Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2321 – 2334
© TÜBİTAK
doi:10.55730/1300-0098.3271

On properties of the Reeb vector field of (α, β) trans-Sasakian structure

Alexander YAMPOLSKY[*]
Department of Pure Mathematics, Faculty of Mathematics and Computer Science,
V.N. Karazin Kharkiv National University, Kharkiv, Ukraine

Received: 07.01.2022 • **Accepted/Published Online:** 06.05.2022 • **Final Version:** 04.07.2022

Abstract: The paper focused on the mean curvature and totally geodesic property of the Reeb vector field ξ on (α, β) trans-Sasakian manifold M of dimension $(2n+1)$ as a submanifold in the unit tangent bundle T_1M with Sasaki metric g_S. We give an explicit formula for the norm of mean curvature vector of the submanifold $\xi(M) \subset (T_1M, g_S)$. As a byproduct, for the Reeb vector field, we get some known results concerning its minimality, harmonicity and the property to define a harmonic map. We prove that on connected proper trans-Sasakian manifold the Reeb vector field does not give rise to totally geodesic submanifold in T_1M. On α-Sasakian the Reeb vector field is totally geodesic only if $\alpha = 1$. On β-Kenmotsu manifold the Reeb vector field is totally geodesic if and only if $\nabla\beta = \frac{\beta^2(1+\beta^2)}{1-\beta^2}\xi$. If M is compact, then $\beta = 0$.

Key words: Mean curvature of vector field, totally geodesic unit vector field, trans-Sasakian manifold, Reeb vector field, minimal vector field, harmonic vector field, harmonic map

1. Introduction

A unit vector field ξ on the Riemannian manifold (M, g) defines a mapping $\xi : M \to T_1M$ into a unit tangent bundle T_1M. The canonic Riemannian metric g_S (the Sasakian one) on the tangent bundle TM gives rise to Riemannian metric on the image $\xi(M) \subset T_1M \subset TM$. In this way $\xi(M)$ endowed with the intrinsic and extrinsic geometry. If ξ is globally defined, then ξ is called minimal if $\xi(M)$ has minimal volume under variation of the vector field. H. Gluck and W. Ziller [9] proved that the Hopf vector field on S^3 is minimal within a class of vector field variations. O. Gil-Medrano [7] proved that if ξ is minimal under vector field variations, then it is minimal under variations of general type. In other words, minimality of ξ is equivalent to minimality of $\xi(M) \subset (T_1M, g_S)$ as a submanifold. It follows that minimality of $\xi(M)$ make sense even if ξ is defined locally. The necessary and sufficient conditions on the vector field to be (locally) minimal was found in [7, 8] in terms of hard-to-check equations. Nevertheless, by using of this criteria a number of examples of minimal unit vector fields were found (see, e.g., [10], [11], [16]). In a wider scope we say that ξ is minimal/totally geodesic/constant mean curvature if $\xi(M) \subset (T_1M, g_S)$ possesses the same property; the vector field ξ is called of constant sectional curvature if $\xi(M) \subset (T_1M, g_S)$ is of constant sectional curvature, etc. A little number of examples of the vector field of constant mean curvature [21] and constant sectional curvature [22] was found.

From the other point of view a unit vector field can be considered as a (nonisometric in general) mapping

[*]Correspondence: a.yampolsky@karazin.ua

2010 *AMS Mathematics Subject Classification:* 53C25

$\xi : (M, g) \to (T_1M, g_S)$. So one can assign to ξ a tension of mapping in a sense of J. Eells and L. Lemaire [4]. It was proved that the tension of mapping $\xi : (M, g) \to (T_1M, g_S)$ can be expressed in terms of the rough Laplacian $\bar{\Delta}\xi$ and $\rho_\xi(X) = trace\big(Y \to R(\xi, \nabla_Y\xi)X\big)$. A unit vector field is said to be harmonic [20], if $\bar{\Delta}\xi = |\nabla\xi|^2\xi$. It defines a harmonic map if, in addition, $\rho_\xi(X) = 0$ for all $X \in \mathscr{X}(M)$.

Minimality and harmonicity for the Hopf vector fields on S^{2n+1} was proved in [17] and their totally geodesic property was proved in [23], minimality of the characteristic/Reeb vector field of the Sasakian structure was proved in [8]. Minimal or harmonic properties of the characteristic vector field on general or three-dimensional contact metric manifolds was treated in [11]. All minimal left-invariant unit vector fields on 3-dimensional Lie groups with a left-invariant metric was described in[16] and subclass of the fields with totally geodesic property was separated in [27]. All harmonic left-invariant unit vector fields which define a harmonic map was found by J.C. González- avila and L. Vanhecke [12]. The totally geodesic (local) unit vector fields on 2-dimensional manifolds was described in [25].

It happened that in case of 3-dimensional Lie group the totally geodesic left-invariant unit vector field defines an almost contact structure on the group. A wide class of so-called (α, β) trans-Sasakian almost contact metric manifolds was introduced by J.A. Oubiña [15]. The $(\alpha, 0)$ - structure is called α-Sasakian. The 1-Sasakian structure is called *Sasakian* [1], the 0-Sasakian structure is called cosymplectic [5]. The $(0, \beta)$-structure is called β- *Kenmotsu*, the 1-Kenmotsu structure is called *Kenmotsu* [13]. The $(\alpha \neq 0, \beta \neq 0)$ trans-Sasakian manifold is called the proper one.

Harmonic properties of the almost contact structures including α-Sasakian, β-Kenmotsu and (α, β) trans-Sasakian structure in case of $dim(M) \geq 5$ was considered by E. Vergara-Diaz and C. M. Wood in [18]. As byproduct, the authors reproved J.C. Marrero result [14] stating that if $dim(M) \geq 5$, then the manifold is either α-Sasakian with $\alpha = const$ or β-Kenmotsu. The 3-dimensional proper trans-Sasakian case was treated in[19]. The minimality, harmonicity of the Reeb vector field, its property to define a harmonic map and their interactions was clarified in details.

In this paper we focus on mean curvature of the Reeb vector field and its totally geodesic property by using the explicit expression for the second fundamental form of the submanifold $\xi(M) \subset (T_1M, g_S)$ and explicit expression for its mean curvature vector. We also treat the interaction between the second fundamental form of mapping $\xi : M \to (T_1M, g_S)$ and the second fundamental form of $\xi(M) \subset (T_1M, g_S)$ in case of (α, β) trans-Sasakian structure. As a byproduct we reprove the results of E. Vergara-Diaz and C. M. Wood [18], Y. Wang [19] concerning minimal and harmonic properties.

The main results are the following Theorems.

Theorem 3.1 *The norm of the mean curvature vector H_ξ for the Reeb vector field ξ on (α, β) trans-Sasakian manifold of $\dim M = 2n + 1$ is of the form*

$$|H_\xi| = \frac{\big|(1 + \alpha^2 + \beta^2)(\varphi^2\nabla\alpha - \varphi\nabla\beta) + (n-1)\varphi^2\big(\alpha\nabla(\alpha^2 + \beta^2) + \beta\varphi\nabla(\alpha^2 + \beta^2)\big)\big|}{(2n+1)(1 + \alpha^2 + \beta^2)^{3/2}}.$$

As a consequence, we get

- if $\dim(M) > 3$, then $\beta = 0$ and $\alpha = const$ and hence $|H_\xi| = \dfrac{(1 + (2n-1)\alpha^2)}{(2n+1)(1 + \alpha^2)^{3/2}}|\varphi^2\nabla\alpha| = 0;$

- if $\dim(M) > 3$ and $\alpha = 0$, then $|H_\xi| = \dfrac{(1+(2n-1)\beta^2)}{(2n+1)(1+\beta^2)^{3/2}}|\varphi\nabla\beta|$;

- if $\dim(M) = 3$, then $|H_\xi| = \frac{|\varphi^2\nabla\alpha - \varphi\nabla\beta|}{3\sqrt{1+\alpha^2+\beta^2}}$.

It is worthwhile to mention that if $\alpha^2+\beta^2 = const$ on 3-dimensional compact simply connected trans-Sasakian manifold, then it is homothetic to Sasakian manifold [3] with $\alpha = const$ and hence $|H_\xi| = 0$.

Theorem 3.4 *The Reeb vector field on connected (α,β) trans-Sasakian manifold M gives rise to totally geodesic submanifold $\xi(M) \subset (T_1M, g_S)$ only in the following cases*

- *$\beta = 0, \alpha = 1$ and hence M is Sasakian or $\beta = 0, \alpha = 0$ and hence M is cosymplectic;*

- *$\alpha = 0$ and $\nabla\beta = \dfrac{\beta^2(\beta^2+1)}{1-\beta^2}\xi$. If $\beta = const$ or M is compact, then $\beta = 0$ and hence M is cosymplectic.*

The paper is organized as follows. Section 2 contains necessary general facts on geometry of unit vector fields and necessary facts from the geometry of trans-Sasakian manifolds. Section 3 contains the proofs of main results. Section 4 contains some remarks concerning interrelations between minimal and harmonic properties of the Reeb vector field, applications of general results on totally geodesic unit vector fields to the Reeb vector field of (α,β) trans-Sasakian manifold.

2. Basic definitions

2.1. Unit vector field as a mapping $\xi : M \to T_1M$

Denote by g_S the Sasaki metric on the tangent bundle TM of the Riemannian manifold (M,g) (see, e.g., [1], Section 9 for details). Denote by $\mathscr{X}(M)$ the Lie algebra of smooth vector fields on M. There exists a special tangent frame on TM consisting of so-called vertical $(\cdot)^v$ and horizontal $(\cdot)^h$ lifts of vector fields on M. The vertical lift is tangent to the fiber of TM while the horizontal is transverse to the fiber. In term of lifts the Sasaki metric completely define by scalar products

$$g_S(X^h,Y^h) = g(X,Y), \quad g_S(X^h,Y^v) = 0, \quad g_S(X^v,Y^v) = g(X,Y)$$

for any vector fields $X, Y \in \mathscr{X}(M)$. The vertical and horizontal distributions are mutually orthogonal with respect to Sasaki metric. At each point $(q,\xi) \in T_1M$ the vertical lift ξ^v is a unit normal to the hypersurface $T_1M \subset TM$ which is defined by equation $g(\xi,\xi) = 1$.

The second fundamental form of the submanifold $\xi(M) \subset (T_1M, g_S)$ is closely related to the second fundamental form of mapping $\xi : M \to (T_1M, g_S)$ but is not the same.

Define the ξ-orthogonal distribution by $\mathcal{D}_\xi = \{X \,|\, g(X,\xi) = 0\}$. The so-called Nomizu operator for a **unit** vector field ξ is defined by

$$A_\xi X = -\nabla_X \xi. \tag{2.1}$$

Evidently, $A_\xi : \mathscr{X}(M) \to \mathcal{D}_\xi$. The conjugate (transposed) Nomizu operator is defined in a standard way by

$$g(A_\xi^t X, Y) = g(X, A_\xi Y). \tag{2.2}$$

The tangent bundle of $\xi(M)$ is generated by differential $\xi_*: TM \to T(T_1M)$ acting as

$$\xi_* X = X^h - (A_\xi X)^v \tag{2.3}$$

for all $X \in \mathscr{X}(M)$.

The Sasaki metric on T_1M gives rise to Riemannian metric on $\xi(M)$ by

$$g_S(\xi_* X, \xi_* Y) = g(X,Y) + g(A_\xi X, A_\xi Y).$$

The normal bundle of $\xi(M)$ is defined [23] by mapping $n: \mathcal{D}_\xi \to T_1M$ acting as

$$n(Z) = (A_\xi^t Z)^h + (Z - g(Z,\xi)\xi)^v \tag{2.4}$$

for all $Z \in \mathscr{X}(M)$. In this way (2.3) and (2.4) define tangent and normal framing for $\xi(M) \subset T_1M$.

In general a second fundamental form of a smooth mapping $f:(M,g) \to (N,h)$ is defined by

$$B_f(X,Y) = \nabla^f_{f_* X}(f_* Y) - f_*(\nabla^g_X Y),$$

where ∇^f is the Levi–Civita connection on $f(M) \subset N$ induced by h and ∇^g is the Levi–Civita connection on M. The tension field $\tau(f)$ of the mapping f is defined by $\tau(f) = trace(B_f) = \sum B_f(e_i, e_i)$ with respect to the orthonormal frame. The mapping $f(M,g) \to (N,h)$ is said to be harmonic if $\tau(f) = 0$.

In application to the unit vector field, the second fundamental form of the mapping $\xi:(M,g) \to (T_1M, g_S)$ is

$$B_\xi(X,Y) = \frac{1}{2}\Big(R(A_\xi X,\xi)Y + R(A_\xi Y,\xi)X\Big)^h + \frac{1}{2}\Big((\nabla^2\xi)(X,Y) + (\nabla^2\xi)(Y,X)\Big)^v\Big\downarrow_{\mathcal{D}^v_\xi}, \tag{2.5}$$

where $(\nabla^2\xi)(X,Y) = X^iY^j\nabla_i\nabla_j\xi = \nabla_X\nabla_Y\xi - \nabla_{\nabla_XY}\xi$ and $(\downarrow)$ means projection on $\mathcal{D}^v_\xi = \{Z^v \mid Z \in \mathcal{D}_\xi\}$. A unit section $\xi:(M,g) \to (T_1M, g_S)$ defines a harmonic map if $trace(B_\xi) = 0$.

The $-trace(\nabla^2\xi)$ is known as the *rough Laplacian* $\bar\Delta\xi$. Therefore, it is natural to say that $-\frac{1}{2}\big((\nabla^2\xi)(X,Y) + (\nabla^2\xi)(Y,X)\big)$ is a rough Hessian for ξ. It is easy to see, that $-(\nabla^2\xi)(X,Y) = (\nabla_X A_\xi)Y$ and we define ξ*-rough Hessian* by [23]

$$Hess_\xi(X,Y) = \frac{1}{2}\big((\nabla_X A_\xi)Y + (\nabla_Y A_\xi)X\big).$$

Introduce a tensor field [23]* by

$$Hm_\xi(X,Y) = \tfrac{1}{2}\big(R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X\big). \tag{2.6}$$

So we may write

$$-B_\xi(X,Y) = \big(Hm_\xi(X,Y)\big)^h + \big(Hess_\xi(X,Y)\big)^v\big\downarrow_{\mathcal{D}^v_\xi}$$

and as a consequence

$$-\tau(\xi) = \big(trace(Hm_\xi)\big)^h + \big(\bar\Delta\xi\big)^v\big\downarrow_{\mathcal{D}^v_\xi}.$$

*Here we change the sign of Hm_ξ comparably with [23].

A unit vector field is said to be harmonic [20] if

$$\bar{\Delta}\xi = |A_\xi|^2\xi, \tag{2.7}$$

where $|A_\xi|^2 = \sum_i |A_\xi e_i|^2$ with respect to some orthonormal frame.

It is clear now that ξ defines a harmonic map $\xi : (M, g) \to (T_1M, g_S)$ if and only if ξ is harmonic and

$$trace\, Hm_\xi = 0. \tag{2.8}$$

In what follows we refer to (2.6) as to ξ-*harmonicity tensor*.

As it was proved in [23], the second fundamental form of the submanifold $\xi(M) \subset T_1M$ with respect to normal vector field (2.4) can be expressed in terms of rough Laplacian and the ξ-harmonicity tensor as follows:

$$\tilde{\Omega}_{n(Z)}(\xi_*X, \xi_*Y) = g(Hess_\xi(X, Y) + A_\xi Hm_\xi(X, Y), Z),$$

where $Z \in \mathcal{D}_\xi$. It follows [23] that $\xi(M) \subset T_1M$ is *totally geodesic* if and only if

$$\Omega_\xi(X, Y) = Hess_\xi(X, Y) + A_\xi Hm_\xi(X, Y) - g(A_\xi X, A_\xi Y)\,\xi = 0 \tag{2.9}$$

for all $X, Y \in \mathscr{X}(M)$.

The minimality conditions for the submanifold $\xi(M) \subset T_1M$ was found in [8] by using the variational approach. These conditions do not allow to get the expression for mean curvature vector of $\xi(M)$. An explicit expression for mean curvature of the $\xi(M)$ and, as a consequence, the alternative minimality conditions was found by using a singular frame for the Nomizu operator [23]. By dimension reasons, there always exists (at least locally) a unit vector field e_0 such that $A_\xi e_0 = 0$. Since $A_\xi^t\xi = 0$, there exist the orthonormal frames $e_0, e_1, \ldots, e_m$ and $f_0 = \xi, f_1, \ldots f_m$ such that

$$Ae_i = \lambda_i f_i, \quad A^t f_i = \lambda_i e_i \quad (i = 0, \ldots, m).$$

The functions $\lambda_0 = 0, \lambda_1, \ldots, \lambda_m$ are square roots of eigenvalues of symmetric $(1, 1)$ tensor field $A_\xi^t A_\xi$. With respect to orthonormal frame

$$n(f_k) = \frac{1}{\sqrt{1+\lambda_i}}\big(\lambda_i e_k^h + f_k^v\big) \quad (k = 1, \ldots, m)$$

the components of mean curvature vector takes the form [21]

$$H_{k|} = \frac{1}{(m+1)\sqrt{1+\lambda_k^2}}\left(\sum_{i=0}^{m}\frac{g\big((\nabla_{e_i}A_\xi)e_i, f_k\big) + \lambda_k\lambda_i g\big(R(\xi, f_i)e_i, e_k\big)}{1+\lambda_i^2}\right) \quad (k = 1, \ldots, m), \tag{2.10}$$

or, equivalently,

$$H_{k|} = \frac{1}{(m+1)\sqrt{1+\lambda_k^2}}\left(\sum_{i=0}^{m}\frac{g\big((\nabla_{e_i}A_\xi)e_i, f_k\big) + g\big(A_\xi R(\xi, A_\xi e_i)e_i, f_k\big)}{1+\lambda_i^2}\right) \quad (k = 1, \ldots, m), \tag{2.11}$$

where R is the curvature tensor.

In case of the Reeb vector field on α-Sasakian, β-Kenmotsu or (α, β) trans-Sasakian manifolds the totally geodesic equation (2.9) and expressions for the mean curvature (2.10), (2.11) become extremely simple.

2.2. Trans-Sasakian structure

According to D. Blair [1], an almost contact metric structure on a smooth differentiable manifold (M^{2n+1}, g) of dimension $2n+1$ consists of a unit vector field ξ, $(1,1)$ tensor field φ and 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \tag{2.12}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.13}$$

for all vector fields on the manifold. It easily follows that φ is skew symmetric and

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad \varphi^t = -\varphi \tag{2.14}$$

and orthogonal being restricted on $\ker \eta$

$$(\varphi^t \varphi)X = (\varphi\varphi^t)X = X \tag{2.15}$$

for any $X \in \ker \eta$.

The unit vector field ξ is called characteristic or the Reeb vector field of contact metric manifold. The almost contact metric structure is denoted by (φ, ξ, η, g).

An almost contact metric structure (φ, ξ, η, g) is called a *trans-Sasakian* (α, β)- structure [15] if

$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X) \tag{2.16}$$

for some smooth functions $\alpha, \beta : M^{2n+1} \to R$.

In trans-Sasakian manifold [6]

$$A_\xi X = -\nabla_X \xi = \alpha\varphi X - \beta(X - \eta(X)\xi) = \alpha\varphi X + \beta\varphi^2 X, \tag{2.17}$$

$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{2.18}$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y), \tag{2.19}$$

$$\begin{aligned} R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y)) &+ 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) \\ &+ Y(\alpha)\varphi X - X(\alpha)\varphi Y + Y(\beta)\varphi^2 X - X(\beta)\varphi^2 Y, \end{aligned} \tag{2.20}$$

$$\begin{aligned} R(\xi, Y)X = (\alpha^2 - \beta^2)\big(g(Y, X)\xi - \eta(X)Y\big) &+ 2\alpha\beta\big(g(\varphi X, Y)\xi + \eta(X)\varphi Y\big) \\ &+ X(\alpha)\varphi Y - X(\beta)\varphi^2 Y + g(\varphi X, Y)\nabla\alpha - g(\varphi Y, \varphi X)\nabla\beta, \end{aligned} \tag{2.21}$$

$$Ric(\xi) = \varphi\nabla\alpha - \varphi^2\nabla\beta - 2n\nabla\beta + 2n(\alpha^2 - \beta^2)\xi. \tag{2.22}$$

The functions α and β are not independent [6, 18] but related by

$$\xi(\alpha) + 2\alpha\beta = 0. \tag{2.23}$$

3. Mean curvature and totally geodesic property of the Reeb vector field

We begin with simplification of (2.10) or (2.11) in application tho the Reeb vector field.

Lemma 3.1 *The norm of the mean curvature vector H_ξ for the Reeb vector field ξ on (α,β) trans-Sasakian manifold is*

$$|H_\xi| = \frac{1}{(2n+1)(1+\alpha^2+\beta^2)^{3/2}}|\varphi\big(\bar\Delta\xi + A_\xi trace(Hm_\xi)\big)|.$$

Proof To apply (2.11), let us find a singular frame. For the Nomizu operator we have $A_\xi\xi = 0$ while for the restriction A_ξ on $\mathcal{D}_\xi$ we have

$$A_\xi|_{\mathcal{D}_\xi} = \alpha\varphi - \beta I.$$

It follows

$$(A_\xi^t A_\xi)|_{\mathcal{D}_\xi} = (\alpha\varphi-\beta I)^t(\alpha\varphi-\beta I) = \alpha^2\varphi^t\varphi - \alpha\beta\varphi^t - \alpha\beta\varphi + \beta^2 I = (\alpha^2+\beta^2)I,$$

since $\varphi^t = -\varphi$ and $\varphi^t\varphi = I$. Therefore, the singular orthormal frame can be chosen by

$$e_0 = f_0 = \xi, \{e_i\}\in\mathcal{D}_\xi, \quad f_i = \frac{1}{\sqrt{\alpha^2+\beta^2}}(\alpha\varphi e_i - \beta e_i) \quad (i = 1,\dots,2n)$$

and all singular numbers $\lambda_k = \sqrt{\alpha^2+\beta^2}$. In this case the (2.11) simplifies to

$$H_{k|} = \frac{1}{(2n+1)(1+\alpha^2+\beta^2)^{3/2}}\Big(g(\bar\Delta\xi, f_k) + g(A_\xi trace(Hm_\xi), f_k)\Big) =$$
$$\frac{1}{(2n+1)(1+\alpha^2+\beta^2)^{3/2}}\Big(g(\bar\Delta\xi + A_\xi trace(Hm_\xi), f_k)\Big). \quad (3.1)$$

Since the frame $\{f_1,\dots,2n\}$ is orthonormal,

$$|H_\xi| = \frac{1}{(2n+1)(1+\alpha^2+\beta^2)^{3/2}}\big|\varphi\big(\bar\Delta\xi + A_\xi trace(Hm_\xi)\big)\big|.$$

□

Now we calculate the ξ-rough Hessian and ξ-harmonicity tensor.

Lemma 3.2 *If ξ is the Reeb vector field on (α,β) trans-Sasakian manifold, then*

$$\begin{aligned}2Hess_\xi(X,Y) = &-\big[Y(\beta) + (\alpha^2-\beta^2)\eta(Y)\big]X - \big[X(\beta)+(\alpha^2-\beta^2)\eta(X)\big]Y\\ &+\big[Y(\alpha) - 2\alpha\beta\eta(Y)\big]\varphi X + \big[X(\alpha)-2\alpha\beta\eta(X)\big]\varphi Y\\ &+\big[\big(X(\beta)-2\beta^2\eta(X)\big)\eta(Y) + \big(Y(\beta)-2\beta^2\eta(Y)\big)\eta(X) + 2(\alpha^2+\beta^2)g(X,Y)\big]\xi \quad (3.2)\end{aligned}$$

$$\begin{aligned}2Hm_\xi(X,Y) = &-\big[\alpha Y(\alpha)+\beta Y(\beta)+\beta\eta(Y)(\alpha^2+\beta^2)\big]X - \big[\alpha X(\alpha)+\beta X(\beta)+\beta\eta(X)(\alpha^2+\beta^2)\big]Y+\\ &\big[\alpha Y(\beta)-\beta Y(\alpha)-\alpha\eta(Y)(\alpha^2+\beta^2)\big]\varphi X + \big[\alpha X(\beta)-\beta X(\alpha)-\alpha\eta(X)(\alpha^2+\beta^2)\big]\varphi Y\\ &+\big[2\beta(\alpha^2+\beta^2)g(X,Y)+\eta(X)(\alpha Y(\alpha)+\beta Y(\beta))+\eta(Y)(\alpha X(\alpha)+\beta X(\beta))\big]\xi\\ &+2g(\varphi X,\varphi Y)[\alpha\nabla\alpha+\beta\nabla\beta] \quad (3.3)\end{aligned}$$

for all $X, Y \in \mathscr{X}(M)$.

Proof Using (2.17)–(2.19) a direct computation yields (cf. [2])

$$(\nabla_X A_\xi)Y = X(\alpha)\varphi Y + X(\beta)\varphi^2 Y - 2\alpha\beta\eta(Y)\varphi X - (\alpha^2 - \beta^2)\eta(Y)X \\ + \big[(\alpha^2+\beta^2)g(X,Y) - 2\beta^2\eta(X)\eta(Y)\big]\xi. \quad (3.4)$$

Therefore,

$$2Hess_\xi(X,Y) = \big[X(\alpha)\varphi Y + Y(\alpha)\varphi X\big] + \big[X(\beta)\varphi^2 Y + Y(\beta)\varphi^2 X\big] \\ - 2\alpha\beta\big[\eta(Y)\varphi X + \eta(X)\varphi Y\big] - (\alpha^2-\beta^2)\big[\eta(Y)X + \eta(X)Y\big] + 2\big[(\alpha^2+\beta^2)g(X,Y) - 2\beta^2\eta(X)\eta(Y)\big]\xi = \\ -\big[Y(\beta) + (\alpha^2-\beta^2)\eta(Y)\big]X - \big[X(\beta) + (\alpha^2-\beta^2)\eta(X)\big]Y + \big[Y(\alpha) - 2\alpha\beta\eta(Y)\big]\varphi X + \big[X(\alpha) - 2\alpha\beta\eta(X)\big]\varphi Y \\ + \Big[\big(X(\beta) - 2\beta^2\eta(X)\big)\eta(Y) + \big(Y(\beta) - 2\beta^2\eta(Y)\big)\eta(X) + 2(\alpha^2+\beta^2)g(X,Y)\Big]\xi.$$

The ξ-harmonicity tensor takes the form

$$2Hm_\xi(X,Y) = \alpha\big(R(\xi,\varphi X)Y + R(\xi,\varphi Y)X\big) - \beta\big(R(\xi,X)Y + R(\xi,Y)X\big).$$

Using (2.21) after routine calculations we get

$$R(\xi,X)Y + R(\xi,Y)X = \\ \big[Y(\beta) - (\alpha^2-\beta^2)\eta(Y)\big]X + \big[X(\beta) - (\alpha^2-\beta^2)\eta(X)\big]Y + \big[Y(\alpha) + 2\alpha\beta\eta(Y)\big]\varphi X + \big[X(\alpha) + 2\alpha\beta\eta(X)\big]\varphi Y + \\ \big[2(\alpha^2-\beta^2)g(X,Y) - X(\beta)\eta(Y) - Y(\beta)\eta(X)\big]\xi - 2g(\varphi X,\varphi Y)\nabla\beta$$

$$R(\xi,\varphi X)Y + R(\xi,\varphi Y)X = \\ -\big[Y(\alpha) + 2\alpha\beta\eta(Y)\big]X - \big[X(\alpha) + 2\alpha\beta\eta(X)\big]Y + \big[Y(\beta) - (\alpha^2-\beta^2)\eta(Y)\big]\varphi X + \big[X(\beta) - (\alpha^2-\beta^2)\eta(X)\big]\varphi Y \\ + 2g(\varphi X,\varphi Y)\nabla\alpha + \big[4\alpha\beta g(X,Y) + X(\alpha)\eta(Y) + Y(\alpha)\eta(X)\big]\xi.$$

Combining, we get what was claimed □

Theorem 3.3 *The norm of the mean curvature vector* H_ξ *for the Reeb vector field* ξ *on* (α,β) *trans-Sasakian manifold of* $\dim M = 2n+1$ *is*

$$|H_\xi| = \frac{\Big|(1+\alpha^2+\beta^2)(\varphi^2\nabla\alpha - \varphi\nabla\beta) + (n-1)\varphi^2\Big(\alpha\nabla(\alpha^2+\beta^2) + \beta\varphi\nabla(\alpha^2+\beta^2)\Big)\Big|}{(2n+1)(1+\alpha^2+\beta^2)^{3/2}}.$$

As a consequence,

- *if* $\dim(M) > 3$, *then* $\beta = 0$ *and* $\alpha = const$, *so* $|H_\xi| = \dfrac{(1+(2n-1)\alpha^2)}{(2n+1)(1+\alpha^2)^{3/2}}|\varphi^2\nabla\alpha| = 0;$

- *if* $\dim(M) > 3$ *and* $\alpha = 0$, *then* $|H_\xi| = \dfrac{(1+(2n-1)\beta^2)}{(2n+1)(1+\beta^2)^{3/2}}|\varphi\nabla\beta|$;

- *if* $\dim(M) = 3$, *then* $|H_\xi| = \dfrac{|\varphi^2\nabla\alpha - \varphi\nabla\beta|}{3\sqrt{1+\alpha^2+\beta^2}}$.

Proof From (3.2) it follows that

$$Hess_\xi(e_0,e_0) = 0, \quad Hess_\xi(e_i,e_i) = e_i(\alpha)\varphi e_i - e_i(\beta)e_i + (\alpha^2+\beta^2)\xi.$$

Taking trace, we get (cf. [18, 19])

$$\bar{\Delta}\xi = \varphi\nabla\alpha + \varphi^2\nabla\beta + 2n(\alpha^2+\beta^2)\xi. \tag{3.5}$$

From (3.3) (or directly from the definition of Hm_ξ using $A_\xi\xi = 0$) we conclude $Hm_\xi(e_0,e_0) = 0$ and

$$Hm_\xi(e_i,e_i) = -(\alpha e_i(\alpha) + \beta e_i(\beta))e_i + (\alpha e_i(\beta) - \beta e_i(\alpha))\varphi e_i + (\alpha\nabla\alpha + \beta\nabla\beta) + \beta(\alpha^2+\beta^2)\xi.$$

Therefore,

$$\begin{aligned} trace(Hm_\xi) &= -\alpha(\nabla\alpha - \xi(\alpha)\xi) - \beta(\nabla\beta - \xi(\beta)\xi) + \alpha\varphi\nabla\beta - \beta\varphi\nabla\alpha + 2n(\alpha\nabla\alpha + \beta\nabla\beta) + 2n\beta(\alpha^2+\beta^2)\xi \\ &= (2n-1)(\alpha\nabla\alpha + \beta\nabla\beta) + \alpha\varphi\nabla\beta - \beta\varphi\nabla\alpha + (2n\beta(\alpha^2+\beta^2) + \alpha\xi(\alpha) + \beta\xi(\beta))\xi. \end{aligned} \tag{3.6}$$

Since $\nabla\alpha = -\varphi^2\nabla\alpha + \xi(\alpha)\xi$ and $\nabla\beta = -\varphi^2\nabla\beta + \xi(\beta)\xi$, one continue

$$\begin{aligned} trace(Hm_\xi) &= -(2n-1)\varphi^2(\alpha\nabla\alpha + \beta\nabla\beta) + \alpha\varphi\nabla\beta - \beta\varphi\nabla\alpha + 2n(\beta(\alpha^2+\beta^2) + \alpha\xi(\alpha) + \beta\xi(\beta))\xi = \\ &-(n-1)\varphi^2\nabla(\alpha^2+\beta^2) - \alpha\varphi^2\nabla\alpha - \beta\varphi^2\nabla\beta + \alpha\varphi\nabla\beta - \beta\varphi\nabla\alpha + 2n\beta(-\alpha^2+\beta^2+\xi(\beta)\xi = \\ &-(n-1)\varphi^2\nabla(\alpha^2+\beta^2) - \alpha\varphi(\varphi\nabla\alpha - \nabla\beta) - \beta\varphi(\varphi\nabla\beta + \nabla\alpha) + 2n\beta(-\alpha^2+\beta^2+\xi(\beta))\xi = \\ &-(n-1)\varphi^2\nabla(\alpha^2+\beta^2) - \alpha\varphi(\varphi\nabla\alpha + \varphi^2\nabla\beta) - \beta\varphi(-\varphi^3\nabla\beta - \varphi^2\nabla\alpha) + 2n\beta(-\alpha^2+\beta^2+\xi(\beta)\xi = \\ &-(n-1)\varphi^2\nabla(\alpha^2+\beta^2) - \alpha\varphi(\varphi\nabla\alpha + \varphi^2\nabla\beta) + \beta\varphi^2(\varphi^2\nabla\beta + \varphi\nabla\alpha) + 2n\beta(-\alpha^2+\beta^2+\xi(\beta)\xi = \\ &-(n-1)\varphi^2\nabla(\alpha^2+\beta^2) - \alpha\varphi\bar{\Delta}\xi + \beta\varphi^2\bar{\Delta}\xi + 2n\beta\big(-\alpha^2+\beta^2+\xi(\beta)\big)\xi. \end{aligned}$$

So finally,

$$trace(Hm_\xi) = -(n-1)\varphi^2\nabla(\alpha^2+\beta^2) - \alpha\varphi\bar{\Delta}\xi + \beta\varphi^2\bar{\Delta}\xi + 2n\beta\big(-\alpha^2+\beta^2+\xi(\beta)\big)\xi. \tag{3.7}$$

As a consequence,

$$\varphi\, trace(Hm_\xi) = (n-1)\varphi\nabla(\alpha^2+\beta^2) - \alpha\varphi^2\bar{\Delta}\xi - \beta\varphi\bar{\Delta}\xi$$
$$\varphi^2\, trace(Hm_\xi) = (n-1)\varphi^2\nabla(\alpha^2+\beta^2) + \alpha\varphi\bar{\Delta}\xi - \beta\varphi^2\bar{\Delta}\xi$$

and hence

$$\begin{aligned} A_\xi trace(Hm_\xi) &= \alpha\varphi\, trace(Hm_\xi) + \beta\varphi^2 trace(Hm_\xi) = \\ &\alpha\big[(n-1)\varphi\nabla(\alpha^2+\beta^2) - \alpha\varphi^2\bar{\Delta}\xi - \beta\varphi\bar{\Delta}\xi\big] + \beta\big[(n-1)\varphi^2\nabla(\alpha^2+\beta^2) + \alpha\varphi\bar{\Delta}\xi - \beta\varphi^2\bar{\Delta}\xi\big] = \\ &(n-1)\varphi\big[\alpha\nabla(\alpha^2+\beta^2) + \beta\varphi\nabla(\alpha^2+\beta^2)\big] - (\alpha^2+\beta^2)\varphi^2\bar{\Delta}\xi. \end{aligned}$$

So we get

$$\bar{\Delta}\xi + A_\xi trace(Hm) = \bar{\Delta}\xi - (\alpha^2+\beta^2)\varphi^2\bar{\Delta}\xi + (n-1)\varphi\big[\alpha\nabla(\alpha^2+\beta^2) + \beta\varphi\nabla(\alpha^2+\beta^2)\big]$$

and finally

$$\varphi(\bar{\Delta}\xi + A_\xi trace(Hm)) = (1+\alpha^2+\beta^2)\varphi\bar{\Delta}\xi + (n-1)\varphi^2\big[\alpha\nabla(\alpha^2+\beta^2) + \beta\varphi\nabla(\alpha^2+\beta^2)\big]. \tag{3.8}$$

After substitution of (3.5) we get the result.

If $n>1$ then $\beta=0$, $\alpha=const$ [14, 18] and (3.5) implies $\varphi^2\nabla\alpha = \varphi\bar{\Delta}\xi$. As a consequence,

$$\varphi(\bar{\Delta}\xi + A_\xi\, trace(Hm_\xi)) = (1+\alpha^2)\bar{\Delta}\xi + (n-1)2\alpha^2\varphi^2\nabla\alpha = (1+(2n-1)\alpha^2)\varphi\bar{\Delta}\xi.$$

So,

$$|H_\xi| = \frac{(1+(2n-1)\alpha^2)}{(2n+1)(1+\alpha^2)^{3/2}}|\varphi\bar{\Delta}\xi| = \frac{(1+(2n-1)\alpha^2)}{(2n+1)(1+\alpha^2)^{3/2}}|\varphi^2\nabla\alpha| = 0. \tag{3.9}$$

In case $n>1$ and $\alpha=0$, the (3.5) implies $\varphi\nabla\beta = -\varphi\bar{\Delta}\xi$. As a consequence,

$$\varphi(\bar{\Delta}\xi + A_\xi\, trace(Hm_\xi)) = (1+\beta^2)\bar{\Delta}\xi - 2(n-1)\beta^2\varphi\nabla\beta = (1+(2n-1)\beta^2)\varphi\bar{\Delta}\xi.$$

So,

$$|H_\xi| = \frac{(1+(2n-1)\beta^2)}{(2n+1)(1+\beta^2)^{3/2}}|\varphi\bar{\Delta}\xi| = \frac{(1+(2n-1)\beta^2)}{(2n+1)(1+\beta^2)^{3/2}}|\varphi\nabla\beta|. \tag{3.10}$$

If $n=1$ then $\varphi(\bar{\Delta}\xi + A_\xi\, trace(Hm_\xi)) = (1+\alpha^2+\beta^2)\varphi\bar{\Delta}\xi$ and hence

$$|H_\xi| = \frac{|\varphi\bar{\Delta}\xi|}{3\sqrt{1+\alpha^2+\beta^2}} = \frac{|\,\varphi^2\nabla\alpha - \varphi\nabla\beta|}{3\sqrt{1+\alpha^2+\beta^2}}. \tag{3.11}$$

□

Theorem 3.4 *The Reeb vector field on connected (α,β) trans-Sasakian manifold M gives rise to totally geodesic submanifold $\xi(M)\subset(T_1M,g_S)$ only in the following cases*

- $\beta=0$, $\alpha=1$ *and hence M is Sasakian; or $\beta=0,\alpha=0$ and hence M is cosymplectic;*
- $\alpha=0$ *and* $\nabla\beta = \dfrac{\beta^2(\beta^2+1)}{1-\beta^2}\xi$. *If $\beta=const$ or M is compact, then $\beta=0$ and hence M is cosymplectic.*

Proof According to (2.9), we need to check

$$\Omega_\xi(X,Y) = Hess_\xi(X,Y) + A_\xi Hm_\xi(X,Y) - g(A_\xi X, A_\xi Y)\,\xi = 0$$

for all $X,Y\in\mathscr{X}(M)$.

The ξ-rough Hessian is given by (3.2). Put $X=\xi, Y\in\mathcal{D}_\xi$ then

$$2Hess_\xi(\xi,Y) = [\xi(\alpha)-2\alpha\beta]\varphi Y - [\xi(\beta)+\alpha^2-\beta^2]Y. \tag{3.12}$$

The ξ-harmonicity tensor is given by (3.3). Then

$$2Hm_\xi(\xi,Y)=(\alpha^2-\beta^2-\xi(\beta))\big(\alpha\varphi Y-\beta Y\big). \tag{3.13}$$

Therefore,

$$2A_\xi Hm_\xi(\xi,Y)=(\alpha^2-\beta^2-\xi(\beta))\big[-(\alpha^2-\beta^2)Y-2\alpha\beta\varphi Y\big].$$

The totally geodesic equation (2.9) takes the form

$$\big[-\xi(\beta)(1+\alpha^2-\beta^2)+(\alpha^2-\beta^2)(1-\alpha^2+\beta^2)\big]Y+\big[\xi(\alpha)-2\alpha\beta+2\alpha\beta(\alpha^2-\beta^2-\xi(\beta))\big]\varphi Y=0.$$

It follows immediately

$$\begin{cases}\xi(\beta)(1+\alpha^2-\beta^2)-(\alpha^2-\beta^2)(1-\alpha^2+\beta^2)=0,\\ \xi(\alpha)-2\alpha\beta+2\alpha\beta(\alpha^2-\beta^2-\xi(\beta))=0.\end{cases} \tag{3.14}$$

If one suppose $\alpha^2-\beta^2=-1$ in $(3.14)_1$, than we get a contradiction. Therefore

$$\xi(\beta)=\frac{(\alpha^2-\beta^2)(\alpha^2-\beta^2-1)}{1+\alpha^2-\beta^2}\quad(1+\alpha^2-\beta^2\neq 0).$$

Substitution of $\xi(\alpha)=-2\alpha\beta$ and $\xi(\beta)$ into $(3.14)_2$ yields $\alpha\beta=0$. If the manifold is connected, then $\alpha=0$ or $\beta=0$.

If $\beta=0$, then $(3.14)_1$ implies $\alpha=1$ (and we come to the Sasakian structure) or $\alpha=0$ (and the structure is cosymplectic).

If $\alpha=0$ then

$$\xi(\beta)=\frac{\beta^2(\beta^2+1)}{1-\beta^2}\quad(\beta\neq 1). \tag{3.15}$$

Take now $X,Y\in\mathcal{D}_\xi$. Then $\varphi^2X=-X,\varphi^2Y=-Y$ and by Lemma 3.2 we have

$$2Hess_\xi(X,Y)=-X(\beta)Y-Y(\beta)X+X(\alpha)\varphi Y+Y(\alpha)\varphi X+2(\alpha^2+\beta^2)g(X,Y)\xi.$$

The ξ-harmonicity tensor takes the form

$$\begin{aligned}2Hm_\xi(X,Y)=&-(\alpha Y(\alpha)+\beta Y(\beta))X-(\alpha X(\alpha)+\beta X(\beta))Y+(\alpha Y(\beta)-\beta Y(\alpha))\varphi X+(\alpha X(\beta)-\beta X(\alpha))\varphi Y\\&+2g(X,Y)(\alpha\nabla\alpha+\beta\nabla\beta)+2\beta(\alpha^2+\beta^2)g(X,Y)\xi.\end{aligned}$$

In case of Sasakian structure $(1,0)$

$$Hess_\xi(X,Y)=g(X,Y),\quad Hm_\xi(X,Y)=0,\quad g(A_\xi X,A_\xi Y)=g(X,Y)$$

and Equation (2.9) is fulfilled. The case $(0,0)$ is trivial.

In case of β -Kenmotsu structure of type $(0,\beta)$ we have

$$\begin{aligned}2Hess_\xi(X,Y)&=-\big(X(\beta)Y+Y(\beta)X\big)+2\beta^2g(X,Y)\xi,\\ 2Hm_\xi(X,Y)&=-\beta Y(\beta)X-\beta X(\beta)Y+2g(X,Y)\beta\nabla\beta+2\beta^3g(X,Y)\xi.\end{aligned}$$

For β-Kenmotsu structure $A_\xi X = \beta\varphi^2 X$. Therefore, $2g(A_\xi X, A_\xi Y) = 2\beta^2 g(X,Y)$ and

$$2A_\xi Hm_\xi(X,Y) = \beta^2\big(Y(\beta)X + X(\beta)Y\big) + 2g(X,Y)\beta^2\varphi^2\nabla\beta.$$

So we get

$$2\Omega_\xi(X,Y) = (\beta^2-1)\big(X(\beta)Y + Y(\beta)X\big) + 2g(X,Y)\beta^2\varphi^2\nabla\beta = 0.$$

In case of $\beta \neq 1$ by taking arbitrary $(X \perp Y) \in \mathcal{D}_\xi$ we get $X(\beta)Y + Y(\beta)X$ which implies $X(\beta) = 0$ for all $X \in \mathcal{D}_\xi$ and hence $\nabla\beta = \xi(\beta)\xi$. In this case $\varphi\nabla\beta = 0$ and the equation is fulfilled.

In case $\beta = const$ the (3.15) implies $\beta = 0$ and the structure is cosymplectic. If M compact, then β attains its global maximum and minimum at some points. As a consequence, $\max(\beta) = \min(\beta) = 0$ and hence $\beta = 0$. □

4. Closing remarks

Remark 4.1 It follows from (3.11) that in 3-dimensional case ξ gives rise to a minimal submanifold if and only if $\nabla\alpha + \varphi\nabla\beta + 2\alpha\beta\xi = 0$ (cf. [19]). As it follows from (3.9), (3.10) and (3.11) in general case the Reeb vector field gives rise to minimal submanifold in T_1M if and only if $\varphi\bar{\Delta}\xi = 0$ which is equivalent to harmonicity of ξ in correspondence with definition (2.7) and expression (3.5) (cf. [18]). If ξ is harmonic, then from (3.7) we get

$$trace(Hm_\xi) = -(n-1)\varphi^2\nabla(\alpha^2+\beta^2) + 2n\beta\big(-\alpha^2+\beta^2+\xi(\beta)\big)\xi.$$

If $n > 1$ then $\beta = 0$ and the Reeb vector field is always minimal, harmonic and defines a harmonic map (cf. [18]). If $n > 1$, or M is β-Kenmotsu, then by (3.5) ξ is harmonic if $\nabla\beta = \xi(\beta)\xi$ and defines a harmonic map if, in addition, $\beta(\xi(\beta) + \beta^2) = 0$ (cf. [18]). It follows easily from (3.5) and (2.22) that in all cases a minimal Reeb vector field is an *eigenvector of the Ricci operator*. A similar results for $n = 1$ was obtained in [19].

Remark 4.2 To define a totally geodesic map ($B_\xi = 0$) the Reeb vector field has to satisfy the equations

$$Hess_\xi(X,Y) = g(A_\xi X, A_\xi Y)\xi, \quad Hm_\xi(X,Y) = 0$$

for all $X, Y \in \mathscr{X}(M)$.

It follows easily from (2.23), (3.12) and (3.13) that if M is connected, then $\alpha = 0$ and $\beta = 0$. It means that the Reeb vector field on connected manifold never defines a totally geodesic map except trivial cosymplectic case.

Remark 4.3 Results of the Theorem 3.4 can be obtained from more general observations. In [24] it was proved that a unit vector field of Riemannian transversally oriented totally umbilical hyperfoliation defines a totally geodesic submanifold in T_1M if

$$(k^2-1)K_\sigma = 2k^2,$$

where K_σ are the eigenvalues of the normal Jacobi operator $X \to R(X,\xi)\xi$ and k is the value of umbilicity of the leaves of the hyperfoliation. The case of β-Kenmotsu is precisely the case with $k^2 = \beta^2$ and (2.20) implies

$$R(X,\xi)\xi = -(\xi(\beta)+\beta^2)X$$

for all $X \in \mathcal{D}_\xi$. Hence all $K_\sigma = -(\xi(\beta) + \beta^2)$ and we get $-(\beta^2 - 1)(\xi(\beta) + \beta^2) = 2\beta^2$. If $\beta = const$ then we get a contradiction. If $\beta \neq 1$, then

$$\xi(\beta) = \frac{\beta^2(1+\beta^2)}{1-\beta^2}.$$

A unit vector field ξ is said to be strongly normal, if $(\nabla_X A_\xi)Y = g(A_\xi X, A_\xi Y)\xi$ for all $X, Y \in \mathcal{D}_\xi$. In our notations it means that $Hess_\xi(X,Y) = g(A_\xi X, A_\xi Y)\xi$. In general [23], if ξ is Killing and strongly normal, then ξ is totally geodesic if and only if

$$K_\sigma(1 - K_\sigma) = 0,$$

where K_σ are the eigenvalues of the Jacobi operator $X \to R(X,\xi)\xi$. The Reeb vector field of α -Sasakian structure is Killing by definition. It is easy to check in (3.2) that if $\beta = 0$ and $\alpha = const$, then the Reeb vector field is strongly normal. From (2.20) it follows that $K_\sigma = \alpha^2$. So we get either $\alpha = 0$ or $\alpha = 1$. In case of 3-dimensional manifold, the converse is also true. If ξ is Killing and totally geodesic, then M^3 is Sasakian and ξ is the Reeb vector field [23].

Acknowledgment

The research was supported by Prospective Plan for Development of Scientific Direction ("Basic funding"), project No BF/32-2021. The author thanks the Pure Mathematics Department of the University of Waterloo (Ontario, Canada) for hospitality during the research visit and the anonymous referee for the corrections.

References

[1] Blair D. Contact Manifolds in Riemannian Geometry. Lecture Notes in Mathematics 509. Springer, New York, 1976.

[2] Deshmukh S. Geometry of 3-dimensional trans-Sasakaian manifolds. Analele Ştiinţifice ale Universităţii "Al.I. Cuza" din Iasi (S.N.), Matematică 2016; LXII(1): 183-192. doi: 10.2478/aicu-2014-0025

[3] Deshmuk S., Al-Solamy F. A Note on Compact Trans-Sasakian Manifolds. Mediterranean Journal of Mathematics 2016, 13: 2099-2104. doi: 10.1007/s00009-015-0582-7

[4] Eells J, Lemaire L. Selected Topics in Harmonic Maps. CBMS Regional Conference Series in Mathematics 50. 1983. doi: 10.1090/cbms/050

[5] Fujimoto A, Muto H. On cosymplectic manifolds. Tensor 1974; 28: 43-52.

[6] De UC. Ricci Tensor in 3-dimensional Trans-Sasakian manifolds. Kyungpook Mathematical Journal 2003; 43: 247-255.

[7] Gil-Medrano O. Relationship between volume and energy of unit vector fields. Differential Geometry and Applications 2001; 15: 137-152. doi: 10.1016/S0926-2245(01)00053-5

[8] Gil-Medrano O, Llinares-Fuster E. Minimal unit vector fields. Tohôku Mathematical Journal 2002; 54 (1): 71-84. doi: 10.2748/tmj/1113247180

[9] Gluck H, Ziller W. On the volume of a unit vector field on the three sphere. Commentarii Mathematici Helvetici 1986; 61: 177-192.

[10] González-D avila JC, Vanhecke L. Examples of minimal unit vector fields. Annals of Global Analysis and Geometry 2000; 18: 385-404. doi: 10.1023/A:1006788819180

[11] González-Dávila JC, Vanhecke L. Minimal and harmonic characteristic vector fields on three-dimensional contact metric manifolds. Journal of Geometry 2001; 72: 65-76. doi: /10.1007/s00022-001-8570-4

[12] González- Dávila JC, Vanhecke L. Invariant harmonic unit vector fields on Lie groups. Bollettino dell' Unione Matematica Italiana 2002; Serie 8, 5-B (2): 377-403.

[13] Kenmotsu KA. A class of almost contact Riemannian manifolds, Tohôku Mathematical Journal 1972; 24: 93-103. doi: 10.2748/tmj/1178241594

[14] Marrero JC. The local structure of trans-Sasakian manifolds, Annali di Matematica Pura ed Applicata 1992; 162 (4): 7-86. doi: 1007/BF01760000

[15] Oubiña J.A. New Classes of almost Contact metric structures, Publicationes Mathematicae Debrecen 1985; 32: 187-193.

[16] Tsukada K, Vanhecke L. Invariant minimal unit vector fields on Lie groups. Periodica Mathematica Hungarica 2000; 40: 123-133. doi: 10.1023/A:1010331408580

[17] Tsukada K, Vanhecke L. Minimality and harmonicity for Hopf vector filds. Illinois Journal of Mathematics 2011; 45 (2): 441-451. doi: 10.1215/ijm/1258138349

[18] Vergara-Diaz E. and Wood C. M. Harmonic Almost Contact Structures. Geometriae Dedicata 2006; 123: 131-151. doi: 10.1007/s10711-006-9112-x

[19] Wang Y. Minimal and harmonic Reeb vector fields on trans-Sasakian 3-manifolds. Journal of the Korean Mathematical Society 2018; 55 (6): 1321-1336. doi: 10.4134/JKMS.j170689

[20] Weigmink G. Total Bending of Vector Fields on Riemannian Manifolds. Mathematische Annalen 1995; 303: 325-344.

[21] Yampolsky A. On the mean curvature of a unit vector field. Publicationes Mathematicae Debrecen 2002; 60 (1-2): 131-155.

[22] Yampolsky A. On intrinsic geometry of a unit vector field. Commentationes Mathematicae Universitatis Carolinae 2002; 43 (2): 131-155.

[23] Yampolsky A. A totally geodesic property of Hopf vector field. Acta Mathematica Hungarica 2003; 101 (1-2): 93-112. doi: 10.1023/B:AMHU.0000003895.90745.38

[24] Yampolsky A. On extrinsic geometry of unit normal vector field of Riemannian hyperfoliation. Publicationes Mathematicae Debrecen 2003, 63 (4): 555-567.

[25] Yampolsky A. Full description of totally geodesic unit vector fields on Riemannian 2-manifold. Journal of Mathematical Physics, Analysis, Geometry 2004; 1 (3): 355-365.

[26] Yampolsky A. On special types of minimal and totally geodesic unit vector fields. In: I. Mladenov (edtitor). Proceedings of the 7-th International Conference in Geometry, Integrability and Quantization June 2-10, 2005. Varna, Bulgaria, pp. 292-306.

[27] Yampolsky A. Invariant totally geodesic unit vector fields on three-dimensional Lie groups. Journal of Mathematical Physics, Analysis, Geometry 2007; 3 (2): 253-276.