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Matrix mappings and compact operators for Schröder sequence spaces

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Abstract: In this paper, we discuss the domain of a recently defined conservative matrix, constructed by means of the Schröder numbers in the spaces of $p-$absolutely summable sequences and bounded sequences. We determine the $\beta-$duals of the Banach spaces, introduced here, and present characterization of some matrix operators. Moreover, we give the characterization of certain compact operators via the Hausdorff measure of noncompactness.

Key words: Schröder numbers, conservative matrix, sequence spaces, matrix transformations, compact operators

1. Introduction

The Schröder numbers S_n, located in a very important position in combinatorics and number theory, can be defined by the recurrence relation

$$S_{n+1} = S_n + \sum_{k=0}^{n} S_k S_{n-k}, \quad \text{for } n \geq 0$$

subject to initial condition $S_0 = 1$. The definition of these numbers can also be given combinatorially as follows: Schröder number S_n counts the number of paths from the southwest corner $(0,0)$ of an $n \times n$ grid to the northeast corner (n,n), using only single steps north, northeast, or east, that do not rise above the southwest–northeast diagonal. The first few Schröder numbers S_n for $0 \leq n \leq 10$ are

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718.$$

Also, the Schröder numbers S_n possess the following representation via hypergeometric function as

$$S_n = 2 {}_2F_1\left(-n+1, n+2; 2; -1\right),$$

where the hypergeometric function ${}_2F_1\left(a, b; c; z\right)$ is defined by

$${}_2F_1\left(a, b; c; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

with the Pochhammer symbol $(x)_n = x(x+1)...(x+n-1)$ for $n \geq 1$, and $(x)_n = 1$ for $n = 0$. For details and applications, we may refer to [10].

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In this study, $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{R}$ denotes the set of all real numbers.

A sequence space is a linear subspace of the set of all real valued sequences ω. We denote ℓ_∞, c, c_0 and ℓ_p as the set of all bounded sequences, the set of all convergent sequences, the set of all convergent to zero sequences and the set of all sequences constituting $p-$absolutely convergent series, respectively. These are Banach spaces with the following norms

$$\|z\|_{\ell_\infty} = \|z\|_c = \|z\|_{c_0} = \sup_{n\in\mathbb{N}} |z_n|$$

and

$$\|z\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |z_k|^p\right)^{1/p}.$$

By Ψ, we denote the set of all finite sequences. A Banach space with all continuous coordinate functionals p_k defined by $p_k(z) = z_k$ is referred as a BK-space while a complete linear metric space with continuous coordinate functionals is referred as a FK-space.

As is known, the theory of sequence spaces plays a central role in functional analysis and summability theory. Indeed, the classical theory concerns with the generalization of the concept of convergence for series and sequences. The purpose is to assign a limit for nonconvergent series and sequences by applying a transformation, represented by means of infinite special matrices. Since the most general linear mapping from a sequence space into another sequence space can be given via an infinite matrix, it is convenient to deal with matrices instead of a general linear mapping.

Let $T = (t_{nk})$ be an infinite matrix with real entries t_{nk} and T_n be the sequence in the nth row of the matrix T for each $n \in \mathbb{N}$. The $T-$transform of a sequence $z = (z_k) \in \omega$ is the sequence Tz obtained by the usual matrix product and its entries are stated as

$$(Tz)_n = \sum_k t_{nk} z_k$$

provided that the series is convergent for each $n \in \mathbb{N}$. $\sum\limits_k$ means $\sum\limits_{k=0}^{\infty}$ briefly. T is called as a matrix mapping from a sequence space Γ to a sequence space Θ if the sequence Tz exists and $Tz \in \Theta$ for all $z \in \Gamma$. The collection of all infinite matrices from Γ to Θ is denoted by (Γ, Θ).

The β-dual of a sequence space Γ is denoted by Γ^β and consists of sequences $x = (x_k) \in \omega$ such that the series $\sum\limits_k x_k z_k$ is convergent for every $z = (z_k) \in \Gamma$. The necessary and sufficient conditions for an infinite matrix T to belong to the class (Γ, Θ) are that $T_n \in \Gamma^\beta$ for each $n \in \mathbb{N}$ and $Tu \in \Theta$ for all $z \in \Gamma$.

Recall that the set

$$\Gamma_T = \{z \in \omega : Tz \in \Gamma\}$$

is referred the domain of the infinite matrix T in the space Γ. For the last two decades, the study of domains of special triangular matrices has been studied intensively by many authors. Some papers in the relevant literature can be given as [2–4, 15–17, 22–24, 28, 33].

Regarding an infinite matrix as a linear operator between two sequence spaces encourages the scholars to deal with theory of matrix transformations in summability theory. More precisely, the theory of matrix transformations provides the necessary and sufficient conditions for a matrix to map a sequence space into another sequence space. The reader can consult [11–13, 18, 19, 30, 31, 37, 38, 40] for further information about matrix operators.

Further details concerning summability theory and its applications can be found in [5].

The notion of Hausdorff measure of noncompactness has been applied to characterize the compact operators between BK-spaces as a very effective tool. Especially, as an application of the Hausdorff measure of noncompactness of a linear operator in the theory of sequence spaces, a number of noteworthy results have been revealed. (See, for instance, [1, 6–9, 20, 21, 25–27, 32, 34–36]).

In this paper, two Banach spaces as the domains of a newly given conservative matrix, whose entries are Schröder numbers, are presented and discussed. Besides this, $\beta-$duals of these spaces are determined and the characterizations of some matrix classes are derived. Furthermore, the compactness of certain matrix operators are characterized by aid of the concept of Hausdorff measure of noncompactness.

2. New Banach sequence spaces $\ell_p\left(\tilde{S}\right)$ and $\ell_\infty\left(\tilde{S}\right)$

Quite recently, the author [14] introduced a new matrix $\tilde{S}=\left(\tilde{S}_{nk}\right)$, whose entries are Schröder numbers S_n as

$$\tilde{S}_{nk}=\begin{cases}\dfrac{S_kS_{n-k}}{S_{n+1}-S_n}, & \text{if } 0\leq k\leq n;\\ 0, & \text{if } k>n;\end{cases}$$

and computed its inverse as

$$\tilde{S}^{-1}_{nk}=\begin{cases}(-1)^{n-k}\dfrac{S_{k+1}-S_k}{S_n}P_{n-k}, & \text{if } 0\leq k\leq n;\\ 0, & \text{if } k>n;\end{cases}$$

where P_n denotes the following determinant for all $n\in\mathbb{N}\setminus\{0\}$

$$\begin{vmatrix} S_1 & S_0 & 0 & \dots & 0\\ S_2 & S_1 & S_0 & \dots & 0\\ S_3 & S_2 & S_1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ S_n & S_{n-1} & S_{n-2} & \dots & S_1 \end{vmatrix}$$

subject to initial condition $P_0=1$.

Now, define new sequence spaces $\ell_p\left(\tilde{S}\right)$ and $\ell_\infty\left(\tilde{S}\right)$ as

$$\ell_p\left(\tilde{S}\right)=\left\{z=(z_n)\in\omega:\sum_{n=0}^{\infty}\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right|^p<\infty\right\}$$

and

$$\ell_\infty\left(\tilde{S}\right)=\left\{z=(z_n)\in\omega:\sup_n\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right|<\infty\right\}.$$

More precisely, if the transformation

$$t_n=\tilde{S}_n(z)=\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k \tag{2.1}$$

is called as $\tilde{S}-$transform of a sequence $z=(z_n)$, then, the spaces $\ell_p\left(\tilde{S}\right)$ and $\ell_\infty\left(\tilde{S}\right)$ consist of all sequences, whose $\tilde{S}-$transforms are in the spaces ℓ_p and ℓ_∞, respectively. Notice that the following relation is valid for all $k\in\mathbb{N}$,

$$z_k=\sum_{i=0}^{k}\left((-1)^{k-i}\frac{S_{i+1}-S_i}{S_k}P_{k-i}\right)t_i. \tag{2.2}$$

Let us start with a theorem concerning the linearity of the spaces $\ell_p\left(\tilde{S}\right)$ and $\ell_\infty\left(\tilde{S}\right)$, whose proof can be reached routinely.

Theorem 2.1 *The spaces $\ell_p\left(\tilde{S}\right)$ and $\ell_\infty\left(\tilde{S}\right)$ are linear spaces which are also BK-spaces with the norms $\|z\|_{\ell_p(\tilde{S})}=\left(\sum_{n=0}^{\infty}\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right|^p\right)^{1/p}$ and $\|z\|_{\ell_\infty(\tilde{S})}=\sup_{n\in\mathbb{N}}\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right|$, respectively.*

Theorem 2.2 *The spaces $\ell_p\left(\tilde{S}\right)$ and $\ell_\infty\left(\tilde{S}\right)$ are linearly isomorphic to ℓ_p and ℓ_∞, respectively.*

Proof For $1\le p\le\infty$, let $L:\ell_p\left(\tilde{S}\right)\to\ell_p$ be a transformation such that $L(z)=\left(\tilde{S}_n(z)\right)$. The one to one property of this mapping is trivial. Proceeding the arguments, used in [14], leads to

$$\begin{aligned}\tilde{S}_n(z)&=\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\\&=\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}\sum_{i=0}^{k}\left((-1)^{k-i}\frac{S_{i+1}-S_i}{S_k}P_{k-i}\right)t_i\\&=\frac{1}{S_{n+1}-S_n}\sum_{i=0}^{n}\left(\sum_{k=0}^{n-i}(-1)^kS_{n-k-i}P_k\right)(S_{i+1}-S_i)\,t_i\\&=t_n,\end{aligned}$$

where we have used that $\sum_{k=0}^{n-i}(-1)^kS_{n-k-i}P_k=0$ for $n\neq i$. So, we have $L_n(z)=t_n$ for all $n\in\mathbb{N}$, where $z=(z_k)$ is given by (2.2) while $t=(t_n)$ is any sequence in ℓ_p. So, from the fact that $Lz=t$, the mapping L is onto. We conclude from Theorem 2.1 that $\|z\|_{\ell_p(\tilde{S})}=\|t\|_{\ell_p}$. Namely, $\ell_p\left(\tilde{S}\right)$ and ℓ_p are linearly isomorphic, as required. □

Now, let us mention a lemma, which is an effective tool to discover the $\beta-$duals of the spaces $\ell_p\left(\tilde{S}\right)$ and $\ell_\infty\left(\tilde{S}\right)$, and the characterizations of some matrix classes. In the sequel, $\mathcal{F}$ denotes the family of all finite subsets of $\mathbb{N}$ and $\mathcal{F}_r$ denotes the subcollection of $\mathcal{F}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than r.

Lemma 2.3 *([39]) $T=(t_{nk})\in(\ell_1,\ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}}|t_{nk}|<\infty. \tag{2.3}$$

$T = (t_{nk}) \in (\ell_1, c)$ *if and only if* (2.3) *holds and*

$$\lim_{n\to\infty} t_{nk} \text{ exists} \tag{2.4}$$

for each $k \in \mathbb{N}$. $T = (t_{nk}) \in (\ell_1, c_0)$ *if and only if* (2.3) *holds and*

$$\lim_{n\to\infty} t_{nk} = 0 \tag{2.5}$$

for each $k \in \mathbb{N}$. $T = (t_{nk}) \in (\ell_1, \ell_p)$ *if and only if*

$$\sup_k \sum_{n=0}^{\infty} |t_{nk}|^p < \infty,$$

where $1 \leq p < \infty$. $T = (t_{nk}) \in (\ell_p, \ell_\infty)$ *if and only if*

$$\sup_k \sum_{n=0}^{\infty} |t_{nk}|^q < \infty, \tag{2.6}$$

where $1 < p < \infty$. $T = (t_{nk}) \in (\ell_p, c)$ *if and only if* (2.4) *and* (2.6) *hold for* $1 < p < \infty$. $T = (t_{nk}) \in (\ell_p, c_0)$ *if and only if* (2.5) *and* (2.6) *hold for* $1 < p < \infty$. $T = (t_{nk}) \in (\ell_p, \ell_1)$ *if and only if*

$$\sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} |t_{nk}|^q \right)^q < \infty \quad \text{for } 1 < p < \infty.$$

$T = (t_{nk}) \in (\ell_\infty, \ell_\infty) = (c, \ell_\infty) = (c_0, \ell_\infty)$ *if and only if*

$$\sup_n \sum_{k=0}^{\infty} |t_{nk}| < \infty.$$

$T = (t_{nk}) \in (\ell_\infty, c)$ *if and only if* (2.4) *holds and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |t_{nk}| = \sum_{k=0}^{\infty} \left| \lim_{n\to\infty} t_{nk} \right|. \tag{2.7}$$

$T = (t_{nk}) \in (\ell_\infty, c_0)$ *if and only if* (2.5) *holds and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |t_{nk}| = 0. \tag{2.8}$$

$T = (t_{nk}) \in (\ell_\infty, \ell_p) = (c, \ell_p) = (c_0, \ell_p)$ *if and only if*

$$\sup_{K\in F} \sum_{n=0}^{\infty} \left| \sum_{k\in K} t_{nk} \right|^p < \infty, \quad \text{for } 1 \leq p < \infty.$$

We are ready to offer the following theorem.

Theorem 2.4 *Let the sets* B_1, B_2, B_3 *and* B_4 *be defined by*

$$B_1 = \left\{ b = (b_k) \in \omega : \lim_{n\to\infty} \sum_{i=k}^{n} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} b_i \text{ exists for each } k \in \mathbb{N} \right\},$$

$$B_2 = \left\{ b = (b_k) \in \omega : \sup_n \sum_k \left| \sum_{i=k}^{n} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} b_i \right|^q < \infty \right\},$$

$$B_3 = \left\{ b = (b_k) \in \omega : \sup_{n,k} \left| \sum_{i=k}^{n} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} b_i \right| < \infty \right\},$$

and

$$B_4 = \left\{ b = (b_k) \in \omega : \lim_{n\to\infty} \sum_k \left| \sum_{i=k}^{n} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} b_i \right| = \sum_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} b_i \right| \right\}.$$

Then, we have $\left(\ell_p\left(\tilde{S}\right)\right)^\beta = B_1 \cap B_2,$ *for* $1 < p < \infty.$ $\left(\ell_1\left(\tilde{S}\right)\right)^\beta = B_1 \cap B_3,$ $\left(\ell_\infty\left(\tilde{S}\right)\right)^\beta = B_1 \cap B_4.$

Proof We prove the first assertion. For $1 < p < \infty,$ $b = (b_k) \in \left(\ell_p\left(\tilde{S}\right)\right)^\beta$ if and only if $\sum_{k=0}^{\infty} b_k z_k$ is convergent for all $z = (z_k) \in \left(\ell_p\left(\tilde{S}\right)\right)$. It is easily read that

$$\begin{aligned}\sum_{k=0}^{n} b_k z_k &= \sum_{k=0}^{n} b_k \left(\sum_{i=0}^{k} (-1)^{k-i} \frac{S_{i+1} - S_i}{S_k} P_{k-i} t_i \right) \\ &= \sum_{k=0}^{n} \left(\sum_{i=k}^{n} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} b_i \right) t_k,\end{aligned}$$

from which we get $b = (b_k) \in \left(\ell_p\left(\tilde{S}\right)\right)^\beta$ if and only if the following matrix

$$d_{nk} = \begin{cases} \sum_{i=k}^{n} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} b_i, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n \end{cases}$$

belongs to the class (ℓ_p, c). Thus, from Lemma 2.3, the following limit exists for each $k \in \mathbb{N}$

$$\lim_{n\to\infty} \sum_{i=k}^{n} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} b_i$$

and the relation

$$\sup_n \sum_{k=0}^{n} \left| \sum_{i=k}^{n} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} b_i \right|^q < \infty$$

is valid. So, we have $b = (b_k) \in B_1 \cap B_2,$ as desired. The proofs of other assertions given in this theorem can be deduced in a similar way. □

3. Certain matrix mappings

In this section, for $\Gamma \in \{\ell_p\}$ with $1 \leq p \leq \infty$ and $\Theta \in \{\ell_1, c_0, c, \ell_\infty\}$, we obtain the characterization of the classes $(\Gamma\left(\tilde{S}\right), \Theta)$.

Theorem 3.1 *Define a matrix $R = (r_{nk})$ by*

$$r_{nk} = \lim_{m\to\infty} \sum_{i=k}^{m} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \tag{3.1}$$

for $n, k \in \mathbb{N}$. Then, we have

$(a):$ $T \in \left(\ell_1\left(\tilde{S}\right), \ell_\infty\right)$ *if and only if*

$$R = (r_{nk}) \text{ is well defined for all } n, k \in \mathbb{N}, \tag{3.2}$$

$$\sup_{m,k} \left| \sum_{i=k}^{m} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \right| < \infty, \text{ for each } n \in \mathbb{N} \tag{3.3}$$

and

$$\sup_{n,k} |r_{nk}| < \infty. \tag{3.4}$$

$(b):$ $T \in \left(\ell_1\left(\tilde{S}\right), c\right)$ *if and only if equations (3.2), (3.3) and (3.4) are valid and*

$$\lim_{n\to\infty} r_{nk} \text{ exists for each } k \in \mathbb{N}. \tag{3.5}$$

$(c):$ $T \in \left(\ell_1\left(\tilde{S}\right), c_0\right)$ *if and only if equations (3.2), (3.3) and (3.4) are valid and*

$$\lim_{n\to\infty} r_{nk} = 0 \text{ for each } k \in \mathbb{N}. \tag{3.6}$$

$(d):$ $T \in \left(\ell_1\left(\tilde{S}\right), \ell_1\right)$ *if and only if equations (3.2) and (3.3) are valid and*

$$\sup_k \sum_n |r_{nk}| < \infty. \tag{3.7}$$

Proof We only prove the assertion (a). It is known that $T \in \left(\ell_1\left(\tilde{S}\right), \ell_\infty\right)$ if and only if $Tz \in \ell_\infty$ for all $z \in \ell_1\left(\tilde{S}\right)$. It follows from the convergence of the series $\sum_{k=0}^{\infty} t_{nk} z_k$ that $t_{nk} \in \left(\ell_1\left(\tilde{S}\right)\right)^\beta$ for each fixed $n \in \mathbb{N}$. From Theorem 2.4, one can readily find that the limit

$$\lim_{m\to\infty} \sum_{i=k}^{m} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni}$$

is available for each $n, k \in \mathbb{N}$ and

$$\sup_{m,k} \left| \sum_{i=k}^{m} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \right| < \infty, \text{ for each } n \in \mathbb{N}$$

from which we arrive that (3.2) and (3.3) hold. Now, consider

$$\sum_{k=0}^{m} t_{nk} z_k = \sum_{k=0}^{m} t_{nk} \sum_{i=0}^{k} \left((-1)^{k-i} \frac{S_{i+1} - S_i}{S_k} P_{k-i} \right) t_i$$
$$= \sum_{i=0}^{m} \left(\sum_{k=i}^{m} t_{nk} (-1)^{k-i} \frac{S_{i+1} - S_i}{S_k} P_{k-i} \right) t_i. \tag{3.8}$$

If we denote a matrix $\tilde{R} = (\tilde{r}_{mi})$ for each $n \in \mathbb{N}$ by

$$\tilde{r}_{mi} = \begin{cases} \sum_{k=i}^{m} (-1)^{k-i} \frac{S_{i+1} - S_i}{S_k} P_{k-i} t_{nk}, & \text{if } 0 \leq i \leq m; \\ 0, & \text{if } i > m; \end{cases}$$

then, we find that (3.2) and (3.3) imply $\tilde{R} = (\tilde{r}_{mi}) \in (\ell_1, c)$. So, the series $\tilde{R}_m(t) = \sum_{i=0}^{\infty} \tilde{r}_{mi} t_i$ converges uniformly in m for all $t \in \ell_1$, from which $lim_{m\to\infty} \tilde{R}_m(t) = \sum_{i=0}^{\infty} lim_{m\to\infty} \tilde{r}_{mi} t_i$. Thus, we conclude from (3.8) that

$$T_n(z) = \lim_{m\to\infty} \tilde{R}_m(t) = \sum_{i=0}^{\infty} (lim_{m\to\infty} \tilde{r}_{mi}) t_i = \sum_{i=0}^{\infty} \tilde{r}_{ni} t_i = R_n(t), \tag{3.9}$$

which gives that for $z \in \ell_1\left(\tilde{S}\right)$, $T(z) \in \ell_\infty$ if and only if for $t \in \ell_1$, $R(t) \in \ell_\infty$. Consequently, $T \in \left(\ell_1\left(\tilde{S}\right), \ell_\infty\right)$ if and only if (3.2), (3.3) and (3.4) hold, which completes the proof. The assertions $(b), (c)$ and (d) can be derived by the similar way. □

Theorem 3.2 *Let $1 < p < \infty$ and let the matrix $R = (r_{nk})$ given by (3.1). Then,*

(a): $T \in \left(\ell_p\left(\tilde{S}\right), \ell_\infty\right)$ if and only if (3.2) holds and

$$\sup_m \sum_{k=0}^{m} \left| \sum_{i=k}^{m} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \right|^q < \infty, \text{ for each } n \in \mathbb{N} \tag{3.10}$$

and

$$\sup_n \sum_k |r_{nk}|^q < \infty. \tag{3.11}$$

(b): $T \in \left(\ell_p\left(\tilde{S}\right), c\right)$ if and only if Equations (3.2), (3.5), (3.10) and (3.11) are valid.

(c): $T \in \left(\ell_p\left(\tilde{S}\right), c_0\right)$ if and only if Equations (3.2), (3.6), (3.10) and (3.11) are valid.

(d): $T \in \left(\ell_p\left(\tilde{S}\right), \ell_1\right)$ if and only if Equations (3.2) and (3.10) are valid and

$$\sup_{N \in F} \sum_k \left| \sum_{n \in N} r_{nk} \right|^q < \infty. \tag{3.12}$$

Proof Let us give the proof of relation (a). $T \in \left(\ell_p\left(\tilde{S}\right), \ell_\infty\right)$ if and only if $Tz \in \ell_\infty$ for all $z \in \ell_p\left(\tilde{S}\right)$. It follows from the convergence of the series $\sum_{k=0}^{\infty} t_{nk} z_k$ that $t_{nk} \in \left(\ell_p\left(\tilde{S}\right)\right)^\beta$ for each fixed $n \in \mathbb{N}$. Applying Theorem 2.4, it yields that the relations (3.2) and

$$\sup_m \sum_{k=0}^{m} \left| \sum_{i=k}^{m} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \right|^q < \infty, \text{ for each } n \in \mathbb{N}$$

are valid. Similarly, Equation (3.9) gives the fact that $Tz \in \ell_\infty$ for all $z \in \ell_p\left(\tilde{S}\right)$ if and only if $R(t) \in \ell_\infty$ for $t \in \ell_p$. Thus, it is satisfied that $T \in \left(\ell_p\left(\tilde{S}\right), \ell_\infty\right)$ if and only if the relations (3.2) and (3.10) hold and $R \in (\ell_p, \ell_\infty)$ namely (3.11) holds. The proofs of other assertions can be reached by proceeding the similar arguments. □

Theorem 3.3 *Let the matrix $R = (r_{nk})$ given by (3.1). Then,*

(a): $T \in \left(\ell_\infty\left(\tilde{S}\right), \ell_\infty\right)$ *if and only if (3.2) holds and*

$$\lim_{m\to\infty} \sum_k \left| \sum_{i=k}^{m} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \right| = \sum_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \right| \tag{3.13}$$

and

$$\sup_n \sum_k |r_{nk}| < \infty. \tag{3.14}$$

(b): $T \in \left(\ell_\infty\left(\tilde{S}\right), c\right)$ *if and only if Equations (3.2), (3.5) and (3.13) are valid and*

$$\lim_{n\to\infty} \sum_k |r_{nk}| = \sum_k \left| \lim_{n\to\infty} r_{nk} \right|. \tag{3.15}$$

(c): $T \in \left(\ell_\infty\left(\tilde{S}\right), c_0\right)$ *if and only if Equations (3.2), (3.6) and (3.13) are valid and*

$$\lim_{n\to\infty} \sum_k |r_{nk}| = 0. \tag{3.16}$$

(d): $T \in \left(\ell_\infty\left(\tilde{S}\right), \ell_1\right)$ *if and only if Equations (3.2) and (3.13) are valid and*

$$\sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} r_{nk} \right| < \infty. \tag{3.17}$$

Proof The proof of (a) comes from taking starting point as $T \in \left(\ell_\infty\left(\tilde{S}\right), \ell_\infty\right)$ if and only if $Tz \in \ell_\infty$ for all $z \in \ell_\infty\left(\tilde{S}\right)$ and from proceedings the manipulations as in the proofs of previous theorems given in this section. Since the proofs of the statements $(b), (c)$ and (d) are similar, we omit them. □

We conclude this section with further characterization of matrix classes $\left(\Gamma, \ell_p\left(\tilde{S}\right)\right)$ with $1 \leq p \leq \infty$, where $\Gamma \in \{\ell_\infty, c, c_0, \ell_1\}$.

Theorem 3.4 *Let* $1 \leq p \leq \infty$. *Then,*

$(a): \ T \in \left(\ell_\infty, \ell_p\left(\tilde{S}\right)\right) = \left(c, \ell_p\left(\tilde{S}\right)\right) = \left(c_0, \ell_p\left(\tilde{S}\right)\right)$ *if and only if*

$$\sup_{K \in F} \sum_{n=0}^{\infty} \left| \sum_{k \in K} \sum_{i=0}^{n} \frac{S_i S_{n-i}}{S_{n+1} - S_n} t_{ik} \right|^p < \infty.$$

$(b): \ T \in \left(\ell_1, \ell_p\left(\tilde{S}\right)\right)$ *if and only if*

$$\sup_{k} \sum_{n=0}^{\infty} \left| \sum_{i=0}^{n} \frac{S_i S_{n-i}}{S_{n+1} - S_n} t_{ik} \right|^p < \infty.$$

$(c): \ T \in \left(\ell_\infty, \ell_\infty\left(\tilde{S}\right)\right) = \left(c, \ell_\infty\left(\tilde{S}\right)\right) = \left(c_0, \ell_\infty\left(\tilde{S}\right)\right)$ *if and only if*

$$\sup_{n} \sum_{k=0}^{\infty} \left| \sum_{i=0}^{n} \frac{S_i S_{n-i}}{S_{n+1} - S_n} t_{ik} \right| < \infty.$$

$(d): \ T \in \left(\ell_1, \ell_\infty\left(\tilde{S}\right)\right)$ *if and only if*

$$\sup_{n,k} \left| \sum_{i=0}^{n} \frac{S_i S_{n-i}}{S_{n+1} - S_n} t_{ik} \right| < \infty.$$

Proof We prove the case $T \in \left(\ell_\infty, \ell_p\left(\tilde{S}\right)\right)$ for $p \geq 1$. Consider the matrix $\tilde{T} = \left(\tilde{t}_{nk}\right)$ as

$$\tilde{t}_{nk} = \sum_{i=0}^{n} \frac{S_i S_{n-i}}{S_{n+1} - S_n} t_{ik}, \quad \text{for all } n, k \in \mathbb{N}.$$

One has

$$\sum_{k=0}^{\infty} \tilde{t}_{nk} z_k = \sum_{i=0}^{n} \frac{S_i S_{n-i}}{S_{n+1} - S_n} \sum_{k=0}^{\infty} t_{ik} z_k, \quad \text{for any } z = (z_k) \in \ell_\infty,$$

from which $\tilde{T}_n(z) = \tilde{S}_n(Tz)$ for all $n \in \mathbb{N}$ and so that $Tz \in \ell_p\left(\tilde{S}\right)$ for $z = (z_k) \in \ell_\infty$ if and only if $\tilde{T}z \in \ell_p$ for $z = (z_k) \in \ell_\infty$. In conclusion, we reach that

$$\sup_{K \in F} \sum_{n=0}^{\infty} \left| \sum_{k \in K} \sum_{i=0}^{n} \frac{S_i S_{n-i}}{S_{n+1} - S_n} t_{ik} \right|^p < \infty.$$

The other expressions can be deduced similarly. □

4. Compact operators on the spaces $\ell_p\left(\tilde{S}\right)$ and $\ell_\infty\left(\tilde{S}\right)$

In this final section, the compactness of certain matrix operators are characterized with the help of Hausdorff measure of noncompactness. Let $z = (z_n) \in \omega$ and $\mathcal{S}_\Gamma$ be unit sphere in the BK-space $\Gamma \supset \Psi$, then, we adopt the following notation in the sequel

$$\|z\|_\Gamma^* = \sup_{s \in \mathcal{S}_\Gamma} \left| \sum_k z_k s_k \right| < \infty$$

under the assumption that the supremum is finite. In this case, observe that $z \in \Gamma^{\beta}$. Also, $\mathcal{C}(\Gamma,\Theta)$ denotes the set of all bounded (continuous) linear operators from Γ into Θ. Now, we list some lemmas, proved by [29].

Lemma 4.1 *We have*

$(a) : \ell_{\infty}^{\beta} = \ell_1$ *and* $\|z\|_{\ell_{\infty}}^{*} = \|z\|_{\ell_1}$ *for all* $z \in \ell_1$.

$(b) : \ell_{1}^{\beta} = \ell_{\infty}$ *and* $\|z\|_{\ell_1}^{*} = \|z\|_{\ell_{\infty}}$ *for all* $z \in \ell_{\infty}$.

$(c) : \ell_{p}^{\beta} = \ell_q$ *and* $\|z\|_{\ell_p}^{*} = \|z\|_{\ell_q}$ *for all* $z \in \ell_q$ *with* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 4.2 *For every* $T \in (\Gamma,\Theta)$ *with the BK-spaces* Γ *and* Θ, *there exists a linear operator* $\mathcal{L}_T \in \mathcal{C}(\Gamma,\Theta)$ *such that* $\mathcal{L}_T(z) = T(z)$ *for all* $z \in \Gamma$.

Lemma 4.3 *Let* $\Gamma \supset \Psi$ *be a BK-space and* $\Theta = \{c_0, c, \ell_{\infty}\}$. *If* $T \in (\Gamma,\Theta)$, *then, we have*

$$\|\mathcal{L}_T\| = \|T\|_{(\Gamma,\Theta)} = \sup_n \|T_n\|_{\Gamma}^{*} < \infty.$$

The Hausdorff measure of noncompactness of a bounded set Z is denoted by $\chi(Z)$ and defined as

$$\chi(Z) = \inf\left\{\delta > 0 : Z \subset \cup_{k=1}^{n} B(x_k, r_k), x_k \in \Gamma, r_k < \delta, n \in \mathbb{N} \setminus \{0\}\right\},$$

where $B(x_k, r_k)$ is the open ball centered at x_k and radius r_k for each $k = 1, 2, ..., n$. For more details about the Hausdorff measure of noncompactness, one can consult [29] and cited references therein.

Theorem 4.4 *([41, Theorem 2.8]) For* $1 \leq p < \infty$, *the Hausdorff measure of noncompactness of a bounded set* Z *in* ℓ_p *is computed as*

$$\chi(Z) = \lim_k \left(\sup_{z \in Z} \|\mathcal{I} - \mathcal{P}_k(z)\|_{\ell_p}\right).$$

Here $\mathcal{P}_k : \ell_p \to \ell_p$ *is the operator defined by* $\mathcal{P}_k(z) = (z_0, z_1, ..., z_k, 0, 0, ...)$ *for each* $k \in \mathbb{N}$ *and* $\mathcal{I} : \ell_p \to \ell_p$ *is the identity operator defined by* $\mathcal{I}(z) = (z_0, z_1, ..., z_k, z_{k+1}, ...)$.

A linear operator $\mathcal{L}$ from a Banach space Γ into another Banach space Θ is called a compact operator if the domain of $\mathcal{L}$ is all of Γ and the image under $\mathcal{L}$ of any bounded subset of Γ is a totally bounded subset of Θ, or equivalently, for every bounded sequence $z = (z_n)$ in Γ, the sequence $(\mathcal{L}(z_n))$ has a convergent subsequence in Θ.

The concepts of the Hausdorff measure of noncompactness and compact operators are linked one another. The necessary and sufficient condition for an operator to be compact is that $\|\mathcal{L}_{\chi}\| = 0$, where $\|\mathcal{L}_{\chi}\|$ denotes the Hausdorff measure of noncompactness of $\mathcal{L}$ and it is defined as $\|\mathcal{L}_{\chi}\| = \chi(\mathcal{L}(\mathcal{S}_{\Gamma}))$.

Let $x = (x_k) \in \left(\ell_p\left(\tilde{S}\right)\right)^{\beta}$ be a sequence, define a sequence $y = (y_k) \in \ell_q$ as

$$y_k = \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} x_i \tag{4.1}$$

for all $k \in \mathbb{N}$. We give some lemmas, needed in our proofs.

Lemma 4.5 *Let* $1 \le p \le \infty$ *and let* $x = (x_k) \in \left(\ell_p\left(\tilde{S}\right)\right)^\beta$. *Then,* $y = (y_k) \in \ell_q$ *and*

$$\sum_k x_k z_k = \sum_k y_k t_k$$

for all $z = (z_k) \in \ell_p\left(\tilde{S}\right)$.

Lemma 4.6 *Let* $y = (y_k)$ *be defined by (4.1), then,*

$(a) : \|x\|^*_{\ell_1(\tilde{S})} = \sup_k |y_k| < \infty$ *for all* $x = (x_k) \in \left(\ell_1\left(\tilde{S}\right)\right)^\beta$.

$(b) : \|x\|^*_{\ell_p(\tilde{S})} = \left(\sum_k |y_k|^q\right)^{1/q} < \infty$ *for all* $x = (x_k) \in \left(\ell_p\left(\tilde{S}\right)\right)^\beta$ *and for* $1 < p < \infty$.

$(c) : \|x\|^*_{\ell_\infty(\tilde{S})} = \sum_k |y_k| < \infty$ *for all* $x = (x_k) \in \left(\ell_\infty\left(\tilde{S}\right)\right)^\beta$.

Proof We only prove the assertion (c). For $x = (x_k) \in \left(\ell_\infty\left(\tilde{S}\right)\right)^\beta$, it follows from Lemma 4.5 that $y = (y_k) \in \ell_1$ and $\sum_k x_k z_k = \sum_k y_k t_k$ for all $z = (z_k) \in \ell_\infty\left(\tilde{S}\right)$. By the fact that $\ell_\infty\left(\tilde{S}\right)$ and ℓ_∞ are isomorphic, one finds $z \in \mathcal{S}_{\ell_\infty(\tilde{S})}$ if and only if $t \in \mathcal{S}_{\ell_\infty}$. Thus, $\|x\|^*_{\ell_\infty(\tilde{S})} = \sup_{z \in \mathcal{S}_{\ell_\infty(\tilde{S})}} \left|\sum_k x_k z_k\right| = \sup_{z \in \mathcal{S}_{\ell_\infty}} \left|\sum_k y_k t_k\right| = \|y\|^*_{\ell_\infty}$. Now, apply Lemma 4.1 to obtain

$$\|x\|^*_{\ell_\infty(\tilde{S})} = \|y\|^*_{\ell_\infty} = \|x\|_{\ell_1} = \sum_k |y_k| < \infty,$$

as desired. The other statements can be achieved similarly. □

Lemma 4.7 *([34, Theorem 3.7 and Theorem 3.11]) For any BK-space* $\Gamma \supset \Psi$, *we have*

$(a) :$ *If* $T \in (\Gamma, \ell_\infty)$, *then* $0 \le \|\mathcal{L}_T\|_\chi \le \limsup_n \|T_n\|^*_\Gamma$ *and* $\mathcal{L}_T$ *is compact if* $\lim_n \|T_n\|^*_\Gamma = 0$.

$(b) :$ *If* $T \in (\Gamma, c_0)$, *then* $\|\mathcal{L}_T\|_\chi \le \limsup_n \|T_n\|^*_\Gamma$ *and* $\mathcal{L}_T$ *is compact if and only if* $\lim_n \|T_n\|^*_\Gamma = 0$.

$(c) :$ *If* $T \in (\Gamma, \ell_1)$, *then*

$$\lim_r \left(\sup_{N \in F_r} \left\|\sum_{n \in N} T_n\right\|^*_\Gamma\right) \le \|\mathcal{L}_T\|_\chi \le 4 \lim_r \left(\sup_{N \in F_r} \left\|\sum_{n \in N} T_n\right\|^*_\Gamma\right)$$

and $\mathcal{L}_T$ *is compact if and only if* $\lim_r \left(\sup_{N \in F_r} \left\|\sum_{n \in N} T_n\right\|^*_\Gamma\right) = 0$.

Now, let us present a lemma, whose proof follows from Lemma 4.5. For this purpose, we define an infinite matrix $\hat{T} = (\hat{t}_{nk})$ as

$$\hat{t}_{nk} = \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \tag{4.2}$$

for all $n, k \in \mathbb{N}$ under the assumption that the series is convergent.

Lemma 4.8 *Let* Γ *be any sequence space and* $T=(t_{ni})$ *be an infinite matrix. If* $T\in\left(\ell_p\left(\tilde{S}\right),\Gamma\right)$, *then* $\hat{T}\in(\ell_p,\Gamma)$ *and* $Tz=\hat{T}t$ *hold for all* $z\in\ell_p\left(\tilde{S}\right)$ *for* $1\leq p\leq\infty$.

Theorem 4.9 *Let* $1<p<\infty$. *Then, we have*

(a) : If $T\in\left(\ell_p\left(\tilde{S}\right),\ell_\infty\right)$, then

$$0\leq\|\mathcal{L}_T\|_\chi\leq\limsup_n\left(\sum_k\left|\sum_{i=k}^{\infty}(-1)^{i-k}\frac{S_{k+1}-S_k}{S_i}P_{i-k}t_{ni}\right|^q\right)^{1/q}$$

and $\mathcal{L}_T$ is compact if

$$\lim_n\left(\sum_k\left|\sum_{i=k}^{\infty}(-1)^{i-k}\frac{S_{k+1}-S_k}{S_i}P_{i-k}t_{ni}\right|^q\right)^{1/q}=0.$$

(b) : If $T\in\left(\ell_p\left(\tilde{S}\right),c_0\right)$, then

$$\|\mathcal{L}_T\|_\chi=\limsup_n\left(\sum_k\left|\sum_{i=k}^{\infty}(-1)^{i-k}\frac{S_{k+1}-S_k}{S_i}P_{i-k}t_{ni}\right|^q\right)^{1/q}$$

and $\mathcal{L}_T$ is compact if and only if

$$\lim_n\left(\sum_k\left|\sum_{i=k}^{\infty}(-1)^{i-k}\frac{S_{k+1}-S_k}{S_i}P_{i-k}t_{ni}\right|^q\right)^{1/q}=0.$$

(c) : If $T\in\left(\ell_p\left(\tilde{S}\right),\ell_1\right)$, then

$$\lim_r\|T\|^r_{(\ell_p(\tilde{S}),\ell_1)}\leq\|\mathcal{L}_T\|_\chi\leq 4\lim_r\|T\|^r_{(\ell_p(\tilde{S}),\ell_1)}$$

and $\mathcal{L}_T$ is compact if and only if

$$\lim_r\|T\|^r_{(\ell_p(\tilde{S}),\ell_1)}=0,$$

where

$$\|T\|^r_{(\ell_p(\tilde{S}),\ell_1)}=\sup_{N\in\mathcal{F}_r}\left(\sum_k\left|\sum_{n\in N}\sum_{i=k}^{\infty}(-1)^{i-k}\frac{S_{k+1}-S_k}{S_i}P_{i-k}t_{ni}\right|^q\right)^{1/q}.$$

Proof We begin with the proof of (a). Let $T\in\left(\ell_p\left(\tilde{S}\right),\ell_\infty\right)$ and $z\in\ell_p\left(\tilde{S}\right)$, then from the convergence of the series $\sum_{k=0}^{\infty}t_{nk}z_k$ that $T_n\in\left(\ell_p\left(\tilde{S}\right)\right)^\beta$ for each $n\in\mathbb{N}$. Using Lemma 4.6 (b), it yields $\|T_n\|^*_{\ell_p(\tilde{S})}=\left(\sum_k\left|\hat{t}_{nk}\right|^q\right)^{1/q}$ for each $n\in\mathbb{N}$, where $\hat{t}_{nk}$ is given by (4.2). So, apply Lemma 4.7 (a) to get

$$0\leq\|\mathcal{L}_T\|_\chi\leq\limsup_n\left(\sum_k\left|\hat{t}_{nk}\right|^q\right)^{1/q}$$

and $\mathcal{L}_T$ is compact if

$$\lim_n \left(\sum_k \left| \hat{t}_{nk} \right|^q \right)^{1/q} = 0.$$

For the proof of (b), let $T \in \left(\ell_p \left(\tilde{S} \right), c_0 \right)$, then, taking into consideration $\| T_n \|^*_{\ell_p(\tilde{S})} = \left(\sum_k \left| \hat{t}_{nk} \right|^q \right)^{1/q}$ and using Lemma 4.7 (b), we obtain that

$$\| \mathcal{L}_T \|_\chi = \limsup_n \left(\sum_k \left| \hat{t}_{nk} \right|^q \right)^{1/q}$$

and $\mathcal{L}_T$ is compact if and only if

$$\lim_n \left(\sum_k \left| \hat{t}_{nk} \right|^q \right)^{1/q} = 0.$$

For the proof of (c), let $T \in \left(\ell_p \left(\tilde{S} \right), \ell_1 \right)$, then, it follows from Lemma 4.6 that $\left\| \sum_{n \in N} T_n \right\|^*_{\ell_p\left(\tilde{S}\right)} = \left\| \sum_{n \in N} \hat{T}_n \right\|^*_{\ell_q}$. So, employing Lemma 4.7 (c), it leads to

$$\lim_r \left(\sup_{N \in \mathcal{F}_r} \sum_k \left| \sum_{n \in N} \hat{t}_{nk} \right|^q \right)^{1/q} \leq \| \mathcal{L}_T \|_\chi \leq 4 \lim_r \left(\sup_{N \in \mathcal{F}_r} \sum_k \left| \sum_{n \in N} \hat{t}_{nk} \right|^q \right)^{1/q}$$

and $\mathcal{L}_T$ is compact if and only if

$$\lim_r \left(\sup_{N \in \mathcal{F}_r} \sum_k \left| \sum_{n \in N} \hat{t}_{nk} \right|^q \right)^{1/q} = 0.$$

Consequently, the proofs are completed. □

We conclude the study with the following two theorems, whose proofs can be made similar to Theorem 4.9.

Theorem 4.10 *We have*

(a) : If $T \in \left(\ell_\infty \left(\tilde{S} \right), \ell_\infty \right)$, then

$$0 \leq \| \mathcal{L}_T \|_\chi \leq \limsup_n \sum_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \right|$$

and $\mathcal{L}_T$ is compact if

$$\limsup_n \sum_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \right| = 0.$$

(b) : If $T \in \left(\ell_\infty \left(\tilde{S} \right), c_0 \right)$, then

$$\| \mathcal{L}_T \|_\chi = \limsup_n \sum_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1} - S_k}{S_i} P_{i-k} t_{ni} \right|$$

and $\mathcal{L}_T$ is compact if and only if

$$\lim_n \sum_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1}-S_k}{S_i} P_{i-k} t_{ni} \right| = 0.$$

(c) : If $T \in \left(\ell_\infty\left(\tilde{S}\right), \ell_1\right)$, then

$$\lim_r \|T\|^r_{\left(\ell_\infty(\tilde{S}),\ell_1\right)} \le \|\mathcal{L}_T\|_\chi \le 4 \lim_r \|T\|^r_{\left(\ell_\infty(\tilde{S}),\ell_1\right)}$$

and $\mathcal{L}_T$ is compact if and only if

$$\lim_r \|T\|^r_{\left(\ell_\infty(\tilde{S}),\ell_1\right)} = 0,$$

where

$$\|T\|^r_{\left(\ell_\infty(\tilde{S}),\ell_1\right)} = \sup_{N \in F_r} \sum_k \left| \sum_{n \in N} \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1}-S_k}{S_i} P_{i-k} t_{ni} \right|.$$

Theorem 4.11 *We have*

(a) : If $T \in \left(\ell_1\left(\tilde{S}\right), \ell_\infty\right)$, then

$$0 \le \|\mathcal{L}_T\|_\chi \le \limsup_n \left(\sup_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1}-S_k}{S_i} P_{i-k} t_{ni} \right| \right)$$

and $\mathcal{L}_T$ is compact if

$$\lim_n \left(\sup_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1}-S_k}{S_i} P_{i-k} t_{ni} \right| \right) = 0.$$

(b) : If $T \in \left(\ell_1\left(\tilde{S}\right), c_0\right)$, then

$$\|\mathcal{L}_T\|_\chi = \limsup_n \left(\sup_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1}-S_k}{S_i} P_{i-k} t_{ni} \right| \right)$$

and $\mathcal{L}_T$ is compact if and only if

$$\lim_n \left(\sup_k \left| \sum_{i=k}^{\infty} (-1)^{i-k} \frac{S_{k+1}-S_k}{S_i} P_{i-k} t_{ni} \right| \right) = 0.$$

5. Conclusion

The approach of constructing a new sequence space by means of a matrix domain has been employed by many scholars. In general, the first aim is to identify its special duals and some matrix transformations and to characterize its compact operators and the Hausdorff measure of noncompactness. In recent years, these topics have been investigated by introducing a new conservative matrix, whose terms are some fascinating special numbers and number theoretic functions. By adopting these considerations, we here deal with the domain of a recently defined conservative matrix, constructed by means of the Schröder numbers in the spaces of $p-$absolutely summable sequences and bounded sequences. We determine the $\beta-$duals of the newly defined Banach spaces and present characterization of some matrix operators and of certain compact operators by means of the Hausdorff measure of noncompactness.

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