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On minimal absolutely pure domain of RD-flat modules

Yusuf ALAGÖZ*
Department of Mathematics, Faculty of Science, Siirt University, Siirt, Turkey

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Abstract: Given modules A_R and ${}_RB$, ${}_RB$ is called absolutely A_R-pure if for every extension ${}_RC$ of ${}_RB$, $A \otimes B \to A \otimes C$ is a monomorphism. The class $\underline{\mathfrak{Fl}}^{-1}(A_R) = \{{}_RB : {}_RB$ is absolutely A_R-pure$\}$ is called the absolutely pure domain of a module A_R. If ${}_RB$ is divisible, then all short exact sequences starting with B is RD-pure, whence B is absolutey A-pure for every RD-flat module A_R. Thus the class of divisible modules is the smallest possible absolutely pure domain of an RD-flat module. In this paper, we consider RD-flat modules whose absolutely pure domains contain only divisible modules, and we referred to these RD-flat modules as rd-indigent. Properties of absolutely pure domains of RD-flat modules and of rd-indigent modules are studied. We prove that every ring has an rd-indigent module, and characterize rd-indigent abelian groups. Furthermore, over (commutative) SRDP rings, we give some characterizations of the rings whose nonprojective simple modules are rd-indigent.

Key words: RD-flat modules, absolutely pure domains, rd-indigent modules, QF-rings

1. Introduction and preliminaries

Throughout, R will denote an associative ring with identity, and modules will be unital right R-modules, unless otherwise stated. For any ring R, we denote by R-Mod, the category of left R-modules. For a module A, the module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is called the character module of A and denoted by A^+.

Some recent works on module theory have focused on relative injectivity rather than considering classical injectivity (see [2, 4]). Given modules A_R and B_R, A is called B-subinjective if all homomorphisms $B \to A$ extends to some $E(B) \to A$, where $E(B)$ is the injective envelope of B. The subinjectivity domain $\underline{\mathfrak{In}}^{-1}(A)$ of A contains exactly all modules B such that A is B-subinjective. It is clear that if a module B is injective, then A is B-subinjective and so, $\underline{\mathfrak{In}}^{-1}(A)$ contains all injective modules. Hence, the modules whose subinjectivity domains contain only injective modules are defined to be indigent in [4]. Presently, it is not known whether indigent module exists for an arbitrary ring, but an affirmative answer is known for Noetherian rings ([13, Proposition 3.4]). Following ideas on subinjectivity domains, in [14], the pure-injective modules whose subinjectivity domains contain only absolutely pure modules are defined to be pi-indigent. In contrast to indigent modules, such pure-injective modules exist over any ring. The dual concepts of these units were studied in [12, 18].

In [13], Durğun is interested in the flat analog of these notions. Namely, given modules A_R and ${}_RB$, ${}_RB$ is called absolutely A_R-pure if for every extension ${}_RC$ of ${}_RB$, $A \otimes B \to A \otimes C$ is a monomorphism. The class

*Correspondence: yusuf.alagoz@siirt.edu.tr
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$\underline{\mathfrak{Fl}}^{-1}(A_R) = \{ {}_RB : {}_RB$ is absolutely A_R-pure$\}$ is called the absolutely pure domain of a module A_R (see [13]). It is clear that a module A_R is flat if and only if $\underline{\mathfrak{Fl}}^{-1}(A_R) = R$-Mod. As absolutely pure domains contains all absolutely pure modules, the authors in [13] considered f-indigent modules as modules whose absolutely pure domain consists of entire class of absolutely pure modules.

A submodule ${}_RA$ of ${}_RB$ is called RD-pure if for all $a \in R$, the induced map $\alpha : (R/aR) \otimes A \to (R/aR) \otimes B$ is a monomorphism, equivalently $\beta : \mathrm{Hom}(R/Ra, B) \to \mathrm{Hom}(R/Ra, B/A)$ is an epimorphism. An R-module ${}_RN$ is said to be RD-projective (resp. RD-injective) if it is projective (resp. injective) with respect to every RD-pure exact sequence ([20, 26]). An R-module F_R is called RD-flat if the map $\alpha : F \otimes A \to F \otimes B$ is monic for all modules ${}_RA$ and ${}_RB$ with ${}_RA$ is RD-pure in ${}_RB$ ([8]). According to Warfields criterions [8, 26], F_R is RD-projective if and only if F_R is RD-flat and pure-projective.

RD-purity is an important example of relative purity. It is the first notion of purity that appeared in the literature. Moreover, RD-purity coincides with purity for some classes of rings not necessarily commutative. On the other side, the RD-flat modules form an example of additive accessible category by [8, Proposition 1.1]. In [20, Proposition 2.3], Mao observed that a left R-module B is divisible if every short exact sequence starting with B is RD-pure.

In this paper, our aim is to reveal the links between the last trends mentioned above by considering questions similar to the seminal work on RD-purity and RD-flatness. Along the way, an easy observation shows that if a module ${}_RB$ is divisible, then ${}_RB$ is absolutey A_R-pure for any RD-flat module A_R. Thus the smallest possible absolutely pure domain of an RD-flat module is the class of divisible modules. We consider in this paper, the RD-flat modules whose absolutely pure domains contain only divisible modules, and we referred to these RD-flat modules as rd-indigent.

In Section 2, we study some properties of rd-indigent modules. We also establish connections between rd-indigent and f-indigent modules. We show that rd-indigent module exists over any ring (Proposition 2.3). For an abelian group H, we prove that H is rd-indigent if and only if $T(H) \neq pT(H)$ for every prime integer p and the torsion submodule $T(H)$ of H. A commutative domain R is shown to be Prüfer if and only if rd-indigent modules coincide with f-indigent modules. Furthermore, we prove that a ring R is left PP if and only if absolutely pure domain of any RD-flat right R-module is closed under quotients (Proposition 2.8). Moreover, a ring R is (von Neumann) regular if and only if there exists a flat rd-indigent right module (Proposition 2.9). Over a left P-coherent ring, R is left divisible if and only if there exists an rd-indigent RD-flat right module which embeds in a flat module (Proposition 2.11).

In Section 3, we address some questions studied and partially answered in [13] and [15]. The first question under consideration here is to give a characterization of a ring over which every simple module is rd-indigent or flat. Over a left Noetherian right C and a right SRDP ring R, we show that every simple right R-module is f-indigent or projective if and only if there exists a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is either (a) right finitely $\sum$-extending right hereditary ring that contains a unique singular simple right R_2-module (up to isomorphism), or (b) n-saturated indecomposable matrix ring over a QF local ring (Theorem 3.6). For a commutative SRDP ring R, we prove that R is a C-ring and every simple R-module is rd-indigent or projective if and only if there exists a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is either (a) finitely $\sum$-extending Noetherian hereditary ring that contains a unique singular simple module (up to isomorphism), or (b) matrix ring over a local QF ring (Theorem 3.8).

2. The absolutely pure domain of an RD-flat module

Following [19], an R-module ${}_RB$ is called divisible if $\mathrm{Ext}^1(R/Ra, B) = 0$ for all $a \in R$. An R-module A_R is said to be torsion-free if $\mathrm{Tor}_1(A, R/Ra) = 0$ for all $a \in R$. Furthermore, a module B is divisible (torsion-free) if and only if any exact sequences starting (ending) with B is an RD-pure exact ([20]). It is clear that a right R-module A is torsion-free if and only if A^+ is divisible. We refer the reader to [20, 21, 24], for more details about torsion-free and divisible modules. A cyclic right R-module $C \cong R/I$ is said to be cyclically presented if $I = aR$ for some $a \in R$. Thus, it is clear by the definitions that every cyclically presented right R-module is RD-flat.

Let A be an RD-flat right R-module. It is clear that A is flat if and only if $= \underline{\mathfrak{Fl}}^{-1}(A_R) = R$-Mod. If a left R-module B is divisible, every short exact sequence starting with B is RD-pure, whence B is absolutey A-pure. Thus the smallest possible absolutely pure domain of an RD-flat module is the class of divisible modules. The next result asserts that absolutely pure domain $\underline{\mathfrak{Fl}}^{-1}(A)$ of an RD-flat right R-module A how small can be. It should contain the divisible modules at least.

Proposition 2.1 *$\bigcap_{A\in\mathscr{RF}} \underline{\mathfrak{Fl}}^{-1}(A) = \{B \in R\text{-Mod} \mid B$ is divisible$\}$, where $\mathscr{RF}$ is the class of all RD-flat right R-modules.*

Proof Let $B \in \underline{\mathfrak{Fl}}^{-1}(A)$ for any $A \in \mathscr{RF}$. Since every cyclically presented right R-module is RD-flat, for each cyclically presented right R-module A, B is absolutely A-pure, which means that $\alpha : A \otimes B \to A \otimes E(B)$ is monic. Thus, B is divisible by [20, Proposition 2.3]. The converse is straightforward. □

In particular, if A is an RD-flat right R-module, then we have the following relations:

$\{$Absolutely pure left modules$\} \subseteq \{$Divisible left modules$\} \subseteq \underline{\mathfrak{Fl}}^{-1}(A) \subseteq R$-mod.

Therefore we wonder about the RD-flat modules whose absolutely pure domain contains only divisible modules.

Definition 2.2 *An RD-flat right R-module A is called* rd-indigent *if $\underline{\mathfrak{Fl}}^{-1}(A) = \{$Divisible left R-modules$\}$.*

In what follows, let $\mathfrak{F} := \bigoplus_{a\in R} R/aR$ for every $a \in R$. Since RD-flat right R-modules are closed under direct sums, $\mathfrak{F}$ is an RD-flat R-module.

The first problem that comes to mind is whether rd-indigent modules exist over all rings. A positive answer to that problem can be given by the following result.

Proposition 2.3 *The module $\mathfrak{F}$ is rd-indigent.*

Proof Recall that a left R-module B is divisible if and only if B is absolutely R/aR-pure for all $a \in R$. Thus Proposition 2.1 and [13, Proposition 2.4] implies that $\underline{\mathfrak{Fl}}^{-1}(\bigoplus_{a\in R} R/aR) = \bigcap_{a\in R} \underline{\mathfrak{Fl}}^{-1}(R/aR) = \{$Divisible left modules$\}$. Hence $\mathfrak{F}$ is rd-indigent. □

Recall that a ring R is said to be RD-ring if RD-pure exact sequence of R-modules is pure exact (see [8]). Serial rings, Dedekind prime rings and two-sided Warfield rings are examples of RD rings (see, [23]). Over an RD-ring R, every divisible left R-module is absolutely pure by [20, Proposition 2.15], but not conversely. Furthermore, if R is two-sided semihereditary such that maximal left and right quotient rings of R

are semisimple, then every torsion-free right R-module is flat by [3, Theorem 5.2]. In this case, every divisible left R-module is absolutely pure.

Corollary 2.4 *Over the following rings, $\mathfrak{F}$ is f-indigent.*

(1) RD-rings.

(2) Two-sided semihereditary such that maximal left and right quotient rings of R are semisimple.

Corollary 2.5 *Let R be a commutative semihereditary ring. The following are equivalent for a module A.*

1. *A is rd-indigent.*
2. *$Z(A)$ is rd-indigent, where $Z(A)$ is the singular submodue of A.*
3. *A is f-indigent.*

Proof (1) $\Leftrightarrow$ (3) Over a commutative semihereditary ring R, every pure projective R-module is RD-projective by [9, Corollary 2.11]. Thus, every R-module is RD-flat by [8, Theorem 1.4] and every divisible R-module is absolutely pure by [20, Proposition 2.15]. Thus (1) $\Leftrightarrow$ (3) follows.

(1) $\Leftrightarrow$ (2) follows by [13, Proposition 5.1]. □

There are several characterizations of Prüfer domains in the literature. A Prüfer domain is exactly a semihereditary integral domain. Over a commutative domain, the ring R is Prüfer if and only if divisible modules are absolutely pure (see [22]). Now, the following is easy by considering Corollary 2.5.

Corollary 2.6 *The following are equivalent for a commutative domain R.*

1. *R is Prüfer.*
2. *rd-indigent modules coincide with f-indigent modules.*
3. *$\mathfrak{F}$ is f-indigent.*

The f-indigent abelian groups are completely characterized in [13]. Using Corollary 2.5, rd-indigent abelian grups coincide with f-indigent groups. As a result of [13, Theorem 5.1] and [13, Corollary 5.1] we can give the characterization of rd-indigent groups.

Corollary 2.7 *The following are equivalent for an abelian group H with the torsion submodule $T(H)$:*

(1) H is rd-indigent.

(2) For every prime integer p, $T(H) \neq pT(H)$.

(3) $T(H) \otimes_R S \neq 0$ for all singular simple modules S.

(4) $\mathrm{Hom}(T(H), S) \neq 0$ for all singular simple modules S.

Recall that, a ring R is left PP if every principal left ideal of R is projective. A ring R is left PP if and only if every quotient of divisible left R-module is divisible (see, [20, Corollary 2.13]). The absolutely pure domain of an RD-flat module needs not be closed under quotients. For example, if we assume that R is not a left PP ring, by Proposition 2.3 we conclude that the absolutely pure domain of $\mathfrak{F}$ is not closed under quotients.

Proposition 2.8 *A ring R is left PP if and only if absolutely pure domain of any RD-flat right R-module A is closed under quotients.*

Proof First suppose that R is left PP and A is an RD-flat right R-module. Let B be a left R-module such that B is absolutely A-pure. For any submodule C of B, we claim that B/C is absolutely A-pure. Consider the commutative diagram below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & B & \longrightarrow & B/C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & C & \longrightarrow & E(B) & \longrightarrow & E(B)/C & \longrightarrow & 0 \end{array}$$

with h is an isomorphism. Applying $A \otimes -$ to the diagram above gives the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \otimes C & \longrightarrow & A \otimes B & \longrightarrow & A \otimes B/C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle f^*} & & \\ 0 & \longrightarrow & A \otimes C & \longrightarrow & A \otimes E(B) & \longrightarrow & A \otimes E(B)/C & \longrightarrow & 0 \end{array}$$

Since h^* and g^* are monomorphisms, f^* is a monomorphism by the Five Lemma. Furthermore, PP condition on R gives that $E(B)/C$ is divisible (see [20, Corollary 2.13]), and so $E(B)/C \in \underline{\mathfrak{Fl}}^{-1}(A)$. On the other hand, consider the following diagram induced by the inclusions $\alpha : B/C \to E(B/C)$ and $\beta : E(B)/C \to E(E(B)/C)$.

$$\begin{array}{ccc} B/C & \xrightarrow{f} & E(B)/C \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ E(B/C) & & E(E(B)/C) \end{array}$$

Since $E(E(B)/C)$ is injective, there exists a homomorphism $\chi : E(B/C) \to E(E(B)/C)$ such that $\chi\alpha = \beta f$. Now, applying $A \otimes -$ to the diagram above gives the following commutative diagram:

$$\begin{array}{ccc} A \otimes B/C & \xrightarrow{f^*} & A \otimes E(B)/C \\ \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \beta^*} \\ A \otimes E(B/C) & \xrightarrow{\chi^*} & A \otimes E(E(B)/C) \end{array}$$

Since $E(B)/C \in \underline{\mathfrak{Fl}}^{-1}(A)$, β^* is monic, and so $\beta^* f^* = \chi^* \alpha^*$ is monic. This means that α^* is monic, whence $B/C \in \underline{\mathfrak{Fl}}^{-1}(A)$ by [13, Proposition 2.2]. For the converse, the hypothesis implies that the absolutely pure domain of $\mathfrak{F}$ is closed under quotients. But $\mathfrak{F}$ is rd-indigent, and so any quotient of a divisible left R-module is divisible, whence R is left PP by [20, Corollary 2.13]. □

A ring R is said to be (von Neumann) regular provided that for any $r \in R$ satisfies $r \in rRr$, equivalently every left R-module is absolutely pure.

Proposition 2.9 *Let R be a ring. The following are equivalent:*

(1) R is (von Neumann) regular.

(2) All left R-modules are divisible.

(3) All RD-flat right R-modules are flat.

(4) All (nonzero) RD-flat right R-modules are rd-indigent.

(5) There exists a flat rd-indigent right R-module.

Proof $(1) \Leftrightarrow (3)$ by [20, Corollary 2.14]. $(2) \Leftrightarrow (3)$, $(2) \Leftrightarrow (4)$ and $(5) \Rightarrow (2)$ are clear.

$(2) \Rightarrow (5)$ By hypothesis, $\underline{\mathfrak{Fl}}^{-1}(\mathfrak{F}) = R$-Mod, and so by Proposition 2.3, $\mathfrak{F}$ is flat and rd-indigent. □

From now on, all rings are supposed to be non (von Neumann) regular, equivalently by Proposition 2.9, there does not exist an rd-indigent flat right R-module.

Proposition 2.10 *If R is a commutative noetherian hereditary ring, then $\bigoplus_{S_i \in \Lambda} S_i$ is both an RD-flat and f-indigent module, where Λ is the set of representatives of simple singular modules.*

Proof Since R is commutative hereditary noetherian, every simple R-module is RD-projective by [5, Theorem 2.14], and so any $S_i \in \Lambda$ is RD-flat and finitely presented by [20, Lemma 2.1]. Since direct sum of any RD-flat modules is again RD-flat, $\bigoplus_{S_i \in \Lambda} S_i$ is RD-flat. Now, let us say that B is absolutely $\bigoplus_{S_i \in \Lambda} S_i$-pure for any module B. By [13, Corollary 2.1], $\bigoplus_{S_i \in \Lambda} S_i$ is B^+-subprojective and so by [13, Proposition 5.2], B^+ is flat. Thus coherence of R gives that B is absolutely pure. □

Recall that a ring R is called left P-coherent if every principal left ideal of R is finitely presented ([21]).

Proposition 2.11 *The following are equivalent for a left P-coherent ring R:*

1. *R is left divisible.*
2. *There exists an rd-indigent RD-flat right module that is embedded in a flat module.*
3. *All flat left modules are divisible.*

Proof $(1) \Rightarrow (2)$ Consider by Proposition 2.3 that $\mathfrak{F} := \bigoplus_{a_i \in R} R/a_iR$ is rd-indigent and RD-flat. Since R is left divisible, by [21, Proposition 4.2], for any $a_i \in R$, every R/a_iR contained in a flat R-module P_i. Set $P = \bigoplus_{P_i \in \Omega} P_i$, for a set Ω of the flat modules P_i. Then $\mathfrak{F}$ is embedded in a flat R-module P.

$(2) \Rightarrow (3)$ Existence of an rd-indigent RD-flat right module C that is embedded in a flat module gives that all flat left modules are absolutely C-pure by [13, Lemma 3.1], whence are divisible by rd-indigence of C.

$(3) \Rightarrow (1)$ is clear by [21, Proposition 4.2]. □

3. Rings whose nonprojective simples are rd-indigent

In this section the rings whose simple modules are rd-indigent or torsion-free are considered. By [20, Corollary 2.5], RD-flat torsion-free right modules are flat. By using this, the following is obvious.

Proposition 3.1 *The following are equivalent for a ring R:*

(1) All RD-flat right modules are rd-indigent or torsion-free.

(2) All RD-flat right modules are rd-indigent or flat.

Following [18], given modules A_R and B_R, A is called B-subprojective if for any epimorphisms $f: C \to B$ and any homomorphism $g: A \to B$, there exists a homomorphism $h: A \to C$ such that $fh = g$. The subprojectivity domain $\underline{\mathfrak{Pr}}^{-1}(A)$ of A contains exactly all modules B such that A is B-subprojective. Clearly, if B is projective, then A is vacuously B-subprojective, and so $\underline{\mathfrak{Pr}}^{-1}(A)$ contains all projective modules. Thus, the modules whose subprojectivity domains contains only projective modules is defined to be p-indigent in [18].

For an RD-projective right R-module A, we have the following relations:

$\{$Flat right modules$\}$ $\subseteq$ $\{$Torsion-free right modules$\}$ $\subseteq \underline{\mathfrak{Pr}}^{-1}(A) \subseteq$ Mod-R.

Recall that, RD-projective modules whose subprojectivity domains contains only torsion-free (flat) modules is defined to be $(s)rdp$*-indigent* in [15].

Lemma 3.2 *Let R be a ring and A a finitely presented RD-flat right R-module. A is rd-indigent if and only if A is rdp-indigent and R is left P-coherent.*

Proof Let A be an rd-indigent right R-module such that A is B-subprojective for a right R-module B. So [13, Corollary 2.1] implies that B^+ is absolutely A-pure. Since A is rd-indigent, B^+ is divisible. Hence B is torsion-free. Also, since A is RD-projective by [20, Lemma 2.1(2)], A is rdp-indigent. Now, let C be a divisible left R-module. Since $C \in \underline{\mathfrak{Fl}}^{-1}(A)$, $C^+ \in \underline{\mathfrak{Pr}}^{-1}(A)$, and so C^+ is torsion-free, whence R is left P-coherent by [21, Theorem 2.7]. Conversely, let us say that B is absolutely A-pure for a left R-module B. Then A is B^+-subprojective by [13, Corollary 2.1] and the hypothesis implies that B^+ is torsion-free. Hence B is divisible by the P-coherence of R. □

Recall by [8, 26] that, a right R-module A is RD-projective if and only if A is RD-flat and pure-projective. A ring R is called right SRDP provided that every simple right R-module is RD-projective, equivalently every simple right R-module is RD-flat and finitely presented. A commutative ring R is SRDP if and only if every simple R-module is both finitely presented and RD-injective. PIR rings and commutative hereditary noetherian rings are SRDP (see [15]).

Corollary 3.3 *Let R be a right SRDP ring that is not left PP. Assume that every simple right R-module is either rd-indigent or projective. Then R has a unique nonprojective simple right module which satisfies the conditions given below:*

(1) R is left divisible.

(2) All flat left R-modules are divisible.

(3) All RD-projective right modules are embedded in a projective module.

(4) R is right Kasch.

Proof By using the Lemma 3.2 and [15, Lemma 3.7], the proof is easy. □

Corollary 3.4 *For a right SRDP ring R that is not left divisible, the following are equivalent.*

(1) All simple right R-modules are rd-indigent or projective.

(2) All simple right R-modules are rdp-indigent or projective.

(3) (i) Torsion-free right modules and the right modules with projective socle coincide.
(ii) R is a left PP ring with a unique nonprojective simple right R-module A (up to isomorphism).
Also, if R is right nonsingular, then the conditions given above are equivalent to:

(4) Singular finitely generated RD-projective right modules are rd-indigent.

Proof (1) ⇒ (2) and (2) ⇔ (3) follows by Lemma 3.2 and [15, Proposition 3.6], respectively.

(3) ⇒ (1) The hypothesis implies that nonprojective simple right R-modules are rdp-indigent by [15, Proposition 3.6]. Since R is a left PP ring, R is left P-coherent, whence S is rd-indigent by Lemma 3.2.

(4) ⇒ (1) is clear.

(2) ⇒ (4) let A be a singular RD-projective and finitely generated right module. So by [15, Proposition 3.2], $\underline{\mathfrak{Pr}}^{-1}(A) \subseteq \{\text{Torsion-free modules}\}$, whence A is rdp-indigent. Since R is left P-coherent, A is rd-indigent by Lemma 3.2.

□

Recall by [7, 10.10] that a ring R is called a right C-ring if for all essential right ideals I of R, $Soc(R/I) \neq 0$. Two-sided noetherian hereditary rings, left perfect rings and right semiartinian rings are trivial examples of such this rings.

Proposition 3.5 *Let R be a right C and right SRDP ring that is not left divisible. Assume that every simple right module is either rd-indigent or projective. Then R has a (up to isomorphism) unique nonprojective simple right module S and it satisfies the following conditions:*

(a) R is right Noetherian right hereditary.

(b) Classes of divisible right R-modules and injective right modules coincide.

(c) Classes of torsion-free right R-modules and nonsingular right modules coincide.

(d) Classes of torsion-free left R-modules and flat left modules coincide.

Proof Corollary 3.4 gives that R is a left PP ring which has a unique nonprojective simple right module S. By our hypothesis, S is rd-indigent, and by Lemma 3.2, S is rdp-indigent. Let A be a nonsingular right R-module. Since S is singular, S is A-subprojective, and so A is torsion-free. Let B be a divisible right module. Being R right SRDP implies that $\text{Ext}(S, B) = 0$ by [20, Proposition 2.3]. By considering the epimorphism $E(B) \to E(B)/B \to 0$, we obtain the epimorphism $\text{Hom}(S, E(B)) \to \text{Hom}(S, E(B)/B) \to 0$. Hence by [7, 10.8 and 10.10], B is a closed submodule of $E(B)$, and so B is injective. In particular, every absolutely pure right module is injective whence R is right Noetherian. In this case, R has no infinite set of orthogonal nonzero idempotents and so R is right PP by [6, Lemma 8.4]. Being right PP implies by [21, Theorem 5.3] that quotients of injective right modules are injective, whence R is right hereditary. We claim that all torsion-free right R-modules are nonsingular. Let A be a torsion-free right R-module and $f : P \to A$ an epimorphism with P projective. Since R is a right C-ring and S is A-subprojective, $Ker(f)$ is closed in P by [16, Theorem 5]. Furthermore, since R is right nonsingular, the projective module P is nonsingular, whence A is nonsingular by [25, Lemma 2.3]. Now, we claim that all torsion-free left R-modules are flat. For a torsion-free left R-module T, T^+ is divisible, whence is injective by (b). Therefore T is flat. □

Recall from [10] that a right R-module A is called *extending* if every closed submodule of A is direct summand. A ring R is called *finitely* $\sum$*-extending* if every finite direct sums of copies of R_R is extending (see [10]).

Theorem 3.6 *For a left Noetherian right C and right SRDP ring R, the following are equivalent.*

(1) All simple right R-modules are f-indigent or projective.

(2) All simple right R-modules are srdp-indigent or projective.

(3) There exists a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is either
(a) Right finitely $\sum$-extending right hereditary ring that contains a unique nonprojective simple right R_2-module (up to isomorphism), or
(b) n-saturated indecomposable matrix ring over a QF local ring.

Proof (1) ⇒ (2) Let S be a nonprojective simple right R-module and S a B-subprojective module for a right R-module B. By [13, Corollary 2.1], B^+ is absolutely S-pure, and by the f-indigence of S, B^+ is absolutely pure. Since R is right C and right SRDP, by the proof of Proposition 3.5, B^+ is injective. Thus, B is flat and S is *srdp*-indigent.

(2) ⇒ (1) Let S be a nonprojective simple right R-module and B an absolutely S-pure module for a right R-module B. This implies that S is B^+-subprojective by [13, Corollary 2.1], and by (2), B^+ is flat. Since B^{++} is injective and B is pure in B^{++}, B is absolutely pure. Thus, S is f-indigent.

(2) ⇒ (3) By using [15, Corollary 3.9], there is a unique nonprojective simple right module S and R is either left absolutely pure or $wD(R) \leq 1$. Since R is right C and right SRDP, every divisible (and so absolutely pure) right R-module is injective, whence R is right Noetherian. In the former case, by [15, Theorem 3.11], there is a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is an n-saturated indecomposable matrix ring over a QF local ring. The latter case implies R is not absolutely pure, and so by Lemma 3.2 and Proposition 3.5, R is right hereditary. Let I denote the sum of injective simple right ideals of R. Right Noetherianity of R implies that I is injective, whence R can be decomposed as $R = I \oplus J$ for some right ideal J of R such that $Soc(I) = I$ and J has no injective simple submodule. From this, by applying the same arguments as in [1, Theorem 1], if there is a nonzero homomorphism $\alpha : I \to J$, then $\alpha(Soc(I)) = \alpha(I) \subseteq Soc(J)$, where $\alpha(I)$ is injective since R is right hereditary. This means that $Soc(J)$ contains an injective simple submodule, a contradiction. Thus, I is also a left ideal by the fact that $\mathrm{Hom}(I, J) = 0$. For reverse order, if there is a nonzero homomorphism $\beta : J \to I$, then $J/Ker(\beta) \cong Im(\beta) \subseteq I$, where $J/Ker(\beta)$ is projective since R is right hereditary. Moreover, $J/Ker(\beta)$ is semisimple and injective as it is isomorphic to a submodule of I. By considering the split exact sequence $0 \to Ker(\beta) \to J \to J/Ker(\beta) \to 0$ we deduce that J contains a copy

of an injective simple R-module, a contradiction. Thus the fact that $\mathrm{Hom}(J,I)=0$ implies J is an ideal, too. Consequently, we get a ring decomposition $R\cong I\times J$ where I is semisimple and J is right hereditary and right Noetherian. Now, we want to prove that J is right finitely $\sum$-extending. Let A be a nonsingular finitely generated right J-module. Since A is a nonsingular right R-module by [17, Proposition 1.28], any epimorphism $P\to A$ is closed exact by [25, Lemma 2.3]. Thus all simple right R-modules are projective relative to epimorphism $P\to A$ by [16, Theorem 5], and so S is A-subprojective. The fact that S is srdp-indigent implies that A is flat, and so A^+ is injective left R-module. This means that A^+ is an injective left J-module by [19, Example 3.11A]. So A is a flat and finitely generated right J-module, whence A is projective by noetherianity of J. Thus J is a right finitely $\sum$-extending ring by [10, Corollary 11.4]. Again by the uniqueness of a nonprojective simple right R-module S and by [17, Proposition 1.28], J has a unique nonprojective simple right J-module (up to isomorphism).

(3) ⇒ (2) Let $R=R_1\times R_2$, where R_1 is semisimple and R_2 is either n-saturated indecomposable matrix ring over a QF local ring, or right finitely $\sum$-extending right hereditary ring with a unique singular simple right B-module (up to isomorphism). In the former case, (2) follows by [12, Theorem 3.1]. Assume the latter case, since R_2 is right hereditary right extending, R_2 cannot have an infinite set of orthogonal nonzero idempotents and so R_2 is right Noetherian by [11, Theorem 3.1]. Moreover, since R_2 is not right divisible, the rest follows by [15, Lemma 3.10]. □

In [15, Corollary 3.12], it is proven that a left Noetherian right PIR ring whose nonprojective simple right R-modules are *srdp*-indigent is a right C-ring. The following is now an easy result of Theorem 3.6 and [15, Corollary 3.12].

Corollary 3.7 *The following are equivalent for a left Noetherian right PIR ring R.*

(1) All simple right R-modules are f-indigent or projective.

(2) All simple right R-modules are srdp-indigent or projective.

(3) There exists a decomposition of rings $R\cong R_1\times R_2$, where R_1 is semisimple and R_2 is either
(a) Right finitely $\sum$-extending right hereditary right C-ring that contains a unique nonprojective simple right B-module (up to isomorphism), or
(b) n-saturated indecomposable matrix ring over a QF local ring.

Recall that if E is an injective cogenerator in R-Mod over a commutative ring R, then for each simple module S, $\mathrm{Hom}(S,E)\cong S$, in particular $S\cong S^+$.

Theorem 3.8 *The following are equivalent for a commutative SRDP ring R.*

(1) R is a C-ring and every simple R-module is rd-indigent or projective.

(2) R is a C-ring and every simple R-module is rdp-indigent or projective.

(3) There exists a decomposition of rings $R\cong R_1\times R_2$, where R_1 is semisimple and R_2 is either
(a) Finitely $\sum$-extending hereditary Noetherian ring that contains a unique nonprojective simple module

(up to isomorphism), or

(b) matrix ring over a QF local ring.

Proof (1) ⇔ (2) is clear by the fact that SRDP C-rings are Noetherian and by Lemma 3.2.

(1) ⇒ (3) Let S be a nonprojective simple R-module. Then by the hypothesis, divisible modules contained in $\underline{\mathfrak{Fl}}^{-1}(S)$. Since R is SRDP and C-ring, by the proof of Proposition 3.5 all divisible modules are injective and so R is Noetherian, i.e. S is f-indigent. By [13, Theorem 5.2], $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is a ring which is either matrix ring over a QF local ring, or noetherian hereditary ring with a unique singular simple module S'. In the later case, R_2 is not divisible, otherwise R_2 would be a semisimple ring. Since S' is rdp-indigent by Lemma 3.2, S' is srdp-indigent by Proposition 3.5(d). Thus, R_2 is finitely $\sum$-extending by Theorem 3.6.

(3) ⇒ (1) In either cases, R_1 and R_2 are C-rings, whence $R \cong R_1 \times R_2$ is a C-ring. Now the rest follows again by [13, Theorem 5.2]. □

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