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On a generalization of Szász-Mirakyan operators including Dunkl-Appell polynomials

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Abstract: In this study, we have introduced a generalization of Szász-Mirakyan operators including Dunkl-Appell polynomials with help of sequences satisfying certain conditions and have derived some approximation properties of this generalization.

Key words: Dunkl exponential, Szász operators, modulus of continuity, Dunkl-Appell polynomials, Lipschitz functions

1. Introduction

Approximation theory, a common scientific field of study where applied mathematics and functional analysis intersect, is based on the Weierstrass Approach Theorem, which was put forward and proved by German mathematician Karl Theodor Wilhelm Weierstrass, known as the father of modern analysis, in 1885 [1]. According to this theorem, there is a sequence of polynomials that are uniformly convergent to every continuous function defined in a closed and finite range. Bernstein operators [2] is one of the best examples of the polynomials for the interval $[0,1]$. The other is an extension of Bernstein operators to the infinite interval and is called as Szász operators defined by Szász [3] as follows:

$$S_m\left(f;x\right)=\frac{1}{\exp(mx)}\sum_{j=0}^{\infty}\frac{\left(mx\right)^j}{j!}f\left(\frac{j}{m}\right),\quad x\geq 0,\ m\in\mathbb{N}$$

where $f\ \in C\left[0,\infty\right).$

Sucu [4] has defined a Dunkl analogue of Szász operators with the help of a generalization of the exponential function given by Rosenblum in [5] as follows

$$S_m^{\mu}\left(f;x\right)=\frac{1}{\exp_{\mu}\left(mx\right)}\sum_{j=0}^{\infty}\frac{\left(mx\right)^j}{\gamma_{\mu}\left(j\right)}f\left(\frac{j+2\mu\theta_j}{m}\right),\quad x\geq 0,\ m\in\mathbb{N}\tag{1.1}$$

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where $\mu \geq 0$ and $f \in C[0,\infty)$. $\exp_\mu(x)$ has been given by Rosenblum [5] in here as follows:

$$\exp_\mu(x) = \sum_{j=0}^{\infty} \frac{x^j}{\gamma_\mu(j)} \tag{1.2}$$

where the coefficients $\gamma_\mu(j)$ are defined by

$$\gamma_\mu(2j) = \frac{2^{2j} j! \Gamma\left(j+\mu+\frac{1}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)} \text{ and } \gamma_\mu(2j+1) = \frac{2^{2j+1} j! \Gamma\left(j+\mu+\frac{3}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)} \tag{1.3}$$

for $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mu > -\frac{1}{2}$ where Γ denotes the Euler's Gamma function. It is known from [5] that γ_μ satisfies the following recurrence relation

$$\gamma_\mu(j) = (j + 2\mu\theta_j)\, \gamma_\mu(j-1), \ j = 1, 2, ... \tag{1.4}$$

where θ_j is defined as

$$\theta_j = \frac{1+(-1)^{j-1}}{2} = \begin{cases} 1 & , \quad j \in 2\mathbb{N}_0 + 1 \\ 0 & , \quad j \in 2\mathbb{N}_0. \end{cases} \tag{1.5}$$

On the other hand, we recall that Dunkl derivative operator has been defined by Rosenblum in [5] with

$$D_{\mu,x}(f(x)) := f'(x) + \mu \frac{f(x) - f(-x)}{x}, \quad x \in \mathbb{C} \tag{1.6}$$

where $\mu \in \mathbb{C} \backslash \{-(2k-1)/2 : \ k \in \mathbb{N}\}$ and f is entire function defined on $\mathbb{C}$. It can be seen that

i) If $\mu = 0$, then $D_{\mu,x}$ is the standard derivative operator,

ii) $D^2_{\mu,x}(f(x)) = f''(x) + 2\frac{\mu}{x} f'(x) - \mu \frac{f(x)-f(-x)}{x^2}$,

iii) $D_{\mu,x}(x^m) = \frac{\gamma_\mu(m)}{\gamma_\mu(m-1)} x^{m-1}, \ m \in \mathbb{N}$

iv) $D_{\mu,x}\left(\exp_\mu(\lambda x)\right) = \lambda \exp_\mu(\lambda x)$,

v) $D_{\mu,x}(f(x)\, g(x)) = f(x)\, D_{\mu,x}(g(x)) + g(-x)\, D_\mu(f(x)) + f'(x)\, [g(x) - g(-x)]$

where f and g are entire functions defined on [5].

Through the above properties, authors have studied various generalizations about Dunkl analogue of Szász operators in [6], [7], [8], [9], [10], [11] and [12]. On the one hand, orthogonal polynomials and other families of polynomials have played an important role in construction of linear positive operators (see [13], [14], [15], [16], [17] and [18]). At the present, we recall Dunkl-Appell polynomials defined by Cheikh and Gaied in [19]. From [19], Dunkl-Appell polynomials are symbolized by $p_j(x)$ and defined by

$$p_j(x) = \sum_{\nu=0}^{j} \binom{j}{\nu}_\mu c_{j-\nu}\, x^\nu, \quad c_0 \neq 0$$

where Dunkl-binomial coefficient is

$$\binom{j}{\nu}_{\mu} = \frac{\gamma_\mu(j)}{\gamma_\mu(\nu)\,\gamma_\mu(j-\nu)}.$$

Moreover, from [19], the polynomials $p_j(x)$ are generated by

$$G(t)\exp_\mu(xt) = \sum_{j=0}^{\infty} \frac{p_j(x)}{\gamma_\mu(j)} t^j \tag{1.7}$$

where G is an analytic function in the disc $|t| < r,\ r > 1$ and

$$G(t) = \sum_{j=0}^{\infty} c_j \ \frac{t^j}{\gamma_\mu(j)}. \tag{1.8}$$

Now, we are going to define our operators thanks to d-Appell polynomials, Dunkl derivative operator and increasing-unbounded sequences of positive real numbers similar to [20] (also see [21] and [22]) in next section.

2. Construction of the operators $L_m^{a_m,b_m,\mu}$

Let $p_j(x)$ be Dunkl-Appell polynomials. We introduce a new generalization of Szász-Mirakyan operators by

$$L_m^{a_m,b_m,\mu}(f;x) = \frac{1}{G(1)\exp_\mu(a_m x)} \sum_{j=0}^{\infty} \frac{p_j(a_m x)}{\gamma_\mu(j)} f\left(\frac{j+2\mu\theta_j}{b_m}\right), \quad x \geq 0,\ m \in \mathbb{N} \tag{2.1}$$

where μ is a real parameter with $|\mu| < 1/2$, $G(t)$ is an analytic function given in (1.8), $f \in C[0,\infty)$, $G(1) \neq 0$, $\frac{c_j}{G(1)} \geq 0$ for $j = 0,1,2,...,$ (a_m) and (b_m) are increasing-unbounded sequences of positive real numbers, satisfying the following conditions

$$\lim_{m\to\infty} \frac{1}{b_m} = 0 \ \text{ and } \ \lim_{m\to\infty} \frac{a_m}{b_m} = 1. \tag{2.2}$$

It is clear that these operators defined in (2.1) are linear positive and if $a_m = b_m = m$, $L_m^{a_m,b_m,\mu}(f;x)$ reduces to the operators given by Sucu in [18].

Lemma 2.1 *For the function given in* (1.7)*, we have the following equalities*

$$i)\quad \sum_{j=0}^{\infty} \frac{p_j(a_m x)}{\gamma_\mu(j)} = G(1)\exp_\mu(a_m x),$$

$$ii)\quad \sum_{j=0}^{\infty} \frac{p_{j+1}(a_m x)}{\gamma_\mu(j)} = \exp_\mu(a_m x)\left[a_m x G(1) + G'(1)\right]$$

$$+\mu\exp_\mu(-a_m x)\left[G(1) - G(-1)\right],$$

$$iii)\quad \sum_{j=0}^{\infty} (-1)^j \frac{p_{j+1}(a_m x)}{\gamma_\mu(j)} = \exp_\mu(-a_m x)\left[a_m x G(-1) + G'(-1)\right]$$
$$+\mu\exp_\mu(a_m x)\left[G(1) - G(-1)\right],$$

$$iv)\quad \sum_{j=0}^{\infty} \frac{p_{j+2}(a_m x)}{\gamma_\mu(j)} = \exp_\mu(a_m x)\left[a_m^2 x^2 G(1) + 2a_m x G'(1) + G''(1)\right]$$

$$+\mu\exp_\mu(-a_m x)\left[2G'(1) - (G(1) - G(-1))\right]$$

where (a_m) *is the positive, increasing and unbounded sequence.*

Proof i) By letting $t \to 1$ and $x \to a_m x$ in the equality (1.7), we obtain the proof of Lemma 2.1-(i).

ii) By applying operator $D_{\mu,x}$ given by (1.6) both sides of the equality (1.7), using the properties (iv) and (v) of Dunkl derivative operator in page 2 and then letting $t \to 1$ and $x \to a_m x$, we have the proof of Lemma 2.1-(ii).

iii) We get the proof of Lemma 2.1-(iii) by applying operator $D_{\mu,x}$ given in (1.6) both sides of the equality (1.7), using the properties (iv) and (v) of Dunkl derivative operator in page 2 and then getting $t \to -1$ and $x \to a_m x$.

iv) By twice applying operator $D_{\mu,x}$ given in (1.6) both sides of the equality (1.7), using the properties (ii), (iv) and (v) of Dunkl derivative operator in page 2 and then getting $t \to 1$ and $x \to a_m x$, we acquire the proof of Lemma 2.1-(iv). □

Lemma 2.2 *Let* $L_m^{a_m,b_m,\mu}(f;x)$ *be the operator introduced in* (2.1)*. Then, we have*

$$i)\quad L_m^{a_m,b_m,\mu}(1;x) = 1,$$

$$ii)\quad L_m^{a_m,b_m,\mu}(t;x) = \frac{a_m}{b_m}x + \frac{1}{b_m}\frac{G'(1)+\mu[G(1)-G(-1)]\xi_m^\mu(x)}{G(1)},$$

$$iii)\quad L_m^{a_m,b_m,\mu}(t^2;x) = \left(\frac{a_m}{b_m}\right)^2 x^2 + \frac{a_m}{b_m^2}\frac{2G'(1)+G(1)+2\mu G(-1)\xi_m^\mu(x)}{G(1)}x$$

$$+\frac{1}{b_m^2}\frac{G''(1)+G'(1)+2\mu^2[G(1)-G(-1)]+2\mu\left[G'(1)+G'(-1)\right]\xi_m^\mu(x)}{G(1)}$$

where $\xi_m^\mu(x) := \frac{\exp_\mu(-a_m x)}{\exp_\mu(a_m x)}$.

Proof i) It is clear from Lemma 2.1-(i).

ii) By using the recursion relation given in (1.4), we acquire the proof of Lemma 2.2-(ii).

iii) We have

$$\begin{aligned} L_m^{a_m,b_m,\mu}(t^2;x) &= \frac{1}{G(1)\exp_\mu(a_m x)}\sum_{j=0}^{\infty}\frac{p_j(a_m x)}{\gamma_\mu(j)}\left(\frac{j+2\mu\theta_j}{b_m}\right)^2 \\ &= \frac{1}{G(1)\exp_\mu(a_m x)}\sum_{j=1}^{\infty}\frac{p_j(a_m x)}{\gamma_\mu(j-1)}\frac{j+2\mu\theta_j}{b_m^2} \\ &= \frac{1}{G(1)\exp_\mu(a_m x)}\sum_{j=0}^{\infty}\frac{p_{j+1}(a_m x)}{\gamma_\mu(j)}\frac{j+1+2\mu\theta_{j+1}}{b_m^2} \end{aligned} \tag{2.3}$$

from the equality (1.4). Furthermore, we can obviously see that

$$\theta_{j+1} = \theta_j + (-1)^j \tag{2.4}$$

where θ_j is a parameter given in (1.5). Using the equalities (1.4) and (2.4) in (2.3), we have

$$\begin{aligned} L_m^{a_m,b_m,\mu}(t^2;x) &= \frac{1}{G(1)\exp_\mu(a_m x)b_m^2} \\ &\times\left[\sum_{j=0}^{\infty}\frac{p_{j+2}(a_m x)}{\gamma_\mu(j)} + \sum_{j=0}^{\infty}\frac{p_{j+1}(a_m x)}{\gamma_\mu(j)} + 2\mu\sum_{j=0}^{\infty}(-1)^j\frac{p_{j+1}(a_m x)}{\gamma_\mu(j)}\right] \end{aligned} \tag{2.5}$$

Consequently, the proof of Lemma 2.2-(iii) is obtained by using Lemma 2.1 in the equality (2.5). □

Throughout the rest of this study, we use

$$\lim_{x\to\infty}\frac{\exp_\mu(-x)}{\exp_\mu(x)} = 0$$

for $|\mu| < 1/2$ and $\left|\frac{\exp_\mu(-x)}{\exp_\mu(x)}\right| \le 1$ for $x \ge 0$ and $\mu > -1/2$ in [12] for approximation properties.

Now, we recall that the r-th central moment of the operators $L_m^{a_m,b_m,\mu}$ in (2.1) is given by

$$M_{m,r}^{\mu}(x) = L_m^{a_m,b_m,\mu}\left((t-x)^r;x\right),\ r = 0,1,2,... \tag{2.6}$$

for $m \in \mathbb{N}$ and $x \ge 0$.

Lemma 2.3 *The first few central moments of $L_m^{a_m,b_m,\mu}$ are given by*

i) $M_{m,0}^{\mu}(x) = 1,$

ii) $M_{m,1}^{\mu}(x) = \left(\frac{a_m}{b_m} - 1\right)x + \frac{1}{b_m}\frac{G'(1)+\mu[G(1)-G(-1)]\xi_m^\mu(x)}{G(1)},$

iii) $$M_{m,2}^{\mu}(x) = \left(\frac{a_m}{b_m} - 1\right)^2 x^2 + \frac{1}{b_m^2}\frac{2[a_m-b_m]G'(1)+[a_m-2\mu b_m\xi_m^\mu(x)]G(1)+2\mu[a_m+b_m]G(-1)\xi_m^\mu(x)}{G(1)}x + \frac{1}{b_m^2}\frac{G''(1)+G'(1)+2\mu^2[G(1)-G(-1)]+2\mu\left[G'(1)+G'(-1)\right]\xi_m^\mu(x)}{G(1)}.$$

Proof From linearity of the operators $L_m^{a_m,b_m,\mu}$, it is that

$$\begin{aligned} M_{m,r}^{\mu}(x) &= L_m^{a_m,b_m,\mu}\left((t-x)^r;x\right) \\ &= L_m^{a_m,b_m,\mu}\left(\sum_{k=0}^{r}\binom{r}{k}t^k(-x)^{r-k};x\right) \\ &= \sum_{k=0}^{r}\binom{r}{k}(-x)^{r-k}L_m^{a_m,b_m,\mu}\left(t^k;x\right). \end{aligned} \tag{2.7}$$

Taking $r=0,1,2$ in the equality (2.7) and using Lemma 2.2, the desired results follow. □

3. Rate of convergence for operators $L_m^{a_m,b_m,\mu}$

In this section, we present the rate of convergence of the operators $L_m^{a_m,b_m,\mu}$ thanks to the definitions of various tools.

Theorem 3.1 *For every* $g\in E^*=\left\{g: g \text{ is continuous on } [0,\infty),\ \frac{g(x)}{1+x^2} \text{ is convergent as } x\to\infty\right\}$, *we have*

$$\lim_{m\to\infty}L_m^{a_m,b_m,\mu}(g;x)=g(x)$$

on each compact subset of $[0,\infty)$.

Proof The proof is based on the well-known universal Korovkin-type theorem in [23]. □

The classical modulus of continuity of $g\in C_B[0,\infty)$, the space of continuous and bounded functions on $[0,\infty)$, is defined by

$$\omega(g;\delta):=\sup_{|h|\leq\delta}\left\{|g(x+h)-g(x)|:\ x\in[0,\infty)\right\} \tag{3.1}$$

where $\delta>0$ [23].

Theorem 3.2 *The operators* $L_m^{a_m,b_m,\mu}$ *defined in* (2.1) *satisfy the following inequality*

$$\left|L_m^{a_m,b_m,\mu}(g;x)-g(x)\right|\leq 2\omega\left(g;\sqrt{M_{m,2}^{\mu}(x)}\right)$$

where $g\in C_B[0,\infty)$ *and* $M_{m,2}^{\mu}(x)$ *has been given in* Lemma 2.3.

Proof The modulus of classical continuity of function $g\in C_B[0,\infty)$ satisfies the below inequality in [23]

$$|g(t)-g(x)|\leq\omega(g;\delta)\left(\frac{|t-x|}{\delta}+1\right). \tag{3.2}$$

From inequality (3.2), we obtain

$$\begin{aligned} \left|L_m^{a_m,b_m,\mu}(g;x)-g(x)\right| &\leq L_m^{a_m,b_m,\mu}\left(|g(t)-g(x)|;x\right) \\ &\leq \omega(g;\delta)\left(1+\tfrac{1}{\delta}L_m^{a_m,b_m,\mu}\left(|t-x|;x\right)\right). \end{aligned} \tag{3.3}$$

It can be clearly written by Cauchy-Schwarz inequality that

$$\begin{aligned} L_m^{a_m,b_m,\mu}\left(|t-x|;x\right) &= \sum_{j=0}^{\infty}\sqrt{\left(\frac{1}{G(1)\exp_\mu(a_m x)}\right)^2\left(\frac{p_j(a_m x)}{\gamma_\mu(j)}\right)^2\left(\frac{j+2\mu\theta_j}{b_m}-x\right)^2} \\ &\le \sqrt{\sum_{j=0}^{\infty}\frac{1}{G(1)\exp_\mu(a_m x)}\frac{p_j(a_m x)}{\gamma_\mu(j)}\left(\frac{j+2\mu\theta_j}{b_m}-x\right)^2} \\ &\times\sqrt{\sum_{j=0}^{\infty}\frac{1}{G(1)\exp_\mu(a_m x)}\frac{p_j(a_m x)}{\gamma_\mu(j)}} \\ &= \sqrt{L_m^{a_m,b_m,\mu}\left((t-x)^2;x\right)}\sqrt{L_m^{a_m,b_m,\mu}(1;x)} \\ &= \sqrt{M_{m,2}^{\mu}(x)}. \end{aligned} \tag{3.4}$$

Using the inequality (3.4) in (3.3), we have

$$\left|L_m^{a_m,b_m,\mu}(g;x)-g(x)\right| \le \omega(g;\delta)\left(1+\frac{1}{\delta}\sqrt{M_{m,2}^{\mu}(x)}\right). \tag{3.5}$$

Choosing $\delta=\sqrt{M_{m,2}^{\mu}(x)}$ in inequality (3.5), the proof is completed. □

Theorem 3.3 *Let* $L_m^{a_m,b_m,\mu}$ *be the operator defined in* (2.1)*. Then, we have*

$$\left|L_m^{a_m,b_m,\mu}(g;x)-g(x)\right| \le \sqrt{M_{m,2}^{\mu}(x)}\left(|g'(x)|+2\omega\left(g';\sqrt{M_{m,2}^{\mu}(x)}\right)\right)$$

where $g\in C_B^1[0,\infty):=\{g:g,g'\in C_B[0,\infty)\}$, $x\in[0,\infty)$ *and* $M_{m,2}^{\mu}(x)$ *have been calculated in Lemma 2.3.*

Proof We can clearly write that

$$g(t)-g(x)=(t-x)g'(x)+\int_x^t\left(g'(s)-g'(x)\right)ds \tag{3.6}$$

for $x,t\in[0,\infty)$. Moreover, because the function $g\in C_B^1[0,\infty)$ verifies the inequality in (3.2), we have

$$\left|\int_x^t\left(g'(s)-g'(x)\right)ds\right| \le \omega(g';\delta)\left(\frac{(t-x)^2}{\delta}+|t-x|\right). \tag{3.7}$$

If we apply $L_m^{a_m,b_m,\mu}$ to (3.6) and then taking the absolute value on both sides of this equality, we can write by using (3.7) and triangle inequality that

$$\begin{aligned} \left|L_m^{a_m,b_m,\mu}(g;x)-g(x)\right| \le\ & |g'(x)|\,L_m^{a_m,b_m,\mu}\left(|t-x|;x\right) \\ & +\omega(g';\delta)\left(L_m^{a_m,b_m,\mu}\left(|t-x|;x\right)+\frac{M_{m,2}^{\mu}(x)}{\delta}\right). \end{aligned} \tag{3.8}$$

Using Cauchy-Schwarz inequality in (3.8), we can write

$$\left|L_m^{a_m,b_m,\mu}(g;x)-g(x)\right| \leq |g'(x)| \sqrt{M_{m,2}^{\mu}(x)}+\omega(g';\delta)\left(\sqrt{M_{m,2}^{\mu}(x)}+\frac{M_{m,2}^{\mu}(x)}{\delta}\right).$$

Choosing $\delta=\sqrt{M_{m,2}^{\mu}(x)}$ in the above inequality, the proof of Theorem 3.3 is completed. □

Now, we remind that Lipschitz class functions of order α is defined as follows

$$Lip_N(\alpha):=\{g\in C[0,\infty): |g(t)-g(x)|\leq N\ |t-x|^{\alpha},\ t,x\in[0,\infty)\} \tag{3.9}$$

where $0<\alpha\leq 1$ and $N>0$ [24].

Theorem 3.4 *For the operators $L_m^{a_m,b_m,\mu}$ given in (2.1), we have*

$$\left|L_m^{a_m,b_m,\mu}(g;x)-g(x)\right| \leq N\left(M_{m,2}^{\mu}(x)\right)^{\frac{\alpha}{2}}$$

where $g\in Lip_N(\alpha)$ and $M_{m,2}^{\mu}(x)$ have been calculated in Lemma 2.3.

Proof Using linearity property of $L_m^{a_m,b_m,\mu}$ and $g\in Lip_N(\alpha)$, we get

$$\left|L_m^{a_m,b_m,\mu}(g;x)-g(x)\right| \leq NL_m^{a_m,b_m,\mu}\left(|t-x|^{\alpha};x\right). \tag{3.10}$$

We can write that

$$\begin{aligned}\left|L_m^{a_m,b_m,\mu}(g;x)-g(x)\right| &\leq NL_m^{a_m,b_m,\mu}\left(|t-x|^{\alpha};x\right)\\ &\leq N\left(L_m^{a_m,b_m,\mu}\left((t-x)^2;x\right)\right)^{\frac{\alpha}{2}}.\left(L_m^{a_m,b_m,\mu}(1;x)\right)^{\frac{2-\alpha}{2}}\\ &\leq N\left(M_{m,2}^{\mu}(x)\right)^{\frac{\alpha}{2}}\end{aligned}$$

from Lemma 2.3 and applying Hölder's inequality for sum with $p=\frac{\alpha}{2}$ and $q=\frac{2-\alpha}{2}$ in (3.10). So, the proof is done. □

Peetre's K-functional of the function $g\in C_B[0,\infty)$ is defined by

$$K(g;\delta)=\inf_{h\in C_B^2[0,\infty)}\left\{\|g-h\|_{C_B[0,\infty)}+\delta\|h\|_{C_B^2[0,\infty)}\right\} \tag{3.11}$$

for $\delta>0$ where $\|g\|_{C_B[0,\infty)}=\sup_{x\in[0,\infty)}|g(x)|$ [25]. Here, $C_B^2[0,\infty):=\{g\in C_B[0,\infty):\ g',g''\in C_B[0,\infty)\}$ is the normed space with following norm

$$\|g\|_{C_B^2[0,\infty)}=\|g\|_{C_B[0,\infty)}+\|g'\|_{C_B[0,\infty)}+\|g''\|_{C_B[0,\infty)} \tag{3.12}$$

for every $g\in C_B^2[0,\infty)$.

The second-order modulus of smoothness of function $g \in C_B^2[0,\infty)$ is given by

$$\omega_2(g;\delta) := \sup_{0<h\leq\delta} \{|g(x+2h) - 2g(x+h) + g(x)| : \ x \in [0,\infty)\} \tag{3.13}$$

for $\delta > 0$ [23]. Moreover, we know that there is a connection between Peetre's K-functional and ω_2 given by

$$K(g;\delta) \leq N \left\{\omega_2\left(g, \sqrt{\delta}\right) + \min(1,\delta) \|g\|_{C_B[0,\infty)}\right\} \tag{3.14}$$

for $\delta > 0$ and N is positive constant [26].

Theorem 3.5 *The operators $L_m^{a_m,b_m,\mu}$ defined in (2.1) have the below inequality*

$$\left|L_m^{a_m,b_m,\mu}(g;x) - g(x)\right| \leq 2N \left\{\omega_2\left(g, \sqrt{\delta_n(x)}\right) + \min(1,\delta_n(x)) \|g\|_{C_B[0,\infty)}\right\}$$

where $g \in C_B[0,\infty)$, N is a positive constant that is independent of n and $\delta_n(x) = \sqrt{M_{m,2}^{\mu}(x)} + \frac{M_{m,2}^{\mu}(x)}{2}$. Here $M_{m,2}^{\mu}(x)$ is calculated in Lemma 2.3.

Proof For $f \in C_B^2[0,\infty)$, we have that

$$f(t) - f(x) = f'(x)(t-x) + f''(c)\frac{(t-x)^2}{2} \tag{3.15}$$

where c between x and t. When we apply the operators $L_m^{a_m,b_m,\mu}$ to both sides of (3.15) and remind the linearity property of the operators $L_m^{a_m,b_m,\mu}$, we acquire

$$|L_m^{a_m,b_m,\mu}(f;x) - f(x)| \leq |f'(x)| L_m^{a_m,b_m,\mu}(|t-x|;x) + \frac{|f''(c)|}{2} L_m^{a_m,b_m,\mu}\left((t-x)^2;x\right).$$

From (3.4), we obtain

$$\begin{aligned}
\left|L_m^{a_m,b_m,\mu}(f;x) - f(x)\right| &\leq |f'(x)|\sqrt{M_{m,2}^{\mu}(x)} + \frac{|f''(c)|}{2} M_{m,2}^{\mu}(x) \\
&\leq \|f'\|_{C_B[0,\infty)} \sqrt{M_{m,2}^{\mu}(x)} + \frac{\|f''\|_{C_B[0,\infty)}}{2} M_{m,2}^{\mu}(x) \\
&\leq \left(\sqrt{M_{m,2}^{\mu}(x)} + \frac{M_{m,2}^{\mu}(x)}{2}\right)\left(\|f'\|_{C_B[0,\infty)} + \|f''\|_{C_B[0,\infty)}\right) \\
&\leq \left(\sqrt{M_{m,2}^{\mu}(x)} + \frac{M_{m,2}^{\mu}(x)}{2}\right)\|f\|_{C_B^2[0,\infty)}.
\end{aligned}$$

Using the above inequality, we can write

$$\begin{aligned}
&\left|L_m^{a_m,b_m,\mu}(g;x) - g(x)\right| \\
&= \left|L_m^{a_m,b_m,\mu}(g;x) - L_m^{a_m,b_m,\mu}(f;x) + L_m^{a_m,b_m,\mu}(f;x) - f(x) + f(x) - g(x)\right| \\
&\leq L_m^{a_m,b_m,\mu}(|g-f|;x) + |g(x) - f(x)| + \left|L_m^{a_m,b_m,\mu}(f;x) - f(x)\right| \\
&\leq 2\|g-f\|_{C_B[0,\infty)} + 2\|f\|_{C_B^2[0,\infty)}\left(\sqrt{M_{m,2}^{\mu}(x)} + \frac{M_{m,2}^{\mu}(x)}{2}\right)
\end{aligned} \tag{3.16}$$

for $g \in C_B[0,\infty)$. Let K be Peetre's K-functional of the function $g \in C_B[0,\infty)$. Consequently, we have

$$\left|L_m^{a_m,b_m,\mu}(g;x) - g(x)\right| \leq 2K\left(g;\left(\sqrt{M_{m,2}^{\mu}(x)} + \frac{M_{m,2}^{\mu}(x)}{2}\right)\right) \tag{3.17}$$

by taking infimum on both sides of the inequality (3.16) for $f \in C_B^2[0,\infty)$. As a result, the desired result is obtained from (3.14) and (3.17). □

4. Weighted approximation

Korovkin's theorem has an important place in approximation theory. According to this theorem, uniformly convergence of a linear positive operator to some test functions is examined, and by using the approximation character of these test functions, it is concluded that this operator uniformly converges to a continuous function in a real closed limited interval [27].

Gadjiev extended to Korovkin's theorem in an unlimited range for weighted function spaces [28]. Let function φ be a monotonous increased function, $\lim_{x\to\infty}\varphi(x) = \infty$ and $\rho(x) = 1 + \varphi^2(x)$ is a weighted function. Then, we recall some function spaces associated with weighted Korovkin theorem with help of function ρ as follows:

$$B_\rho[0,\infty) := \left\{g : [0,\infty) \longrightarrow \mathbb{R} \mid \ |g(x)| \leqslant M_f\left(1+\varphi^2(x)\right)\right\},$$

$$C_\rho[0,\infty) := \{g \in \ B_\rho[0,\infty) \mid f \textit{ is continuous on } [0,\infty)\},$$

$$C_\rho^\xi[0,\infty) := \left\{g \in \ C_\rho[0,\infty) \mid \lim_{x\to\infty}\tfrac{|g(x)|}{1+\varphi^2(x)} = \xi_f < \infty\right\},$$

(cf.[28]). $B_\rho[0,\infty), C_\rho[0,\infty)$ and $C_\rho^\xi[0,\infty)$ are normed spaces and the weighted norm on the space $B_\rho[0,\infty)$ is given as

$$\|g\|_\rho := \sup_{x\geq 0}\frac{|g(x)|}{\rho(x)}$$

for $g \in B_\rho[0,\infty)$. It is obviously $C_\rho^\xi[0,\infty) \subset C_\rho[0,\infty) \subset B_\rho[0,\infty)$, (cf.[28]).

Theorem 4.1 *(cf. [28]) If a sequence of linear positive operators $\{L_n\}_{n\geq 1}$ satisfying two conditions*

J1) the operators L_n act from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$,

J2) $\lim_{n\to\infty}\|L_n(e_k;\cdot) - e_k\|_\rho = 0,\ k = 0,1,2,$

then we have

$$\lim_{n\to\infty}\|L_n(f;\cdot) - f\|_\rho = 0$$

for $f \in C_\rho^\xi[0,\infty)$ where $e_k(t) := t^k$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

Lemma 4.2 *The operators $L_m^{a_m,b_m,\mu}$ defined in* (2.1) *have the following inequality*

$$L_m^{a_m,b_m,\mu}(\rho;x) \leq K\rho(x), K > 0$$

where $\rho(t) = 1 + t^2$.

We use the inequality $|\xi_m^\mu(x)| \le 1$ to prove the following theorem.

Theorem 4.3 *The operators* $L_m^{a_m,b_m,\mu}$ *given in* (2.1) *verify*

$$\lim_{m\to\infty} \left\| L_m^{a_m,b_m,\mu}(g;\cdot) - g \right\|_\rho = 0$$

for $g \in C_\rho^\xi[0,\infty)$ *where* $\rho : \rho(x) = 1 + x^2$.

Proof J1) Let $g \in C_\rho[0,\infty)$. For $\rho(t) = 1 + t^2$, we have

$$L_m^{a_m,b_m,\mu}(g;x) = L_m^{a_m,b_m,\mu}\left(\frac{g}{\rho}\rho;x\right) \le \|g\|_\rho L_m^{a_m,b_m,\mu}(\rho;x) \le \|g\|_\rho K\rho(x) \le M_g\rho(x)$$

where $M_g > 0$ thanks to Lemma 4.2. From the above inequality, it is $L_m^{a_m,b_m,\mu}(g;\cdot) \in B_\rho[0,\infty)$. So, we obtain that the operators $L_m^{a_m,b_m,\mu}$ act from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$.

J2) Let $e_k(t) := t^k$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It is clear that $\left\| L_m^{a_m,b_m,\mu}(e_0;\cdot) - e_0 \right\|_\rho = 0$. Furhermore, by using Lemma 2.2, we can write that

$$\begin{aligned} \left\| L_m^{a_m,b_m,\mu}(e_1;\cdot) - e_1 \right\|_\rho &\le \left| \frac{a_m}{b_m} - 1 \right| \sup_{x\ge0} \frac{x}{1+x^2} + \frac{1}{b_m} \sup_{x\ge0} \frac{\left| \frac{G'(1)+\mu[G(1)-G(-1)]\xi_m^\mu(x)}{G(1)} \right|}{1+x^2} \\ &\le \frac{1}{2}\left| \frac{a_m}{b_m} - 1 \right| + \frac{C_1}{b_m} \to 0, \ (m\to\infty) \end{aligned}$$

where the constant C_1 verifies

$$\left| \frac{G'(1) + \mu[G(1) - G(-1)]\xi_m^\mu(x)}{G(1)} \right| \le C_1, \ C_1 > 0$$

thanks to that $|\xi_m^\mu(x)| \le 1$ and the function G is an analytic function in the disc $|t| < r$, $r > 1$.

From Lemma 2.2, we obtain that

$$\begin{aligned} \left\| L_m^{a_m,b_m,\mu}(e_2;\cdot) - e_2 \right\|_\rho &\le \left| \left(\frac{a_m}{b_m}\right)^2 - 1 \right| \sup_{x\ge0} \frac{x^2}{1+x^2} + \frac{a_m}{b_m^2} \sup_{x\ge0} \frac{\left| \frac{2G'(1)+G(1)+2\mu G(-1)\xi_m^\mu(x)}{G(1)} \right| x}{1+x^2} \\ &\quad + \frac{1}{b_m^2} \sup_{x\ge0} \frac{\left| \frac{G''(1)+G'(1)+2\mu^2[G(1)-G(-1)]+2\mu\left[G'(1)+G'(-1)\right]\xi_m^\mu(x)}{G(1)} \right|}{1+x^2} \\ &\le \left| \left(\frac{a_m}{b_m}\right)^2 - 1 \right| + \frac{C_2}{2}\frac{a_m}{b_m^2} + \frac{C_3}{b_m^2} \to 0, \ (m\to\infty) \end{aligned}$$

where the constants C_2 and C_3 satisfy

$$\left| \frac{2G'(1)+G(1)+2\mu G(-1)\xi_m^\mu(x)}{G(1)} \right| \le C_2, \quad C_2 > 0$$

$$\left| \frac{G''(1)+G'(1)+2\mu^2[G(1)-G(-1)]+2\mu\left[G'(1)+G'(-1)\right]\xi_m^\mu(x)}{G(1)} \right| \le C_3, \quad C_3 > 0$$

thanks to that $|\xi_m^\mu(x)| \leq 1$ and the function G is an analytic function in the disc $|t| < r,\ r > 1$. As a result, in the light of all the above procedures, we obtain

$$\lim_{m\to\infty} \left\| L_m^{a_m,b_m,\mu}(e_k;\cdot) - e_k \right\|_\rho = 0,\ k = 0,1,2.$$

Finally, since the conditions J1 and J2 are verified, the desired result is obtained by using Theorem 4.1 and so, the proof is completed. □

5. Conclusion

We introduce a generalization of Szász-Mirakyan operators including Dunkl-Appell polynomials and sequences satisfying certain conditions and obtain some approximation properties. Similarly, the generalizations of Kantorovich, Durrmeyer and Stancu type of these operators can be constructed. These could be considered for future research.

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