xi_1 \notin \{\xi_m + \xi_i : i = 0, 1, \ldots, m\}$. By *Lemma 2.5*, we obtain $H_1^2(z) \equiv 0$, a contradiction.

Case 1.2.1.2. If $h_0 \in \{0, 1, \ldots, m-1\}$, since $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m-1} < \xi_m$ and $2\xi_i \neq \xi_j$, $i, j = 0, 1, \ldots, m$, then for $m > h_0$,

$$2\xi_m - \xi_{h_0} + \xi_m > \max\{2\xi_m - \xi_{h_0} + \xi_i : i = 0, 1, \ldots, m-1\}.$$

Also, $2\xi_m = \max\{\xi_i + \xi_j : i, j = 0, 1, \ldots, m\}$. In view of *Lemma 2.5*, we obtain $q(z)e^{Q_{t-1}(z)}\mathcal{A}_m \equiv 0$, i.e. $q(z) \equiv 0$, a contradiction.

Case 1.2.2. Let $\mathcal{A}_0(z) = 0$ and $H_0(z) \neq 0$. Then (2.1) becomes

$$\sum_{i,j=0}^{m} H_i(z)H_j(z)e^{(\xi_i+\xi_j)\omega z^t} + a_1 \sum_{l=0}^{m} H_l(z)e^{\xi_l\omega z^t}$$
$$+q(z)e^{Q_{t-1}(z)} \sum_{h=1}^{m} \mathcal{A}_h(z)e^{(v_t+\xi_h\omega)z^t} = P(z). \tag{2.2}$$

Then similar as *Case 1.2.1*, by *Lemma 2.5*, from (2.2), we have there exists one $h_0 \in \{1, 2, \ldots, m\}$ such that $2\xi_m\omega = v_t + \xi_{h_0}\omega$ and proceeding similarly as done in *Case 1.2.1*, we can get a contradiction.

Case 1.2.3. Let $\mathcal{A}_0(z) = 0$ and $H_0(z) = 0$. Then (2.1) becomes

$$\sum_{i,j=1}^{m} H_i(z)H_j(z)e^{(\xi_i+\xi_j)\omega z^t} + a_1 \sum_{l=1}^{m} H_l(z)e^{\xi_l\omega z^t}$$
$$+q(z)e^{Q_{t-1}(z)} \sum_{h=1}^{m} \mathcal{A}_h(z)e^{(v_t+\xi_h\omega)z^t} = P(z). \tag{2.3}$$

Next, similar as *Case 1.2.1*, by *Lemma 2.5*, from (2.3), we have there exists one $h_0 \in \{1, 2, \ldots, m\}$ such that $2\xi_m\omega = v_t + \xi_{h_0}\omega$ and adopting the same method as done in *Case 1.2.1*, we get a contradiction.

Case 2. $\xi_m < 0$. Note that $\min\{\xi_i + \xi_j : i, j = 0, 1, \ldots, m\} = 2\xi_1$. Similar as *Case 1*, we divide the following cases.

Case 2.1. Let all $\mathcal{A}_h(z) = 0$, $h = 1, 2, \ldots, m$. Then $\mathcal{A}_0(z) \not\equiv 0$., i.e. $H_0(z) \not\equiv 0$. If $2\xi_1\omega \neq v_t$, applying *Lemma 2.5* on (2.1), we obtain $H_1^2(z) \equiv 0$, a contradiction. Next, let $2\xi_1\omega = v_t$. Since, $\omega_i \neq 2\omega_j$, applying *Lemma 2.5* on (2.1), we obtain $H_m^2(z) \equiv 0$, a contradiction.

Case 2.2. Let at least one of $\mathcal{A}_h(z) \neq 0$ for $h = 1, 2, \ldots, m$.
Case 2.2.1. Let $\mathcal{A}_0(z) \neq 0$. Then by *Lemma 2.5*, there exists one $h_0 \in \{0, 1, \ldots, m\}$ such that $2\xi_1\omega = v_t + \xi_{h_0}\omega$. Otherwise, we have $H_1^2(z) \equiv 0$, a contradiction.
Case 2.2.1.1. If $h_0 = 1$, then we have $v_t = \xi_1\omega$.
Since, $2\xi_i \neq \xi_j$ and $2\xi_m \notin \{\xi_i + \xi_j : 0 \leq i, j \leq m, (i, j) \neq (m, m)\}$ and $2\xi_m \notin \{\xi_1 + \xi_i : i = 0, 2, \ldots, m\}$. By *Lemma 2.5*, we obtain $H_m^2(z) \equiv 0$, a contradiction.
Case 2.2.1.2. If $h_0 \in \{0, 2, 3, \ldots, m\}$, since $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m-1} < \xi_m$ and $2\xi_i \neq \xi_j$, $i, j = 0, 1, \ldots, m$, then

$$2\xi_1 - \xi_{h_0} + \xi_1 < \min\{2\xi_1 - \xi_{h_0} + \xi_i : i = 0, 2, 3, \ldots, m\}.$$

Also, $\min\{\xi_i + \xi_j : i, j = 0, 1, \ldots, m\} = 2\xi_1$. By *Lemma 2.5*, we obtain $q(z)e^{Q_{t-1}}\mathcal{A}_1(z) \equiv 0$, i.e. $q(z) \equiv 0$, a contradiction.
Case 2.2.2. Let $\mathcal{A}_0(z) = 0$ and $H_0(z) \neq 0$. Then in this case, we get Equation (2.2). Similar as *Case 2.2.1*, by *Lemma 2.5*, there exists one $h_0 \in \{2, 3, \ldots, m\}$ such that $2\xi_1\omega = v_t + \xi_{h_0}\omega$ and proceeding similarly as adopted in *Case 2.2.1*, we can get a contradiction.
Case 2.2.3. Let $\mathcal{A}_0(z) = 0$ and $H_0(z) = 0$. Then, we have Equation (2.3). Similar as *Case 2.2.1*, by *Lemma 2.5*, there exists one $h_0 \in \{2, 3, \ldots, m\}$ such that $2\xi_1\omega = v_t + \xi_{h_0}\omega$. Next, adopting the same method as executing in *Case 2.2.1*, we get a contradiction. □

Lemma 2.9 *If $m \geq 2$ and $\omega_i \neq 2\omega_j$ for any $i \neq j$, then f of the form (1.2) is not a solution of (1.8) for $n = 2$.*

Proof Suppose on the contrary to the assertion that, $m \geq 2$. Substituting f of the form (1.2) into (1.8), we get

$$\begin{aligned} F(z) &= f^2(z) + a_1 f(z) - P(z) \\ &= G(z) + \sum_{\substack{i,\, j\, =\, 0 \\ \omega_i + \omega_j \neq 0}}^{m} H_i(z)H_j(z)e^{(\omega_i+\omega_j)z^t} + a_1 \sum_{l=1}^{m} H_l(z)e^{\omega_l z^t}, \end{aligned} \tag{2.4}$$

where $G(z) = H_0(z)(H_0(z) + a_1) - P(z)$ is either an exponential polynomial of degree $< t$ or a polynomial in z.
Therefore, also

$$F(z) = -q(z)e^{Q(z)}L(z, f) = -q(z)e^{Q(z)} \sum_{h=0}^{m} \mathcal{A}_h(z)e^{\omega_h z^t} \tag{2.5}$$

such that $\mathcal{A}_h(z)$ is defined as in(2.1).
Now we set

$$\begin{aligned} X_1 &= \{\overline{\omega}_1, \ldots, \overline{\omega}_m, \overline{\omega}_i + \overline{\omega}_j : \overline{\omega}_i + \overline{\omega}_j \neq 0, i, j = 1, \ldots, m\}, \\ X_2 &= \{\overline{\omega}_1, \ldots, \overline{\omega}_m, 2\overline{\omega}_1, \ldots, 2\overline{\omega}_m\}, \\ X_3 &= \{2\overline{\omega}_1, \ldots, 2\overline{\omega}_m\}. \end{aligned}$$

Clearly, by the theory of convexity, we have $\overline{\omega}_i + \overline{\omega}_j = \frac{1}{2} \cdot 2\overline{\omega}_i + (1-\frac{1}{2}) \cdot 2\overline{\omega}_j$, i.e. $co(X_1) = co(X_2)$. Since, $X_3 \subset X_2$, we have $co(X_3) \leq co(X_2)$, respectively.
Next, we consider the following cases to show a contradiction.

Case 1. If all $\mathcal{A}_h(z) = 0$ for $h = 1, \ldots, m$, then we have $\mathcal{A}_0(z) \neq 0$, which implies $H_0(z) \neq 0$. Then (2.5) becomes $F(z) = -q(z)e^{Q(z)}\mathcal{A}_0(z)$. Then applying *Lemma 2.7*, we get

$$N\left(r, \frac{1}{F(z)}\right) = N\left(r, \frac{1}{\mathcal{A}_0(z)}\right) = o(r^t). \tag{2.6}$$

Subcase 1.1. Let $G(z) \equiv 0$. Applying *Lemma 2.7* on (2.4), we have

$$N\left(r, \frac{1}{F(z)}\right) = C(co(X_1))\frac{r^t}{2\pi} + o(r^t). \tag{2.7}$$

Therefore, (2.6) and (2.7) yields a contradiction.
Subcase 1.2. Let $G(z) \not\equiv 0$. Applying *Lemma 2.7* on (2.4), we have $m\left(r, \frac{1}{F(z)}\right) = o(r^t)$ and then

$$\begin{aligned} m\left(r, \frac{1}{F(z)}\right) + N\left(r, \frac{1}{F(z)}\right) &= T(r, F(z)) + O(1) = 2T(r, f(z)) + S(r, f) \\ &= 2\left(C(co(W_0))\frac{r^t}{2\pi} + o(r^t)\right) + S(r, f) \\ \Longrightarrow N\left(r, \frac{1}{F(z)}\right) &= 2C(co(W_0))\frac{r^t}{2\pi} + o(r^t). \end{aligned} \tag{2.8}$$

Therefore, using (2.6) and (2.8), we get a contradiction.

Case 2. Let there exists some $h_0 \in \{1, 2, \ldots, m\}$ such that $\mathcal{A}_{h_0}(z) \neq 0$. Now, we denote the following set as

$$V = \{\overline{\omega}_{h_0} : h_0 \in \{1, 2, \ldots, m\} \text{ for which } \mathcal{A}_{h_0}(z) \neq 0\} \text{ and } V_0 = V \cup \{0\}.$$

Since, $V \subseteq W$ and $V_0 \subseteq W_0$, then $C(co(V)) \leq C(co(W))$ and $C(co(V_0)) \leq C(co(W_0))$, respectively.
Case 2.1. Let $\mathcal{A}_0(z) \equiv 0$ and $H_0(z) \equiv 0$. Then using *Lemma 2.7* on (2.5), we have

$$N\left(r, \frac{1}{F(z)}\right) = N\left(r, \frac{1}{L(z, f)}\right) + O(\log r) = C(co(V))\frac{r^t}{2\pi} + o(r^t). \tag{2.9}$$

Case 2.1.1. If $G(z) \not\equiv 0$. Then similar as *Subcase 1.2*, we have Equation (2.8). From (2.8) and (2.9), we obtain a contradiction by $C(co(W_0)) \geq C(co(V_0)) \geq C(co(V)) = 2C(co(W_0))$.
Case 2.1.2. If $G(z) \equiv 0$, using *Lemma 2.7* on (2.4), we have Equation (2.7). Using (2.7) and (2.9), from $C(co(W)) \geq C(co(V)) = C(co(X_1)) = C(co(X_2)) \geq C(co(X_3)) = 2C(co(W))$, we get a contradiction.
Case 2.2. Let $\mathcal{A}_0(z) \equiv 0$ and $H_0(z) \not\equiv 0$. Then proceeding similarly as done in *Case 2.1*, we get a contradiction.
Case 2.3. Let $\mathcal{A}_0(z) \neq 0$, which implies $H_0(z) \not\equiv 0$. Then using *Lemma 2.7* on (2.5), we have $m\left(r, \frac{1}{L(z,f)}\right) =$

$o(r^t)$ and then

$$\begin{aligned} N\left(r, \frac{1}{F(z)}\right) &= N\left(r, \frac{1}{L(z,f)}\right) + O(\log r) \\ &= T(r, L(z,f)) + o(r^t) = C(co(V_0))\frac{r^t}{2\pi} + o(r^t). \end{aligned} \tag{2.10}$$

<u>Case 2.3.1.</u> If $G(z) \not\equiv 0$. Then, from (2.8) and (2.10), we get $C(co(W_0)) \geq C(co(V_0)) = 2C(co(W_0))$, a contradiction.

<u>Case 2.3.2.</u> If $G(z) \equiv 0$. Using (2.7) and (2.10), we get

$$C(co(V_0)) = C(co(X_1)). \tag{2.11}$$

Now, since $m \geq 2$, by *Lemma 2.8*, $co(W_0)$ cannot be a line-segment. Therefore, $co(W_0)$ must be a polygon with nonempty interior. If 0 is not a boundary point of $co(W_0)$, then we have $co(W_0) = co(W)$. Then we have $C(co(W)) = C(co(W_0)) \geq C(co(V_0)) = C(co(X_1)) = C(co(X_2)) \geq C(co(X_3)) = 2C(co(W))$, a contradiction. So, 0 is a boundary point of $co(W_0)$. We choose the other nonzero corner points of $co(W_0)$ among the points $\overline{\omega}_1, \ldots, \overline{\omega}_m$ are $u_1, \ldots, u_t$, $t \leq m$ such that $0 \leq \arg(u_i) \leq \arg(u_{i+1}) \leq 2\pi$ for $1 \leq i \leq t-1$. Hence,

$$C(co(W_0)) = |u_1| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + |u_t|. \tag{2.12}$$

Let $X_4 = \{u_1, 2u_1, 2u_2, \ldots, 2u_t, u_t\}$. Therefore, the points $2u_1, 2u_2, \ldots, 2u_t$ are the corner points of $co(X_4)$. However, since $t \leq m$, $co(X_4)$ may have more corner points. Then, using (2.12), we have

$$\begin{aligned} C(co(X_3)) > C(co(X_4)) &> |2u_1 - u_1| + |2u_2 - 2u_1| + \cdots + |2u_t - 2u_{t-1}| + |u_t - 2u_t| \\ &= |u_1| + 2|u_2 - u_1| + \cdots + 2|u_t - u_{t-1}| + |u_t| \\ &> |u_1| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + |u_t| \\ &= C(co(W_0)). \end{aligned}$$

Therefore,

$$C(co(X_1)) = C(co(X_2)) \geq C(co(X_3)) > C(co(W_0)) \geq C(co(V_0)),$$

which contradicts (2.11).

Hence, the proof is completed. □

Lemma 2.10 *Let f be given by (1.2), which is a solution of (1.8) for $n = 2$, then f takes the form,*

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z^t},$$

i.e. $f \in \Gamma_1'$. In this case,

(I) either $t = 1$, $\rho(f) = 1$ and $H_0(z)$, $H_1(z)$ are polynomials and $Q(z)$ is a polynomial of degree 1

(II) or $H_0(z) = -\frac{a_1}{2}$, $P(z) = -\frac{a_1^2}{4}$, $H_1^2(z) = \frac{b_0 a_1}{2} q(z) e^{Q_{t-1}(z)}$ and $L(z,f) = b_0 H_0(z)$

(III) or $H_0(z) = -\frac{a_1}{2}$, $P(z) = -\frac{a_1^2}{4}$, $H_1^2(z) = -q(z)e^{Q_{t-1}(z)}\mathcal{A}_1(z)$ and $L(z,f) = \mathcal{A}_1(z)e^{\omega_1 z^t}$, where $\mathcal{A}_1(z) = \sum_{i=0}^{k} b_i \tilde{H}_1(z+c_i)e^{\omega_1(z+c_i)^t - \omega_1 z^t}$ such that $\tilde{H}_1(z+c_i)$ are the delay-differential polynomial of $H_1(z)$.

Proof For $n=2$, (1.8) becomes

$$f^2(z)+a_1f(z)+q(z)e^{Q(z)}L(z,f)=P(z). \tag{2.13}$$

By *Lemma 2.9*, we have $m=1$, i.e. (1.2) becomes

$$f(z)=H_0(z)+H_1(z)e^{\omega_1 z^t}, \tag{2.14}$$

where $H_0(z)$, $H_1(z)(\not\equiv 0)$ are either exponential polynomials of order $<t$ or ordinary polynomials in z. Substituting (2.14) in (2.13), we have

$$\begin{aligned}&H_1(z)(2H_0(z)+a_1)e^{\omega_1 z^t}+H_1^2(z)e^{2\omega_1 z^t}+q(z)e^{Q_{t-1}(z)}\mathcal{A}_0(z)e^{v_t z^t}\\ &+q(z)e^{Q_{t-1}(z)}\mathcal{A}_1(z)e^{(v_t+\omega_1)z^t}=P(z)-H_0(z)(H_0(z)+a_1),\end{aligned} \tag{2.15}$$

where $Q_{t-1}(z)=Q(z)-v_tz^t$ with $\deg Q_{t-1}(z)\leq q-1$ and $\mathcal{A}_0(z)=\sum_{i=0}^{k} b_iH_0^{(r_i)}(z+c_i)$, $\mathcal{A}_1(z)=\sum_{i=0}^{k} b_i\tilde{H}_1(z+c_i)e^{\omega_1(z+c_i)^t-\omega_1 z^t}$ such that $\tilde{H}_h(z+c_i)$ are the delay-differential polynomial of $H_h(z)$ for $h=1,2$. Since, $L(z,f)\not\equiv 0$, then at least one of $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$ is nonvanishing. Next, we divide the following cases to prove our result.

Case 1. Let $\mathcal{A}_1(z)\equiv 0$. Then $\mathcal{A}_0(z)\not\equiv 0$, which implies $H_0(z)\not\equiv 0$.
If $v_t\neq\omega_1,2\omega_1$ or $v_t=\omega_1$, applying *Lemma 2.5* on (2.15), we have $H_1^2(z)\equiv 0$, a contradiction.
If $v_t=2\omega_1$, then, applying *Lemma 2.5* on (2.15), we have

$$H_1(2H_0(z)+a_1)=0,$$

$$H_1^2(z)+q(z)e^{Q_{t-1}(z)}\mathcal{A}_0(z)=0,$$

$$P(z)-H_0(z)(H_0(z)+a_1)=0.$$

Since, $H_1(z)\not\equiv 0$. Therefore, solving these three equations, we have $H_0(z)=-\frac{a_1}{2}$, $P(z)=-\frac{a_1^2}{4}$ and $H_1^2(z)=\frac{b_0a_1}{2}q(z)e^{Q_{t-1}}$. Therefore, in this case $L(z,f)=b_0H_0(z)$.

Case 2. Let $\mathcal{A}_0(z)\equiv 0$. Then $\mathcal{A}_1(z)\not\equiv 0$.

Subcase 2.1. Let $H_0(z)\equiv 0$. If $v_t=\pm\omega_1$, using *Lemma 2.5* on (2.15), we have $H_1\equiv 0$, a contradiction.

Subcase 2.2. Let $H_0(z)\not\equiv 0$. If $v_t=-\omega_1$, in view of *Lemma 2.5*, from on (2.15), we have $H_1\equiv 0$, a contradiction. If $v_t=\omega_1$, similar as *Case 1*, we have $H_0(z)=-\frac{a_1}{2}$, $P(z)=-\frac{a_1^2}{4}$ and $H_1^2(z)=-q(z)e^{Q_{t-1}}\mathcal{A}_1$. Also, in this case $L(z,f)=\mathcal{A}_1e^{\omega_1 z^t}$.

Case 3. Let $\mathcal{A}_0(z)\not\equiv 0$ and $\mathcal{A}_1(z)\not\equiv 0$, which implies $H_0(z)\not\equiv 0$.
If $v_t=-\omega_1$ or $v_t\neq\pm\omega_1$, using *Lemma 2.5* on (2.15), we have $H_1\equiv 0$, a contradiction. If $v_t=2\omega_1$, by *Lemma 2.5*, from (2.15), we have $\mathcal{A}_1\equiv 0$, a contradiction.
If $v_t=\omega_1$, applying *Lemma 2.5* on (2.15), we have

$$H_1(z)(2H_0(z)+a_1)+q(z)e^{Q_{t-1}(z)}\sum_{i=0}^{k} b_iH_0^{(r_i)}(z+c_i)=0, \tag{2.16}$$

$$H_1^2(z)+q(z)e^{Q_{t-1}(z)}\sum_{i=0}^{k} b_i\tilde{H}_1(z+c_i)e^{\omega_1(z+c_i)^t-\omega_1 z^t}=0, \tag{2.17}$$

$$P(z)-H_0(z)(H_0(z)+a_1)=0. \tag{2.18}$$

Now, we show that $H_0(z)$ is a polynomial. If possible let, $H_0(z)$ is transcendental. Then from (2.18), we have

$$2T(r,H_0(z))+S(r,H_0(z))=T(r,P)=O(\log r),$$

a contradiction.
Next, from (2.16), we have

$$H_1(z)=\beta(z)e^{Q_{t-1}(z)}, \tag{2.19}$$

where $\beta(z)=-\frac{q(z)\sum_{i=0}^{k} b_i H_0^{(r_i)}(z+c_i)}{2H_0(z)+a_1}$. Since, f is entire, then $\beta(z)$ is a polynomial. Substituting (2.19) in (2.17), we have

$$\beta^2(z)+q(z)\sum_{i=0}^{k} b_i\tilde{\beta}(z+c_i)e^{Q_{t-1}(z+c_i)-Q_{t-1}(z)+\omega_1(z+c_i)^t-\omega_1 z^t}=0, \tag{2.20}$$

where $\tilde{\beta}(z+c_i)$ is a delay-differential polynomial in $H_0(z)$ and $Q_{t-1}(z)$.
Note that $\deg(\omega_1(z+c_i)^t-\omega_1 z^t)=t-1$ and $\deg(Q_{t-1}(z+c_i)-Q_{t-1}(z))\leq t-2$. If $t\geq 2$, applying *Lemma 2.5* on (2.20), we have $q(z)=0$, a contradiction. Therefore, $t=1$, i.e. $H_1(z)$ is a polynomial. Hence, $f(z)$ reduces to the form

$$f(z)=H_0(z)+H_1(z)e^{\omega_1 z},$$

where $H_0(z)$ and $H_1(z)$ are polynomials. So, $f\in\Gamma_1'$. □

3. Proof of the Theorem 1.6 (i)

Suppose that f be a finite order nonvanishing entire solution of (1.8). Using *Lemma 2.5*, we have f is transcendental. Otherwise, we will get $L(z,f)\equiv 0$, which yields a contradiction. In view of *Lemma 2.3*, from (1.8), we get

$$\begin{aligned}
n\,T(r,f)+S(r,f) &= m\left(r, f^n(z)+\sum_{i=1}^{n-1} a_i f^i(z)\right)\\
&= m(r,P(z))-q(z)e^{Q(z)}L(z,f))\\
&= m(r,e^{Q(z)})+m(r,L(z,f))+O(1)\\
&= m(r,e^{Q(z)})+m\left(r,\frac{L(z,f)}{f(z)}\right)+m(r,f(z))+O(1)\\
&= T(r,e^{Q(z)})+T(r,f(z))+S(r,f)\\
\Longrightarrow (n-1)\,T(r,f) &\leq T(r,e^{Q(z)})+S(r,f). \qquad (3.1)
\end{aligned}$$

Therefore, for $n \geq 2$, $\rho(f) \leq \deg Q(z)$. If $\rho(f) < \deg Q(z)$, then comparing order of growth of (1.8), we get a contradiction. So, $\rho(f) = \deg Q(z)$. Now, from the definition of type, we have

$$\tau(f) = \overline{\lim}_{r\to\infty} \frac{T(r,f)}{r^{\rho(f)}} = \overline{\lim}_{r\to\infty} \frac{T(r,f)}{r^{\deg Q(z)}} \in (0,\infty),$$

i.e. f is of mean type.

4. Proof of the Theorem 1.6 (ii)

First, we prove that if zero is a Borel exceptional value of $f(z)$, then we have $a_{n-1} = \cdots = a_1 = 0 \equiv P(z)$. Adopting the similar process as done in the proof of Theorem 1.2(b) in [10], using *Lemmas 2.3, 2.4, 2.6* and replacing $f(z+c)$ by $L(z,f)$, we can prove our result. In this regards, only the equation (10) of [10] is replaced by the following lines

$$\begin{aligned} N\left(r, \frac{1}{G(z)}\right) &= N\left(r, \frac{1}{q(z)e^{Q(z)}L(z,f)}\right) \\ &\leq N\left(r, \frac{1}{q(z)}\right) + N\left(r, \frac{1}{L(z,f)}\right) + S(r,f) \\ &\leq (k+1)\, N\left(r, \frac{1}{f(z)}\right) + S(r,f) = S(r,f). \end{aligned}$$

Next, we prove the converse part of *Theorem 1.6 (ii)*. For this, using *Lemmas 2.3, 2.6* and replacing $f(z+c)$ by $L(z,f)$, we proceed similar up to equation (17) in the proof of Theorem 1.2(c) in [10]. Here, the equations (15), (16) and (17) of [10], respectively, will be

$$T\left(r, \frac{L(z,f)}{f(z)}\right) = S(r,f), \tag{4.1}$$

$$\left(f(z) + \frac{a_{n-1}}{n-1}\right)^{n-1} = f^{n-1}(z) + \sum_{i=1}^{n-1} a_i f^{i-1}(z) = -q(z)e^{Q(z)}\frac{L(z,f)}{f(z)} \tag{4.2}$$

and

$$a_{i+1} = \frac{(n-1)!}{i!(n-1-i)!}\left(\frac{a_{n-1}}{n-1}\right)^{n-1-i}, \quad i = 0,1,...,n-2. \tag{4.3}$$

<u>**Case 1.**</u> If there exists $i_0 \in \{1,\ldots,n-1\}$ such that $a_{i_0} = 0$, then from (4.3), we have all a_i must be equal to zero for $i = 1,2,\ldots,n-1$. Therefore, (4.2) becomes

$$f(z)^{n-1} = -q(z)e^{Q(z)}\frac{L(z,f)}{f(z)}.$$

By using (4.1), for each $\epsilon > 0$, we have

$$
\begin{aligned}
(n-1)N\left(r, \frac{1}{f(z)}\right) &= N\left(r, \frac{1}{q(z)\frac{L(z,f)}{f(z)}}\right) \\
&\leq N\left(r, \frac{1}{q(z)}\right) + N\left(r, \frac{1}{\frac{L(z,f)}{f(z)}}\right) + S(r,f) = S(r,f).
\end{aligned}
$$

Therefore, $\lambda(f) < \rho(f)$.

<u>Case 2.</u> If there exists no $i_0 \in \{1, \ldots, n-1\}$ such that $a_{i_0} = 0$, then from (4.1) and (4.2), we have

$$
\overline{N}\left(r, \frac{1}{f(z) + \frac{a_{n-1}}{n-1}}\right) \leq \overline{N}\left(r, \frac{1}{q(z)}\right) + \overline{N}\left(r, \frac{1}{\frac{L(z,f)}{f(z)}}\right) + S(r,f) = S(r,f).
$$

Using the second main theorem, we have

$$
\begin{aligned}
T(r,f) &\leq \overline{N}\left(r, \frac{1}{f(z) + \frac{a_{n-1}}{n-1}}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, f(z)\right) \\
&= \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r,f).
\end{aligned}
$$

Therefore, $\rho(f) \leq \lambda(f)$ but we know that $\lambda(f) \leq \rho(f)$. Therefore, $\lambda(f) = \rho(f)$.

5. Proof of the Theorem 1.6 (iii)

Suppose that f be a nonvanishing finite order entire solution of (1.8). Similar as *Theorem 1.6 (i)*, f is transcendental.

First suppose that f belongs to Γ_0', which means that 0 is a Borel exceptional value of f. Thus, from *Theorem 1.6 (ii)*, we have $a_{n-1} = \cdots = a_1 = 0 \equiv P(z)$.

Next, we suppose that $P(z) \equiv 0$ and there exists an $i_0 \in \{1, \ldots, n-1\}$ such that $a_{i_0} = 0$, then from the converse part of *Theorem 1.6 (ii)*, we have all of the $a_i (i = 1, \ldots, n-1)$ must be zero as well and $\lambda(f) < \rho(f)$. From Hadamard factorization theorem, we can see that

$$
f(z) = h(z)e^{\alpha(z)}, \tag{5.1}
$$

where $\alpha(z)$ is a polynomial and $h(z)$ is the canonical product of zeros of f with $\deg \alpha(z) = \rho(f) = \deg Q(z) = t$ and $\rho(h) = \lambda(h) = \lambda(f) < \rho(f)$.

Substituting (5.1) in (1.8) with all $a_i = 0$, we have

$$
h^n(z)e^{n\alpha(z)} + q(z)e^{Q(z)+\alpha(z)}\left(\sum_{i=0}^{k} L_i(z,h)e^{\Delta_{c_i}\alpha(z)}\right) = 0, \tag{5.2}
$$

where

$$\begin{aligned}L_i(z,h) \quad = \quad & b_i \Big[h(z+c_i)M_{k_i}(\alpha'(z+c_i),\alpha''(z+c_i),\ldots,\alpha^{(k_i)}(z+c_i)) \\ & +h'(z+c_i)M_{k_i-1}(\alpha'(z+c_i),\alpha''(z+c_i),\ldots,\alpha^{(k_i-1)}(z+c_i)) \\ & +\cdots+h^{(k_i-1)}(z+c_i)M_1(\alpha'(z+c_i))+h^{(k_i)}(z+c_i)\Big].\end{aligned}$$

Clearly, $\rho(L_i(z,h)) < t$. Rewriting (5.2), we have

$$h^n(z)e^{n\alpha_{t-1}(z)}e^{nu_tz^t}+q(z)e^{\alpha_{t-1}(z)+Q_{t-1}(z)}\left(\sum_{i=0}^{k}L_i(z,h)e^{\Delta_{c_i}\alpha(z)}\right)e^{(u_t+v_t)z^t}=0 \tag{5.3}$$

such that $\alpha(z)=u_tz^t+\alpha_{t-1}(z)$ and $Q(z)=v_tz^t+Q_{t-1}(z)$, where u_t, v_t are nonzero constants and $\alpha_{t-1}(z)$, $Q_{t-1}(z)$ are of degree $\leq t-1$.
In view of *Lemma 2.5*, we can easily say that (5.3) is possible only when $(n-1)u_t=v_t$. Therefore, (5.3) becomes

$$h^n(z)+q(z)e^{(1-n)\alpha_{t-1}(z)+Q_{t-1}(z)}\left(\sum_{i=0}^{k}L_i(z,h)e^{\Delta_{c_i}\alpha(z)}\right)=0. \tag{5.4}$$

Here, the following cases arise.

Case 1: Let $\rho(h) < t-1 > 0$. If $\deg\{(1-n)\alpha_{t-1}(z)+Q_{t-1}(z)\}=t-1$, applying *Lemma 2.5*, we have $q(z)=0$, a contradiction. If $\deg\{(1-n)\alpha_{t-1}(z)+Q_{t-1}(z)\} < t-1$, from $\deg\{\Delta_{c_i}\alpha(z)\}=t-1$, by using *Lemma 2.5*, again we have $q(z)=0$, a contradiction.

Case 2: Let $\rho(h)\geq t-1 > \rho(h)-1$, $t-1>0$. By logarithmic derivative lemma [5, Corollary 2.5], for each $\epsilon>0$, we have

$$m\left(r,\frac{L_i(z,h)}{h(z)}\right)=O(r^{\rho(h)-1+\epsilon})+O(\log r). \tag{5.5}$$

Since, h is entire, using (5.4) and (5.5), we have

$$\begin{aligned}&T\left(r,\sum_{i=0}^{k}\frac{L_i(z,h)}{h}e^{\Delta_{c_i}\alpha(z)}\right)\\ &\quad=m\left(r,\sum_{i=0}^{k}\frac{L_i(z,h)}{h}e^{\Delta_{c_i}\alpha(z)}\right)+N\left(r,\sum_{i=0}^{k}\frac{L_i(z,h)}{h}e^{\Delta_{c_i}\alpha(z)}\right)\\ &\quad\leq\sum_{i=0}^{k}T\left(r,e^{\Delta_{c_i}\alpha(z)}\right)+O(r^{\rho(h)-1+\epsilon})+O(\log r).\end{aligned} \tag{5.6}$$

Therefore, using (5.4) and (5.6), we obtain for each $\epsilon > 0$

$$\begin{aligned}(n-1)N\left(r,\frac{1}{h(z)}\right) &\leq N\left(r,\frac{1}{\sum_{i=0}^{k}\frac{L_i(z,h)}{h}e^{\Delta_{c_i}\alpha(z)}}\right)+O(\log r)\\ &\leq T\left(r,\sum_{i=0}^{k}\frac{L_i(z,h)}{h}e^{\Delta_{c_i}\alpha(z)}\right)+O(\log r)\\ &\leq \sum_{i=0}^{k}T\left(r,e^{\Delta_{c_i}\alpha(z)}\right)+O(r^{\rho(h)-1+\epsilon})+O(\log r). \qquad (5.7)\end{aligned}$$

Now, if $c_i = c_j$ for all $1 \leq i, j \leq k$, say, c, then (5.7) becomes

$$(n-1)N\left(r,\frac{1}{h(z)}\right) \leq T\left(r,e^{\Delta_c\alpha(z)}\right)+O(r^{\rho(h)-1+\epsilon})+O(\log r).$$

Thus, from the above equation, we have $\lambda(h) \leq t-1$. But in this case, $\lambda(h) = \rho(h) \geq t-1$. Therefore, $\lambda(f) = \lambda(h) = t-1 = \rho(f)-1$.

If $t-1=0$, we get $\rho(h) = \lambda(h) < \rho(f) = t = 1$. If $h(z)$ is transcendental, then $h(z)$ has infinitely many zeros. Now noting that $\Delta_{c_i}\alpha(z)$ is of degree $t-1$, from (5.7), we get $N\left(r,\frac{1}{h(z)}\right) = O(\log r)$, a contradiction. Therefore, $h(z)$ will be a polynomial. So, f belongs to Γ'_0.

<u>Case 3:</u> Let $\rho(h) \geq t$. Then from (5.7), we obtain $\lambda(h) < \rho(h)$, a contradiction. So, $h(z)$ will be a polynomial. Therefore, f belongs to Γ'_0.

6. Proof of the Theorem 1.6 (iv)

Suppose that f is a nonvanishing finite order entire solution of (1.8). Similarly, f is transcendental. If possible, let $P(z) \not\equiv 0$. By the assumption, $card\{z : p(z) = p'(z) = p''(z) = 0\} \geq 1$ or $card\{z : p(z) = p'(z) = 0\} \geq 2$, where $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z$, we mean that $p(z)$ has at least one zero with multiplicity at least three or at least 2 zeros with multiplicities at least two. In view of *Lemma 2.3* and the second main theorem, we have

$$\begin{aligned}
n\,T(r,f) &= T(r,f^n(z)+a_{n-1}f^{n-1}(z)+\cdots+a_1f(z))\\
&\leq \overline{N}\left(r,\frac{1}{f^n+a_{n-1}f^{n-1}+\cdots+a_1f-P(z)}\right)\\
&\quad +\overline{N}\left(r,\frac{1}{f^n+a_{n-1}f^{n-1}+\cdots+a_1f}\right)+\overline{N}(r,f^n+a_{n-1}f^{n-1}+\cdots+a_1f)\\
&\leq \overline{N}\left(r,\frac{1}{q(z)L(z,f)}\right)+(n-2)\,T(r,f)+S(r,f)\\
&\leq T\left(r,q(z)L(z,f)\right)+(n-2)\,T(r,f)+S(r,f)\\
&\leq N\left(r,q(z)L(z,f)\right)+m\left(r,q(z)L(z,f)\right)+(n-2)\,T(r,f)+S(r,f)\\
&\leq m\left(r,q(z)\frac{L(z,f)}{f(z)}\right)+m(r,f(z))+(n-2)\,T(r,f)+S(r,f)\\
&\leq T(r,f)+(n-2)\,T(r,f)+S(r,f)\\
&\leq (n-1)\,T(r,f)+S(r,f),
\end{aligned}$$

a contradiction. Therefore, $P(z)\equiv 0$.

Since, at least one $a_{i_0}=0$ $(i_0=1,2,\ldots,n-1)$, using *Theorem 1.6 (iii)*, we have $f\in\Gamma_0'$. Note that, since, $P(z)\equiv 0$ and at least one $a_{i_0}=0$ $(i_0=1,2,\ldots,n-1)$, from *Theorem 1.6 (ii)*, we have all of a_j's $(j=1,\ldots,n-1)$ must be zero. Therefore $p(z)$ is of the form z^n, which implies $card\{z:p(z)=p'(z)=0\}\geq 2$ is not possible.

7. Proof of the Theorem 1.6 (v)

From *Theorem 1.6 (i)*, we have $\rho(f)=\deg(Q(z))$. Using *Lemmas 2.8–2.10*, we can prove the result.

8. Concluding remarks

From the paper we know that, taking the notion of convexity in background, the key idea was to use the value distribution theory of exponential polynomials introduced by Steinmetz [14], that makes this direction of research more interesting. Particularly, it was shown that any exponential polynomial solution must reduce to a specific form.

In view of the discussions in the paper, the existence of solutions of another equation namely

$$f^n(z)+\sum_{i=1}^{n-1}a_if^i(z)+\sum_{j=1}^{n-1}q_j(z)e^{Q_j(z)}L_j(z,f)=P(z),$$

where $n\geq 2$, $L_i(z,f)$'s are defined as in (1.7), would also be of highly interesting.

Further, in this regard, the matter of concern will be weather it is possible to find out the conclusions *(i)–(v)* in *Theorem 1.6*. In that case, the value of m in (1.2) as a solution f of the above equation also deserves attention. The possibility to find out any other conclusions under some new conditions might also be explored.

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