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Unitary equivalence to truncated Hankel operators

Xi ZHAO, Tao YU*

School of Mathematical Sciences, Faculty of Science, Dalian University of Technology, Dalian, China

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Abstract: In this paper, we characterize the operators which are unitarily equivalent to truncated Hankel operators. We show that every rank one operator and every 2×2 matrix is unitarily equivalent to a truncated Hankel operator. Furthermore, we get that certain sum of tenser products of truncated Hankel operators is unitarily equivalent to a truncated Hankel operator.

Key words: Truncated Hankel operator, unitary equivalence, model space

1. Introduction

The model theory plays an important role in the study of contraction operators. It is particularly powerful when dealing with $C_0(1)$-operators. It is shown in [1] that every $C_0(1)$-operator is unitarily equivalent to a compressed shift on a model space. Hence, studying operators on a model space is of great significance. In 2007, Sarason introduces the truncated Toeplitz operator on the model space in [13]. Afterwards, truncated Toeplitz operators are extensively studied (see [5, 8] and their references). However, the truncated Hankel operators are not studied well. R. V. Bessonov [2] proved that every compact truncated Hankel operator has a continuous symbol. C. Gu characterized zero and rank one truncated Hankel operators on the model space, respectively [10]. It is reasonable to predict that truncated Hankel operators will also become an interesting class of operators. In this paper, we consider the problem what kind of operator is unitarily equivalent to a truncated Hankel operator.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ its boundary, the unit circle. Let L^2 denote the space of Lebesgue square integrable functions on the unit circle $\mathbb{T}$, and let H^2 denote the standard Hardy space which is the subspace of L^2 consisting of functions whose negative Fourier coefficients vanish.

Let P be the projection from L^2 onto H^2. For $\varphi \in L^2$, the Toeplitz operator T_φ with symbol φ is densely defined on H^2 by

$$T_\varphi f = P(\varphi f), \text{ for } f \in H^\infty.$$

The Hankel operator H_φ with symbol $\varphi \in L^2$ is densely defined on H^2 by

$$H_\varphi f = J(I-P)(\varphi f), \text{ for } f \in H^\infty,$$

*Correspondence: tyu@dlut.edu.cn

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where J is the unitary operator on L^2 defined by

$$[Jf](z) = \overline{z}\hat{f}, \text{ with } \hat{f}(z) = f(\overline{z}), \; f \in L^2.$$

Clearly, $H_\varphi^* = H_{\varphi^\#}$, where $\varphi^\#(z) = \overline{\varphi(\overline{z})}$.

A function $u \in H^2$ is called an inner function if $|u| = 1$ a.e. on $\mathbb{T}$. If u is a nonconstant inner function, then the model space $K_u = H^2 \ominus uH^2$ is a proper nontrivial invariant subspace of T_z^*. Let M_u be the multiplication operator on L^2 defined by $M_u f = uf$ for $f \in L^2$, and let P_u be the orthogonal projection from L^2 onto K_u. It is easy to see

$$P_u = P - M_u P M_{\overline{u}}.$$

For $\varphi \in L^2$, the truncated Toeplitz operator A_φ^u with symbol φ is densely defined on K_u by

$$A_\varphi^u f = P_u(\varphi f) \text{ for } f \in K_u \cap H^\infty.$$

For $\varphi \in L^2$, a truncated Hankel operator B_φ^u with symbol φ is densely defined on K_u by

$$B_\varphi^u f = P_u J(I-P)\varphi f = P_u J\varphi f, \text{ for } f \in K_u \cap H^\infty.$$

Obviously, $(B_\varphi^u)^* = B_{\varphi^\#}^u$.

Unitary equivalence to the truncated Toeplitz operator has been studied extensively. In [4], the authors show that all rank one operators, two by two matrices, normal operators and inflations of finite Toeplitz matrices are unitarily equivalent to truncated Toeplitz operators. The authors in [15] and [7] continue to study the problem. In [15], the authors obtain the theorem that a sum of some tensor products of truncated Toeplitz operators is unitarily equivalent to some truncated Toeplitz operator, and then give some operators which are unitarily equivalent to truncated Toeplitz operators. In [7], the authors give a necessary and sufficient condition when a n by n matrix is unitarily equivalent to an analytic truncated Toeplitz operator.

Now a natural problem is what operators are unitarily equivalent to truncated Hankel operators.

In this paper, we will generalize several results mentioned above to the setting of truncated Hankel operators. Firstly, we prove every rank one operator and every two by two matrix is unitarily equivalent to the truncated Hankel operator. Then we show that a sum of some tensor products of truncated Hankel operators with the form $\sum_j B_{u^j\psi}^u \otimes B_{\overline{z}^{j+1}\varphi}^\theta$ is unitarily equivalent to a truncated Hankel operator, and then obtain some concrete particular situations. We introduce when the direct sum of the truncated Hankel operator with zero operators and the tensor product of the truncated Hankel operator with the rank one operator are unitarily equivalent to truncated Hankel operators. We give two examples to show that some truncated Hankel operators are unitarily equivalent to block Hankel matrices.

2. Preliminaries

The reproducing kernels of the K_u are

$$K_\lambda^u(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z}, \; \lambda \in \mathbb{D}.$$

K_λ^u with $\lambda \in \mathbb{D}$ is a reproducing kernel at $\mathbb{D}$. Then we introduce the reproducing kernel at $\mathbb{T}$. To understand the reproducing kernel at $\mathbb{T}$, we need to know the angular derivative. In [13], we know a function u is said to have an angular derivative in the sense of Carathéodory (an ADC) at the point η of $\mathbb{T}$ if u has a nontangential limit $u(\eta)$ of unit modulus at η, and u' has a nontangential limit $u'(\eta)$ at η. There is a fundamental theorem about ADC.

Theorem 2.1 *(The Julia-Carathé odory Theorem) Suppose φ is a holomorphic self-map of $\mathbb{D}$, and $\zeta \in \mathbb{T}$. Then the following three statements are equivalent:*

(1) $0 < \liminf\limits_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} = \delta < \infty$;

(2) $\angle \lim\limits_{z\to\zeta} \frac{\eta-\varphi(z)}{\zeta-z}$ *exists for some* $\eta \in \mathbb{T}$;

(3) $\angle \lim\limits_{z\to\zeta} \varphi(z) = \eta \in \mathbb{T}$, $\angle \lim\limits_{z\to\zeta} \varphi'(z)$ *exists and equal to* $\zeta\overline{\eta}\delta$.

The $\angle \lim$ *means the nontangential limit.*

If u has an ADC at $\eta \in \mathbb{T}$, then the function

$$K_\eta^u(z) = \frac{1-\overline{u(\eta)}u(z)}{1-\overline{\eta}z} = \eta\overline{u(\eta)}(\frac{u(z)-u(\eta)}{z-\eta})$$

is a reproducing kernel at η. And $K_\lambda^u \to K_\eta^u$ as λ approaches η nontangentially from $\mathbb{D}$.

The norm of reproducing kernel is

$$\|K_\lambda^u\|^2 = \frac{1-|u(\lambda)|^2}{1-|\lambda|^2}.$$

Let $k_\lambda^u := \frac{K_\lambda^u}{\|K_\lambda^u\|}$ denote the normalized reproducing kernel.

Define a conjugation C on L^2 by $(Cf)(\zeta) = u(\zeta)\overline{\zeta f(\zeta)}$, for $f \in L^2$. We write $\widetilde{f}$ for Cf. A short calculation reveals that

$$\widetilde{K_\lambda^u}(z) = \frac{u(z)-u(\lambda)}{z-\lambda}.$$

Now we introduce several properties of truncated Hankel operators. It is easy to see that B_φ^u does not depend on the analytic part of the symbol function φ. Therefore, we often assume $\varphi \in \overline{zH^2}$. In [10], C. Gu gives the symbols of truncated Hankel operators which are zero.

Theorem 2.2 *[10] Suppose u is an inner function. A bounded truncated Hankel operator B_φ^u with $\varphi \in \overline{zH^2}$ is zero operator if and only if*

$$\varphi(z) = \overline{u(z)}u(\overline{z})\overline{h(z)} - |u(0)|^2\overline{h(0)},\ h \in H^2.$$

If x, y are two elements in some Hilbert space $\mathcal{H}$, let $x \odot y$ denote the rank one operator defined by $(x \odot y)h = \langle h, y\rangle x$, for $h \in \mathcal{H}$. Gu gives the rank one truncated Hankel operators.

Theorem 2.3 *[10] Let u be an inner function and $\theta(z) := u(z)\overline{u(\overline{z})}$.*

(1) For $\lambda \in \mathbb{D}$, the operator $K_{\overline{\lambda}}^u \odot K_\lambda^u$ and $\widetilde{K_{\overline{\lambda}}^u} \odot \widetilde{K_\lambda^u}$ belong to $\mathcal{G}_u$. And $K_{\overline{\lambda}}^u \odot K_\lambda^u$ has the symbol $\frac{1}{z-\lambda}$, $\widetilde{K_{\overline{\lambda}}^u} \odot \widetilde{K_\lambda^u}$ has the symbol $\frac{\overline{\theta}}{\overline{z}-\overline{\lambda}}$.

(2) If θ has an ADC at the point η of $\mathbb{T}$, then the operator $K_{\overline{\eta}}^u \odot K_\eta^u$ belongs to $\mathcal{G}_u$. And $\widetilde{K_{\overline{\eta}}^u} \odot \widetilde{K_\eta^u}$ has the symbol $\overline{K_\eta^\theta}$.

(3) The only rank one operators in $\mathcal{G}_u$ are nonzero scalar multiples of the operators in (1) and (2).

3. Rank one operators and 2×2 matrices

In this section, we show that every rank one operator and every 2×2 matrix is unitarily equivalent to a truncated Hankel operator.

The following lemma is well known.

Lemma 3.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Two operators of rank one $x_i \odot y_i \in L(\mathcal{H}_i)$, $i = 1, 2$, are unitarily equivalent if and only if the following are satisfied:*

(1) dim $\mathcal{H}_1$ = dim $\mathcal{H}_2$;

(2) $\|x_1\|\|y_1\| = \|x_2\|\|y_2\|$;

(3) $\langle x_1, y_1\rangle = \langle x_2, y_2\rangle$.

Theorem 3.2 *Every rank one operator is unitarily equivalent to a truncated Hankel operator.*

Proof Let $R = x \odot y$ on $\mathcal{H}$ where $\mathcal{H}$ is a Hilbert space. Without loss of generality, we assume that $\|x\| = \|y\| = 1$, then $0 \le |\langle x, y\rangle| \le 1$. We need to show that there exists an inner function θ and an appropriate point ζ such that $x \odot y$ is unitarily equivalent to $\rho k_{\overline{\zeta}}^\theta \odot k_\zeta^\theta$ or $\rho \widetilde{k_{\overline{\zeta}}^\theta} \odot \widetilde{k_\zeta^\theta}$ for some constant ρ with $|\rho| = 1$. Applying Lemma 3.1, it suffices to claim that

$$|\langle x, y\rangle| = |\langle k_\zeta^\theta, k_{\overline{\zeta}}^\theta\rangle|. \tag{3.1}$$

Suppose that $|\langle x, y\rangle| = 1$. Choosing any inner function θ and any real number $\lambda \in \mathbb{D}$, clearly we have

$$\langle k_\lambda^\theta, k_\lambda^\theta\rangle = |\langle x, y\rangle| = 1.$$

Suppose that $0 \le |\langle x, y\rangle| < 1$, we divide the discussion into two cases.

Case 1. If $\dim \mathcal{H} = n < \infty$, we choose $\theta(z) = z^n$. Take $\zeta \in \mathbb{T}$, then

$$\begin{aligned} |\langle k_\zeta^\theta, k_{\overline{\zeta}}^\theta\rangle| &= \left|\frac{1-\overline{\zeta}^{2n}}{n(1-\overline{\zeta}^2)}\right| \\ &= \left|\frac{1+\overline{\zeta}^2+\cdots+\overline{\zeta}^{2n-2}}{n}\right|. \end{aligned}$$

We know that $|\frac{1+\overline{\zeta}^2+\cdots+\overline{\zeta}^{2n-2}}{n}|$ can achieve 0 and 1 with $\zeta \in \mathbb{T}$, and it is a continuous function. Hence, the equation (3.1) is true by selecting appropriate ζ.

Case 2. If $\dim \mathcal{H} = \infty$. Suppose that $\langle x, y\rangle = 0$. Let θ be the singular inner function $\exp\left(\frac{z^2+1}{z^2-1}\right)$. It is clear that θ has ADC at $\zeta \in \mathbb{T}\backslash\{1, -1\}$. Taking $\zeta = \pm i$, we have

$$\begin{aligned}\langle k_\zeta^\theta, k_{\overline{\zeta}}^\theta\rangle &= \frac{1}{\sqrt{|\theta'(\zeta)\theta'(\overline{\zeta})|}}\frac{1-\overline{\theta(\zeta)}\theta(\overline{\zeta})}{1-\overline{\zeta}^2}\\ &= 0.\end{aligned}$$

Suppose that $0 < |\langle x, y\rangle| < 1$. Let $\theta_1(z) = \exp\left(\alpha\frac{z^2+1}{z^2-1}\right)$, $\alpha > 0$, and $\theta(z) = z\theta_1(z)$. Then θ has ADC at $\mathbb{T}\backslash\{1, -1\}$. Taking $\zeta^2 = -1$, we have $\theta(\overline{\zeta}) = \overline{\theta(\zeta)} = \overline{\zeta}$ and $|\theta_1'(\zeta)| = |\theta_1'(\overline{\zeta})| = \alpha$. Then

$$\begin{aligned}\langle k_\zeta^\theta, k_{\overline{\zeta}}^\theta\rangle &= \frac{1-\theta(\overline{\zeta})\overline{\theta(\zeta)}}{(1-\overline{\zeta^2})\sqrt{|\theta'(\zeta)||\theta'(\overline{\zeta})|}}\\ &= \frac{1}{\sqrt{|\theta'(\zeta)||\theta'(\overline{\zeta})|}}.\end{aligned}$$

By Theorem 2.1, we have

$$\begin{aligned}\liminf_{z\to\zeta}\frac{1-|\theta(z)|}{1-|z|} &= \liminf_{z\to\zeta}\frac{1-|z\theta_1(z)|}{1-|z|}\\ &= 1+\liminf_{z\to\zeta}\frac{1-|\theta_1(z)|}{1-|z|}\\ &= 1+\alpha.\end{aligned}$$

Similarly,

$$\liminf_{z\to\overline{\zeta}}\frac{1-|\theta(z)|}{1-|z|} = 1+\alpha > 1.$$

By Theorem 2.1, we get $|\theta'(\zeta)| = |\theta'(\overline{\zeta})| = 1+\alpha$. Hence, $\langle k_\zeta^\theta, k_{\overline{\zeta}}^\theta\rangle = \frac{1}{1+\alpha} < 1$, and it equals $|\langle x, y\rangle|$ by choosing appropriate α. □

Theorem 3.3 *Every 2×2 matrix is unitarily equivalent to a truncated Hankel operator.*

Proof Let T be a 2×2 matrix. In fact, every 2×2 matrix is unitarily equivalent to a complex symmetric matrix (see [3]). There is a unitary operator U such that $T = URU$, where R is a complex symmetric matrix. Hence, R is a Hankel matrix. Take $\theta = z^2$. It is easy to see that R is a truncated Hankel operator on K_θ. □

4. Tensor products of truncated Hankel operators

In this section, we prove that a sum of some tensor products of truncated Hankel operators is unitarily equivalent to a truncated Hankel operator. Suppose X, Y are Hilbert spaces, let $X\otimes Y$ denote a tensor product (see [11]). In the paper [15], a unitary operator from $K_u\otimes L^2$ onto L^2 is defined as the following.

Lemma 4.1 *[15] Suppose that u is an inner function. The mapping*

$$\Omega: \ h \otimes f \mapsto h(f \circ u), \quad h \in K_u, \ f \in L^2,$$

can be extended linearly to define a unitary operator from $K_u \otimes L^2$ onto L^2, denoted by Ω_u. It is clear that the image of $K_u \otimes H^2$ under Ω_u is H^2.

Lemma 4.2 *[15] Suppose that u and θ are inner functions, Ω_u is the unitary operator defined in Lemma 4.1. Then*

$$\Omega_u(K_u \otimes \theta H^2) = (\theta \circ u)H^2, \qquad \Omega_u(K_u \otimes K_\theta) = K_{\theta \circ u}.$$

Lemma 4.3 *Suppose that θ is an inner function and $\varphi \in L^2$. If B^θ_φ is bounded then $B^\theta_{z^j\varphi}$ is bounded for any $j \in \mathbb{Z}$.*

The proof of Lemma 4.3 is similar to that of Lemma 2.1 in [15], and we omit it.

In the following of the paper, we let u and θ denote inner functions satisfying $u^\# = u$. Let us now show that the sum of some tensor products of truncated Hankel operators is unitarily equivalent to a truncated Hankel operator.

Theorem 4.4 *Suppose $\psi, \varphi \in L^2$ are subjected to the following conditions,*

(1)$B^u_{u^j\psi}$ are bounded and nonzero only for a finite number of j,

(2)B^θ_φ is bounded,

(3)$\psi(\varphi \circ u) \in L^2$.

Then $B^{\theta\circ u}_{\psi(\varphi\circ u)}$ is bounded, and

$$B^{\theta\circ u}_{\psi(\varphi\circ u)}\Omega_u = \Omega_u\left(\sum_j B^u_{u^j\psi} \otimes B^\theta_{\overline{z}^{j+1}\varphi}\right). \tag{4.1}$$

Proof Let $h \in K_u^\infty$, $f \in K_\theta^\infty$. We have $J\psi(\varphi\circ u)\Omega_u(h\otimes f) = (J\psi h)(J\overline{z}(\varphi\circ u)(f\circ u))$. By the assumption $\psi \in L^2$, clearly $J\psi h \in L^2 = \oplus_{j\in\mathbb{Z}} u^j K_u$. It follows that

$$\begin{aligned} J\psi h &= \sum_j P_{u^jK_u} J\psi h \\ &= \sum_j u^j P_u \overline{u}^j J\psi h \\ &= \sum_j u^j P_u J u^j \psi h \\ &= \sum_j u^j B^u_{u^j\psi} h. \end{aligned}$$

Since $B^u_{\overline{u}^j\psi} \neq 0$ only for a finite number of j, then the sum above is finite. We can write

$$\begin{aligned} J\psi h(\varphi\circ u)(f\circ u) &= (J\psi h)J\overline{z}(\varphi\circ u)(f\circ u) \\ &= \sum_j u^j(B^u_{u^j\psi}h)J\overline{z}(\varphi\circ u)(f\circ u) \\ &= \sum_j (B^u_{u^j\psi}h)J\overline{u}^j\overline{z}(\varphi\circ u)(f\circ u) \\ &= \sum_j \Omega_u(B^u_{u^j\psi}h\otimes J\overline{z}^{j+1}\varphi f). \end{aligned}$$

Note that $\Omega_u(K_u\otimes K_\theta)=K_{\theta\circ u}$, we have

$$P_{\theta\circ u}=\Omega_u(P_u\otimes P_\theta)\Omega_u^*.$$

Then

$$\begin{aligned} B^{\theta\circ u}_{\psi(\varphi\circ u)}\Omega_u(h\otimes f) &= P_{\theta\circ u}J\psi(\varphi\circ u)(f\circ u)h \\ &= \Omega_u(P_u\otimes P_\theta)\Omega_u^*J\psi h(\varphi\circ u)(f\circ u) \\ &= \Omega_u(P_u\otimes P_\theta)(\sum_j (B^u_{u^j\psi}h\otimes J\overline{z}^{j+1}\varphi f)) \\ &= \Omega_u(\sum_j (B^u_{u^j\psi}h\otimes B^\theta_{\overline{z}^{j+1}\varphi}f)). \end{aligned}$$

By Lemma 4.3, $B^\theta_{z^j\varphi}$ is bounded for any j. The proof is now complete. □

Remark 4.5 *The proof of Theorem 4.4 is kind of similar to that of theorem 4.2 in [15] which shows that a truncated Toeplitz operator* $A^{\theta\circ B}_{\psi(\varphi\circ B)}$ *is unitarily equivalent to* $\sum_j (A^B_{\overline{u}^j\psi}\otimes A^\theta_{z^j\varphi})$*, where* θ *and* B *are inner functions. However, in the proof of Theorem 4.4, according to the definition of the truncated Hankel operator* $B^u_\psi h=P_uJ\psi h$*, we need to consider the operator* J *that is different from the truncated Toeplitz operator. And the discussion is closely related to the properties of* u*.*

Then we investigate some special cases. By Theorems 4.4 and 2.2, we take $\psi\in\overline{K_{u^2}}$ and get the following corollary:

Corollary 4.6 *If* $\psi\in\overline{K_{u^2}}$*,* $\varphi\in L^2$ *and* $B^u_{u^j\psi}$*,* B^θ_φ *are bounded for* $j=-1,0,1$*, then* $B^{\theta\circ u}_{\psi(\varphi\circ u)}$ *is bounded and*

$$B^{\theta\circ u}_{\psi(\varphi\circ u)}\cong B^u_{u\psi}\otimes B^\theta_{\overline{z}^2\varphi}+B^u_\psi\otimes B^\theta_{\overline{z}\varphi}+B^u_{\overline{u}\psi}\otimes B^\theta_\varphi. \tag{4.2}$$

Proof Note that $K_{u^2}=K_u\oplus uK_u$. And it is easy to see that $u\overline{K_u}=zK_u$.

If $j\geq 2$,

$$u^j\psi\in u^{j-2}u^2\overline{K_{u^2}}=u^{j-2}u^2(\overline{K_u}\oplus\overline{uK_u})=u^{j-1}zK_u\oplus u^{j-2}zK_u;$$

thus, $B^u_{u^j\psi}=0$.

If $j \leq -2$, it is easy to check that $u^j\psi \in \overline{u^2H^2}$. By Theorem 2.2, we get $B^u_{u^j\psi} = 0$.

Combining equation (4.1), we get that equation (4.2) holds. □

According to Corollary 4.6, we prove that the tenser product of a truncated Hankel operator and a self-adjoint rank one operator is unitarily equivalent to a truncated Hankel operator. The conclusions are analogous with truncated Toeplitz operators in [15], but the details of proof are different.

Theorem 4.7 *Let $d = \dim K_u$, take $\psi \in \overline{K_{u^2}}$ and $n \in \mathbb{N}\cup\{\infty\}$. Suppose that $B^u_{u^j\psi}$ is bounded for $j = -1, 0, 1$, then for any real constant $\lambda \in \mathbb{D}$, the operator $B^u_{\overline{u}\psi+\lambda\psi+\lambda^2u\psi} \oplus 0_{nd}$ is unitarily equivalent to a truncated Hankel operator.*

Proof Let θ be an inner function with order $n+1$, by (1) of Theorem 2.3, for any real constant $\lambda \in \mathbb{D}$

$$\widetilde{K^\theta_\lambda} \odot \widetilde{K^\theta_\lambda} = B^\theta_{\frac{\overline{\theta(z)}\theta(\overline{z})}{\overline{z}-\lambda}}.$$

Take $\varphi = \frac{\overline{\theta(z)}\theta(\overline{z})}{\overline{z}-\lambda}$, then $\overline{z}\varphi - \lambda\varphi = \overline{\theta(z)}\theta(\overline{z})$ and $\overline{z}^2\varphi - \lambda\overline{z}\varphi = \overline{z}\overline{\theta(z)}\theta(\overline{z})$. Theorem 2.2 states

$$B^\theta_{\overline{z}^2\varphi} = \lambda B^\theta_{\overline{z}\varphi} = \lambda^2 B^\theta_\varphi.$$

By equation (4.2) of Corollary 4.6,

$$B^{\theta\circ u}_{\psi(\varphi\circ u)} \cong B^u_{\overline{u}\psi+\lambda\psi+\lambda^2u\psi} \otimes B^\theta_\varphi = B^u_{\overline{u}\psi+\lambda\psi+\lambda^2u\psi} \otimes (\widetilde{K^\theta_\lambda} \odot \widetilde{K^\theta_\lambda}).$$

Since $\widetilde{K^\theta_\lambda} \odot \widetilde{K^\theta_\lambda}$ is a self-adjoint operator of rank one, $\frac{1}{\|K^\theta_\lambda\|^2} B^u_{\overline{u}\psi+\lambda\psi+\lambda^2u\psi} \otimes (\widetilde{K^\theta_\lambda} \odot \widetilde{K^\theta_\lambda})$ is unitarily equivalent to $B^u_{\overline{u}\psi+\lambda\psi+\lambda^2u\psi} \oplus 0_d \oplus \cdots \oplus 0_d$, where 0_d is repeated n times.

Therefore, $B^u_{\overline{u}\psi+\lambda\psi+\lambda^2u\psi} \oplus 0_{nd}$ is unitarily equivalent to the truncated Hankel operator on $K_{\theta\circ u}$ with the symbol $\frac{1}{\|K^\theta_\lambda\|^2}\psi\frac{\overline{\theta(u)}\theta(\overline{u})}{\overline{u}-\lambda}$.

□

Theorem 4.7 gives when the direct sum of a truncated Hankel operator with the zero operator is unitarily equivalent a truncated Hankel operator, and by Theorems 4.7 and 3.2, we get the following conclusion.

Corollary 4.8 *Suppose that $\psi \in \overline{K_{u^2}}$ and $B^u_{u^j\psi}$ is bounded for $j = -1, 0, 1$. Then for any selfadjoint rank one operator R and any real constant $\lambda \in \mathbb{D}$, the operator $B^u_{\overline{u}\psi+\lambda\psi+\lambda^2u\psi} \otimes R$ is unitarily equivalent to a truncated Hankel operator.*

Then we take $\psi \in \overline{K_u}$ and get the following corollary.

Corollary 4.9 *If $\psi \in \overline{K_u}$, $\varphi \in L^2$ and B^u_ψ, $B^u_{\overline{u}\psi}$, B^θ_φ are bounded, then $B^{\theta\circ u}_{\psi(\varphi\circ u)}$ is bounded and*

$$B^{\theta\circ u}_{\psi(\varphi\circ u)} \cong B^u_\psi \otimes B^\theta_{\overline{z}\varphi} + B^u_{\overline{u}\psi} \otimes B^\theta_\varphi. \tag{4.3}$$

The proof of the Corollary 4.9 is similar to that of Corollary 4.6.

Using Corollary 4.9, we obtain that the tenser product of a truncated Hankel operator and a nonself-adjoint rank one operator is unitarily equivalent to a truncated Hankel operator.

Theorem 4.10 *Suppose that θ has ADC at ζ and $\overline{\zeta}$, where $\zeta \in \mathbb{T}$, and $\zeta \neq \pm 1$. If $\psi \in \overline{K_u}$ such that B^u_ψ and $B^u_{\overline{u}\psi}$ are bounded, then $B^u_{\zeta\psi+\overline{u}\psi} \otimes (\widetilde{K^\theta_{\overline{\zeta}}} \odot \widetilde{K^\theta_\zeta})$ is unitarily equivalent to a truncated Hankel operator.*

Proof Take

$$\varphi(z) = \frac{\theta(\overline{z})\overline{\theta(z)} - \theta(\overline{\zeta})\overline{\theta(\zeta)}}{\overline{z} - \overline{\zeta}}.$$

In [10], we know that $B^\theta_\varphi = \widetilde{K^\theta_{\overline{\zeta}}} \odot \widetilde{K^\theta_\zeta}$. A simple calculation shows that $\zeta\overline{z}\varphi - \varphi = \zeta(\theta(\overline{z})\overline{\theta(z)} - \theta(\overline{\zeta})\overline{\theta(\zeta)})$, it follows that $B^\theta_{\zeta\overline{z}\varphi} = B^\theta_\varphi$. Applying Corollary 4.9, we obtain that

$$B^{\theta\circ u}_{\psi(\varphi\circ u)} \cong B^u_{\zeta\psi+\overline{u}\psi} \otimes (\widetilde{K^\theta_{\overline{\zeta}}} \odot \widetilde{K^\theta_\zeta}).$$

$\square$

By Theorems 4.10 and 3.2, the following Corollary is immediately obtained.

Corollary 4.11 *If $\psi \in \overline{K_u}$, such that B^u_ψ and $B^u_{\overline{u}\psi}$ are bounded, and $\zeta, \overline{\zeta} \in \mathbb{T}, \zeta \neq \pm 1$, then for any nonself-adjoint rank one operator R, we have that $B^u_{\zeta\psi+\overline{u}\psi} \otimes R$ is unitarily equivalent to a truncated Hankel operator.*

Then we take $\psi = 1$, equation (4.1) gives the following conclusion.

Corollary 4.12 *Suppose that $\varphi \in L^2$ and u is a Blaschke product of order k, $k = 1, 2, \cdots, \infty$. Assume that B^θ_φ is bounded. Then $B^{\theta\circ u}_{\varphi\circ u}$ is bounded and unitarily equivalent to $B^u_{\overline{u}} \otimes B^\theta_\varphi$, and $(B^{\theta\circ u}_{\varphi\circ u})^2$ is unitarily equivalent to $I_k \otimes (B^\theta_\varphi)^2$.*

Proof Take $\psi = 1$. If $j \geq 0$, u^j is analytic, clearly $B^u_{u^j} = 0$. If $j \leq -2$, we have $u^j \in \overline{u^2 H^2}$, Theorem 2.2 implies $B^u_{u^j} = 0$. Then equation (4.1) gives $B^{\theta\circ u}_{\varphi\circ u} \cong B^u_{\overline{u}} \otimes B^\theta_\varphi$. We have $(B^u_{\overline{u}})^2 = I$. Therefore, $(B^{\theta\circ u}_{\varphi\circ u})^2 \cong (B^u_{\overline{u}} \otimes B^\theta_\varphi)^2 = I_k \otimes (B^\theta_\varphi)^2$. $\square$

Remark 4.13 *Unlike the truncated Toeplitz operator, we note that the identity operator is not a truncated Hankel operator. Therefore, the result in Corollary 4.12 is somewhat different from the setting of the truncated Toeplitz operator.*

5. Block Hankel matrices

Now we give two examples which show that some truncated Hankel operator is unitarily equivalent to a block Hankel matrix.

Example 5.1 *Suppose $u(z) = z^n$, then*

$$B^u_{\overline{z}^m} = (a_{i,j}) = \begin{cases} 1, & if\ i + j = m + 1, \\ 0, & otherwise. \end{cases}$$

Take $\varphi_m \in L^2$, $m = 0, \cdots, n-1$, such that $B^\theta_{\varphi_m}$ are bounded. Define

$$h(z) = \sum_{m=0}^{n-1} \overline{z}^m(\varphi_m(z^n)) = \sum_{m=0}^{n-1} \overline{z}^m(\varphi_m \circ u).$$

By corollary 4.9, we have

$$B_h^{\theta\circ u} \cong \sum_{m=0}^{n-1} (B^u_{\overline{z}^m} \otimes B^\theta_{\overline{z}\varphi_m} + B^u_{\overline{z}^{m+n}} \otimes B^\theta_{\varphi_m}).$$

Then $B_h^{\theta\circ u}$ *is bounded and unitarily equivalent to the matrix*

$$\begin{pmatrix} B^\theta_{\overline{z}\varphi_1} & B^\theta_{\overline{z}\varphi_2} & \cdots & B^\theta_{\overline{z}\varphi_{n-1}} & B^\theta_{\varphi_0} \\ B^\theta_{\overline{z}\varphi_2} & B^\theta_{\overline{z}\varphi_3} & \cdots & B^\theta_{\varphi_0} & B^\theta_{\varphi_1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ B^\theta_{\overline{z}\varphi_{n-1}} & B^\theta_{\varphi_0} & \cdots & B^\theta_{\varphi_{n-3}} & B^\theta_{\varphi_{n-2}} \\ B^\theta_{\varphi_0} & B^\theta_{\varphi_1} & \cdots & B^\theta_{\varphi_{n-2}} & B^\theta_{\varphi_{n-1}} \end{pmatrix}.$$

Example 5.2 *Suppose* $\theta = z^n$. *Take* $\psi \in \overline{K_u}$, $i = 0, \cdots, 2n-1$. *Assume* $B^u_{\psi_i}$ *and* $B^u_{\overline{u}\psi_i}$ *are bounded for* $i = 0, \cdots, 2n-1$. *Define*

$$h(z) = \sum_{m=1}^{2n-1} \psi_m \overline{u}^m = \sum_{m=1}^{2n-1} \psi_m (\overline{z}^m \circ u).$$

We have

$$B_h^{\theta\circ u} \cong \sum_{m=0}^{2n-1} (B^u_{\psi_m} \otimes B^\theta_{\overline{z}^{m+1}} + B^u_{\overline{u}\psi_m} \otimes B^\theta_{\overline{z}^m}).$$

Then $B_h^{\theta\circ u}$ *is bounded and unitarily equivalent to the matrix*

$$\begin{pmatrix} B^u_{\psi_0+\overline{u}\psi_1} & B^u_{\psi_1+\overline{u}\psi_2} & \cdots & B^u_{\psi_{n-2}+\overline{u}\psi_{n-1}} & B^u_{\psi_{n-1}+\overline{u}\psi_n} \\ B^u_{\psi_1+\overline{u}\psi_2} & B^u_{\psi_2+\overline{u}\psi_3} & \cdots & B^u_{\psi_{n-1}+\overline{u}\psi_n} & B^u_{\psi_n+\overline{u}\psi_{n+1}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ B^u_{\psi_{n-2}+\overline{u}\psi_{n-1}} & B^u_{\psi_{n-1}+\overline{u}\psi_n} & \cdots & B^u_{\psi_{2n-4}+\overline{u}\psi_{2n-3}} & B^u_{\psi_{2n-3}+\overline{u}\psi_{2n-2}} \\ B^u_{\psi_{n-1}+\overline{u}\psi_n} & B^u_{\psi_n+\overline{u}\psi_{n+1}} & \cdots & B^u_{\psi_{2n-3}+\overline{u}\psi_{2n-2}} & B^u_{\psi_{2n-2}+\overline{u}\psi_{2n-1}} \end{pmatrix}.$$

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References

[1] Bercovici H. Operator Theory and Arithmetic in H^∞. Mathematical Surveys and Monographs, No. 26, Providence, RI, USA: American Mathematical Society, 1988.

[2] Bessonov RV. Fredholmness and compactness of truncated Toeplitz and Hankel operators. Integral Equations and Operator Theory 2015; 82: 451-467. doi: 10.1007/s00020-014-2177-2

[3] Chevrot N, Fricain E, Timotin D. The characteristic function of a complex symmetric contraction. Proceedings of the American Mathematical Society 2007; 135 (9): 2877-2886. doi: 10.1090/S0002-9939-07-08803-X

[4] Cima JA, Garcia SR, Ross WT, Wogen WR. Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity. Indiana University Mathematics Journal 2010; 59 (2): 595-620. doi: 10.1512/iumj.2010.59.4097

[5] Garayev M, Guediri H, Sadraoui H. Applications of reproducing kernels and Berezin symbols. New York Journal of Mathematics 2016; 22: 583-604.

[6] Garnett JB. Bounded Analytic Functions. New York, NY, USA: Academic Press, 1981.

[7] Garcia SR, Poore DE, Ross WT. Unitary equivalence to a truncated Toeplitz operator: analytic symbols. Proceedings of the American Mathematical Society 2012; 140 (4): 1281-1295. doi: 10.1090/S0002-9939-2011-11060-8

[8] Garcia SR, Ross WT. Recent progress on truncated Toeplitz operators. In Blaschke Products and Their Applications, Fields Institute Communication, vol. 65, New York, NY, USA: Springer, 2013, pp. 275-391.

[9] Garcia SR, Ross WT, Wogen WR. Spatial isomorphisms of algebras of truncated Toeplitz operators. Indiana University Mathematics Journal 2010; 59 (6): 1971-2000. doi: 10.1512/iumj.2010.59.4549

[10] Gu C. Algebraic properties of truncated Hankel operators. preprint.

[11] Lang S. Fundamentals of Differential Geometry. volume 191 of Graduate Texts in Mathematics, New York, NY, USA: Springer-Verlag, , 1999.

[12] Nikolskii NK. Operators, Functions, and Systems: An Easy Reading, Vol. 1: Hardy, Hankel, and Toeplitz. Providence, RI, USA: American Mathematical Society, 2002.

[13] Sarason D. Algebraic properties of truncated Toeplitz operators. Operators and Matrices 2007; 1 (4): 491-526. doi: 10.7153/oam-01-29

[14] Sarason D. Angular derivatives via Hilbert space. Complex Variables, Theory and Application1988; 10: 1-10. doi: 10.1080/17476938808814282

[15] Strouse E, Timotin D, Zarrabi M. Unitary equivalence to truncated Toeplitz operators. Indiana University Mathematics Journal 2012; 61 (2): 525-538. doi: 10.1512/iumj.2012.61.4562