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Semisymmetric hypersurfaces in complex hyperbolic two-plane Grassmannians

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Abstract: In this paper, we introduce new notions of symmetric operators such as semisymmetric shape operator and structure Jacobi operator in complex hyperbolic two-plane Grassmannians. Next we prove that there does not exist a Hopf real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$ with such notions.

Key words: Real hypersurfaces, complex hyperbolic two-plane Grassmannians, Hopf hypersurface, semisymmetric shape operator, semisymmetric structure Jacobi operator

1. Introduction

It is one of the main topics in submanifold geometry to investigate an immersed real hypersurface of homogeneous type in Hermitian symmetric spaces of rank 2 (HSS2) with certain geometric condition. Understanding and classifying real hypersurfaces in HSS2 is an important problem in differential geometry. One of these spaces is the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. For indefinite complex Euclidean spaces, we give a definition of dual space of $G_2(\mathbb{C}^{m+2})$ denoted by $G_2^*(\mathbb{C}^{m+2})$ as the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$. $G_2^*(\mathbb{C}^{m+2})$ has homogeneous expression as $SU_{2,m}/S(U_2 \cdot U_m)$ and is called the complex hyperbolic two-plane Grassmannian which is the unique noncompact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold. Thanks to Berndt and Suh [2], comparing to $G_2(\mathbb{C}^{m+2})$ with compact type, we have investigated geometry of submanifolds in $SU_{2,m}/S(U_2 \cdot U_m)$.

Let N be a local unit normal vector field on M. Since $SU_{2,m}/S(U_2 \cdot U_m)$ has a Kähler structure J, we may define the *Reeb vector field* $\xi = -JN$ and a 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$.

Let $\mathcal{C}$ be a distribution which stands for the orthogonal complement of $[\xi]$ in T_xM at $x \in M$. It becomes the complex maximal subbundle of T_xM. Thus the tangent space of M consists of the direct sum of $\mathcal{C}$ and $\mathcal{C}^{\perp}(:= [\xi])$ as follows: $T_xM = \mathcal{C} \oplus \mathcal{C}^{\perp}$. The real hypersurface M is said to be *Hopf* if $A\mathcal{C} \subset \mathcal{C}$, or equivalently, the Reeb vector field ξ is principal with principal curvature $\alpha = g(A\xi, \xi)$. In this case, the principal curvature $\alpha = g(A\xi, \xi)$ is said to be the *Reeb curvature* of M.

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From the quaternionic Kähler structure $\mathfrak{J} = \text{span}\{J_1, J_2, J_3\}$ of $SU_{2,m}/S(U_2 \cdot U_m)$, there naturally exist *almost contact 3-structure* vector fields $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Put $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$, which is a 3-dimensional distribution on M. In addition, $\mathcal{Q}$ stands for the orthogonal complement of $\mathcal{Q}^\perp$ in T_xM. It becomes the quaternionic maximal subbundle of T_xM. Thus the tangent space of M consists of the direct sum of $\mathcal{Q}$ and $\mathcal{Q}^\perp$ as follows: $T_xM = \mathcal{Q} \oplus \mathcal{Q}^\perp$.

Thus, we considered as geometric conditions for real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ that the subbundles $\mathcal{C}$ and $\mathcal{Q}$ of TM are both invariant under the shape operator. By using them, Berndt and Suh proved the following:

Theorem 1.1 ([3]) *Let M be a connected real hypersurface in $SU_{2,m}/S(U_2 \cdot U_m), m \geq 3$. Then the following statements are equivalent:*

- *The maximal complex subbundle $\mathcal{C}$ and the maximal quaternionic subbundle $\mathcal{Q}$ of the tangent bundle of M are invariant under the shape operator A of M;*
- *M is congruent to an open part of one of the following hypersurfaces:*

1. *A tube with radius $r \in \mathbb{R}_+$ around the complex and quaternionic totally geodesic embedding of $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ into $SU_{2,m}/S(U_2 \cdot U_m)$;*
2. *A tube with radius $r \in \mathbb{R}_+$ around the complex and totally complex totally geodesic embedding of the complex hyperbolic space $\mathbb{C}H^m$ into $SU_{2,m}/S(U_2 \cdot U_m)$;*
3. *(Only if $m = 2l$ is even) A tube with radius $r \in \mathbb{R}_+$ around the real and quaternionic totally geodesic embedding of the quaternionic hyperbolic space $\mathbb{H}H^l$ into $SU_{2,m}/S(U_2 \cdot U_m)$;*
4. *A horosphere with singular point at infinity of type $\xi \in \mathcal{Q}^\perp$;*
5. *A horosphere with singular point at infinity of type $\xi \in \mathcal{Q}$;*
6. *The normal vector field N of M is singular of type $\xi \in \mathcal{Q}$ and M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = 2, \gamma = 0, \lambda = 1,$$

with corresponding principal curvature spaces

$$T_\alpha = (\mathcal{C} \cap \mathcal{Q})^\perp, T_\gamma = J\mathcal{Q}^\perp, T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then $JT_\mu \subseteq T_\lambda$ and $\mathcal{J}T_\mu \subseteq T_\lambda$.

Certain parallelisms on the other symmetric operators such as Ricci operator, structure Jacobi and normal Jacobi operators for real hypersurfaces in Hermitian symmetric spaces are extensively studied. Among them, shape operator was studied by Niebergall and Ryan [12]. The Ricci operator were considered by Lee, Suh and Woo [10] and Suh [15, 16] and Suh and Woo [17]. Moreover, the structure Jacobi and normal Jacobi operators for real hypersurfaces in Hermitian symmetric spaces have been undertaken by Hwang, Lee and Woo [6], Panagiotidou and Tripathi [13] and Pérez and Santos [14].

Semisymmetry for an operator T on M means the Riemannian curvature tensor R act as a derivation of T on M as follows: $R \cdot T = 0$. Hwang, Lee and Woo [6] proved nonexistence of Hopf hypersufaces in $G_2(\mathbb{C}^{m+2})$ with semisymmetric T (T denotes either shape operator, structure Jacobi operator or normal Jacobi operator) as follows:

Theorem 1.2 ([6]) *There does not exist any Hopf real hypersurface in complex two-plane Grassmannians* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, *with semiparallel shape operator and* $\xi\alpha = 0$.

Theorem 1.3 ([6]) *There does not exists any Hopf real hypersurface in complex two-plane Grassmannians* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, *with semiparallel structure Jacobi operator and* $\xi\alpha = 0$.

Based on these results, in this paper, we will consider the notion of *semisymmetric operator* for real hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$ in the class of Hermitian symmetric spaces.

Theorem 1.4 *There does not exist any Hopf real hypersurface in complex hyperbolic two-plane Grassmannians* $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, *with semisymmetric shape operator.*

By a result in [4], we know that if a tensor field is recurrent, it is always semisymmetric. Hence, as a corollary we obtain the following:

Corollary 1.5 *There does not exists any Hopf real hypersurface in complex hyperbolic two-plane Grassmannians* $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, *with parallel or recurrent shape operator.*

On the other hand, related to the structure Jacobi operator R_ξ, we have the following results:

Theorem 1.6 *There does not exists any Hopf real hypersurface in complex hyperbolic two-plane Grassmannians* $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, *with semisymmetric structure Jacobi operator.*

Also using [4] again, we have the following:

Corollary 1.7 *There does not exist a connected Hopf real hypersurface in the complex hyperbolic two-plane Grassmannians* $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, *with parallel or recurrent structure Jacobi operator.*

2. The complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$

In this section we summarize basic material about complex hyperbolic two-plane Grassmann manifolds $SU_{2,m}/S(U_2 \cdot U_m)$, for details we refer to [2, 3, 15, 16].

The Riemannian symmetric space $SU_{2,m}/S(U_2 \cdot U_m)$, which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, becomes a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank two. Let $G = SU_{2,m}$ and $K = S(U_2 \cdot U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{A}ut(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m} A I_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}.$$

I_2 and I_m denote the identity (2×2)-matrix and $(m \times m)$-matrix, respectively. Then $< X, Y >= -B(X, \theta Y)$ becomes a positive definite $\mathrm{A}d(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2 {\cdot} U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2 {\cdot} U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2 {\cdot} U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2 {\cdot} U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2 {\cdot} U_m)$. By construction, J commutes with each almost Hermitian structure J_ν in $\mathfrak{J}$ for $\nu = 1, 2, 3$. Recall that a canonical local basis J_1, J_2, J_3 of a quaternionic Kähler structure $\mathfrak{J}$ consists of three almost Hermitian structures J_1, J_2, J_3 in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is to be taken modulo 3. The tensor field JJ_ν, which is locally defined on $SU_{2,m}/S(U_2 {\cdot} U_m)$, is selfadjoint and satisfies $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$, where I is the identity transformation. For a nonzero tangent vector X we define $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

We identify the tangent space $T_o SU_{2,m}/S(U_2 {\cdot} U_m)$ of $SU_{2,m}/S(U_2 {\cdot} U_m)$ at o with $\mathfrak{p}$ in the usual way. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since $SU_{2,m}/S(U_2 {\cdot} U_m)$ has rank two, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_o SU_{2,m}/S(U_2 {\cdot} U_m) \cong \mathfrak{p}$ is contained in some maximal abelian subspace of $\mathfrak{p}$. Generically this subspace is uniquely determined by X, in which case X is called regular. If there exist more than one maximal abelian subspaces of $\mathfrak{p}$ containing X, then X is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

3. Real hypersurfaces in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$

The complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 {\cdot} U_m)$ is the unique noncompact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold. Remarkably, it is equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ (not containing J) satisfying $JJ_\nu = J_\nu J$ $(\nu = 1, 2, 3)$, where $\{J_\nu\}_{\nu=1,2,3}$ is an orthonormal basis of $\mathfrak{J}$.

Let M be a real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$, that is, a submanifold in $SU_{2,m}/S(U_2 \cdot U_m)$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Levi–Civita covariant derivative of (M, g). From the quaternionic Kähler structure $\mathfrak{J}$ of $SU_{2,m}/S(U_2 \cdot U_m)$, there naturally exist *almost contact 3-structure* vector fields defined by $\xi_\nu = -J_\nu N$, $\nu = 1,2,3$. We denote by $\mathcal{C}$ and $\mathcal{Q}$ the maximal complex and quaternionic subbundle of the tangent bundle TM of M, respectively. Now let us put

$$JX = \varphi X + \eta(X)N, \quad J_\nu X = \varphi_\nu X + \eta_\nu(X)N \quad (\nu = 1,2,3) \tag{3.1}$$

for any tangent vector field X on M in $SU_{2,m}/S(U_2 \cdot U_m)$, where φX (resp., $\varphi_\nu X$) denotes the tangential component of JX (resp., $J_\nu X$) and N a unit normal vector field of M in $SU_{2,m}/S(U_2 \cdot U_m)$.

From the Kähler structure J of $SU_{2,m}/S(U_2 \cdot U_m)$ there exists an almost contact metric structure (φ, ξ, η, g) induced on M in such a way that

$$\varphi^2 X = -X + \eta(X)\xi, \; \eta(\xi) = 1, \; \varphi\xi = 0, \quad \text{and} \quad \eta(X) = g(X, \xi) \tag{3.2}$$

for any vector field X on M and $\xi = -JN$.

If M is orientable, then the vector field ξ is globally defined and it is said to be the induced *Reeb vector field* on M. Then each J_ν induces a local almost contact metric structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1,2,3$, on M. Locally, $\mathcal{C}$ is the orthogonal complement in TM of the real span of ξ, and $\mathcal{Q}$ the orthogonal complement in TM of the real span of $\{\xi_1, \xi_2, \xi_3\}$. Moreover, the almost contact metric 3-structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$ on M satisfies

$$\begin{aligned} &\varphi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \; \varphi_\nu\xi_\nu = 0, \; \eta_\nu(\xi_\nu) = 1 \\ &\varphi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \varphi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \end{aligned} \tag{3.3}$$

for any vector field X tangent to M. The tangential and normal components of the commuting identity $JJ_\nu X = J_\nu JX$ give

$$\varphi\varphi_\nu X - \varphi_\nu\varphi X = \eta_\nu(X)\xi - \eta(X)\xi_\nu \text{ and } \quad \eta_\nu(\varphi X) = \eta(\varphi_\nu X). \tag{3.4}$$

The last equation implies $\varphi_\nu\xi = \varphi\xi_\nu$.

From the parallelism of Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$, together with Gauss and Weingarten formulas it follows that

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \varphi AX, \tag{3.5}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \varphi_\nu AX, \tag{3.6}$$

where q_ν are 1-forms associated to the derivatives of $\{J_\nu\}_{\nu=1,2,3}$.

Finally, using the explicit expression for the Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2 \cdot U_m)$ in [2] the Codazzi equation takes the form

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = -\frac{1}{2}\Big[&\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi \\ &+ \sum_{\nu=1}^3 \big\{\eta_\nu(X)\varphi_\nu Y - \eta_\nu(Y)\varphi_\nu X - 2g(\varphi_\nu X, Y)\xi_\nu\big\} \\ &+ \sum_{\nu=1}^3 \big\{\eta_\nu(\varphi X)\varphi_\nu\varphi Y - \eta_\nu(\varphi Y)\varphi_\nu\varphi X\big\} \\ &+ \sum_{\nu=1}^3 \big\{\eta(X)\eta_\nu(\varphi Y) - \eta(Y)\eta_\nu(\varphi X)\big\}\xi_\nu\Big] \end{aligned} \tag{3.7}$$

for any vector fields X and Y on M. Moreover, we have the equation of Gauss as follows:

$$\begin{aligned} R(X,Y)Z = &-\frac{1}{2}\Big[g(Y,Z)X - g(X,Z)Y \\ &+ g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z \\ &+ \sum_{\nu=1}^{3}\{g(\varphi_\nu Y,Z)\varphi_\nu X - g(\varphi_\nu X,Z)\varphi_\nu Y - 2g(\varphi_\nu X,Y)\varphi_\nu Z\} \\ &+ \sum_{\nu=1}^{3}\{g(\varphi_\nu\varphi Y,Z)\varphi_\nu\varphi X - g(\varphi_\nu\varphi X,Z)\varphi_\nu\varphi Y\} \\ &- \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\varphi_\nu\varphi X - \eta(X)\eta_\nu(Z)\varphi_\nu\varphi Y\} \\ &- \sum_{\nu=1}^{3}\{\eta(X)g(\varphi_\nu\varphi Y,Z) - \eta(Y)g(\varphi_\nu\varphi X,Z)\}\,\xi_\nu\Big] \\ &+ g(AY,Z)AX - g(AX,Z)AY \end{aligned} \tag{3.8}$$

for any tangent vector fields X, Y and Z on M. Hereafter, unless otherwise stated, we want to use these basic equations mentioned above frequently without referring to them explicitly.

With the assumption of Hopf hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$, we also have the following equations

$$Y\alpha = (\xi\alpha)\eta(Y) + 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\varphi Y). \tag{3.9}$$

$$\begin{aligned} A\varphi AY = &\frac{\alpha}{2}(A\varphi + \varphi A)Y + \sum_{\nu=1}^{3}\big\{\eta(Y)\eta_\nu(\xi)\varphi\xi_\nu + \eta_\nu(\xi)\eta_\nu(\varphi Y)\xi\big\} \\ &-\frac{1}{2}\varphi Y - \frac{1}{2}\sum_{\nu=1}^{3}\big\{\eta_\nu(Y)\varphi\xi_\nu + \eta_\nu(\varphi Y)\xi_\nu + \eta_\nu(\xi)\varphi_\nu Y\big\} \end{aligned} \tag{3.10}$$

for any vector field Y on M (see [2]).

4. Semisymmetric shape operator

In this section, let M represent a Hopf real hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, and R denote the Riemannian curvature tensor of M. Hereafter unless otherwise stated, we consider that X, Y, and Z are any tangent vector fields on M. Let W be any tangent vector field in $\mathcal{Q}$.

We first give the fundamental equation for the semisymmetry of a tensor field T of type (1,1) in M and prove our Theorem 1.4.

As mentioned in Introduction, a tensor field T on M is said to be semisymmetric, if T satisfies $R\cdot T = 0$. It is equal to

$$(R(X,Y)T)Z = 0. \tag{†}$$

Since $(R(X,Y)T)Z = R(X,Y)(TZ) - T(R(X,Y)Z)$, the equation (†) is equivalent to the following

$$R(X,Y)(TZ) = T(R(X,Y)Z). \tag{‡}$$

Therefore from (3.8), it becomes

$$
\begin{aligned}
& g(Y,TZ)X - g(X,TZ)Y + g(\varphi Y,TZ)\varphi X - g(\varphi X,TZ)\varphi Y \\
& - 2g(\varphi X,Y)\varphi TZ - 2g(AY,TZ)AX + 2g(AX,TZ)AY \\
& + \sum_{\nu}\Big\{g(\varphi_\nu Y,TZ)\varphi_\nu X - g(\varphi_\nu X,TZ)\varphi_\nu Y - 2g(\varphi_\nu X,Y)\varphi_\nu TZ\Big\} \\
& + \sum_{\nu}\Big\{g(\varphi_\nu\varphi Y,TZ)\varphi_\nu\varphi X - g(\varphi_\nu\varphi X,TZ)\varphi_\nu\varphi Y\Big\} \\
& - \sum_{\nu}\Big\{\eta(Y)\eta_\nu(TZ)\varphi_\nu\varphi X - \eta(X)\eta_\nu(TZ)\varphi_\nu\varphi Y\Big\} \\
& - \sum_{\nu}\Big\{\eta(X)g(\varphi_\nu\varphi Y,TZ) - \eta(Y)g(\varphi_\nu\varphi X,TZ)\Big\}\xi_\nu \\
= \; & g(Y,Z)TX - g(X,Z)TY + g(\varphi Y,Z)T\varphi X - g(\varphi X,Z)T\varphi Y \\
& - 2g(\varphi X,Y)T\varphi Z - 2g(AY,Z)TAX + 2g(AX,Z)TAY \\
& + \sum_{\nu}\Big\{g(\varphi_\nu Y,Z)T\varphi_\nu X - g(\varphi_\nu X,Z)T\varphi_\nu Y - 2g(\varphi_\nu X,Y)T\varphi_\nu Z\Big\} \\
& + \sum_{\nu}\Big\{g(\varphi_\nu\varphi Y,Z)T\varphi_\nu\varphi X - g(\varphi_\nu\varphi X,Z)T\varphi_\nu\varphi Y\Big\} \\
& - \sum_{\nu}\Big\{\eta(Y)\eta_\nu(Z)T\varphi_\nu\varphi X - \eta(X)\eta_\nu(Z)T\varphi_\nu\varphi Y\Big\} \\
& - \sum_{\nu}\Big\{\eta(X)g(\varphi_\nu\varphi Y,Z) - \eta(Y)g(\varphi_\nu\varphi X,Z)\Big\}T\xi_\nu,
\end{aligned}
\tag{4.1}
$$

where $\sum_\nu$ moves from $\nu = 1$ to $\nu = 3$.

Using this discussion, let us prove our Theorem 1.4 given in the Introduction. In order to do this, suppose that M has semisymmetric shape operator, that is, the shape operator A of M satisfies the condition $(R(X,Y)A)Z = 0$.

Putting $T = A$ and putting $Y = Z = \xi$ and using the condition of Hopf, the equation (4.1) can be reduced to

$$
\begin{aligned}
& AX - 2\alpha A^2X \\
& \quad - \sum_{\nu}\Big\{\big(\eta_\nu(X) - \eta(X)\eta_\nu(\xi)\big)A\xi_\nu + 3\eta_\nu(\varphi X)A\varphi_\nu\xi + \eta_\nu(\xi)A\varphi_\nu\varphi X\Big\} \\
= \; & \alpha X - 2\alpha^2 AX \\
& \quad - \alpha\sum_{\nu}\Big\{\big(\eta_\nu(X) - \eta(X)\eta_\nu(\xi)\big)\xi_\nu + 3\eta_\nu(\varphi X)\varphi_\nu\xi + \eta_\nu(\xi)\varphi_\nu\varphi X\Big\}.
\end{aligned}
\tag{4.2}
$$

Our first purpose is to show that ξ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$.

Lemma 4.1 *Let M be a Hopf hypersurface with semisymmetric shape operator in $SU_{2,m}/S(U_2\cdot U_m)$, $m \geq 3$. Then ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.*

Proof We consider that ξ satisfies

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{*}$$

for some unit vectors $X_0 \in \mathcal{Q}$, $\xi_1 \in \mathcal{Q}^\perp$, and $\eta(X_0)\eta(\xi_1) \neq 0$.

Lee and Loo [8] show that if M is Hopf, then the Reeb curvature α is constant along the direction of structure vector field ξ, that is, $\xi\alpha = 0$. Also in [9], we see that $\xi\alpha = 0$ yields that the distribution $\mathcal{Q}$ and the $\mathcal{Q}^\perp$-component of the Reeb vector field ξ are invariant by the shape operator A, that is,

$$AX_0 = \alpha X_0, \quad \text{and} \quad A\xi_1 = \alpha\xi_1. \tag{4.3}$$

In the case of $\alpha = 0$, using the equation in (3.9), we obtain that ξ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$.

We next consider the case $\alpha \neq 0$.

Substituting $X = \varphi X_0$ in (4.2) and using basic formulas including (*), we get

$$\begin{aligned} &A\varphi X_0 - 4\eta(X_0)\eta_1(\xi)A\varphi_1\xi + \eta_1(\xi)A\varphi_1 X_0 - 2\alpha A^2\varphi X_0 \\ &= \alpha\varphi X_0 - 4\alpha\eta(X_0)\eta_1(\xi)\varphi_1\xi + \alpha\eta_1(\xi)\varphi_1 X_0 - 2\alpha^2 A\varphi X_0. \end{aligned} \tag{4.4}$$

From (*) and $\varphi\xi = 0$, we obtain that $\varphi_1\xi = \eta(X_0)\varphi_1 X_0$ and $\varphi X_0 = -\eta(\xi_1)\varphi_1 X_0$. In addition, substituting X by X_0 into (3.10) and applying $AX_0 = \alpha X_0$, we see that both vector fields φX_0 and $\varphi_1 X_0$ are principal with same corresponding principal curvature $k = \frac{\alpha^2 - 2\eta^2(X_0)}{\alpha}$. From this, (4.4) gives

$$2k\eta^2(X_0)\varphi X_0 - \alpha k^2\varphi X_0 = 2\alpha\eta^2(X_0)\varphi X_0 - \alpha^2 k\varphi X_0.$$

Multiplying this equation by α, we obtain

$$\eta^2(X_0)(\alpha^2 + 4\eta^2(X_0))\varphi X_0 = 0.$$

By our assumptions, we get $\eta^2(X_0)(\alpha^2 + 4\eta^2(X_0)) > 0$ which means $\varphi X_0 = 0$. This makes a contradiction. Accordingly, we get a complete proof of our Lemma. □

From Lemma 4.1, we only have two cases, $\xi \in \mathcal{Q}$ or $\xi \in \mathcal{Q}^\perp$, under our assumptions. Next we study the case $\xi \in \mathcal{Q}^\perp$.

Lemma 4.2 *Let M be a Hopf hypersurface with semisymmetric shape operator in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$, then the shape operator of M satisfies $A\mathcal{Q} \subset \mathcal{Q}$, that is, M is a $\mathcal{Q}^\perp$-invariant hypersurface.*

Proof Since $\xi \in \mathcal{Q}^\perp$, we may put $\xi = \xi_1 \in \mathcal{Q}^\perp$ for the sake of our convenience. Differentiating $\xi = \xi_1$ along any direction $X \in TM$ and using fundamental formulae in [9], it gives us

$$\varphi AX = 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \varphi_1 AX. \tag{4.5}$$

Taking the inner product of (4.5) with $W \in \mathcal{Q}$ and taking symmetric part, we also have

$$A\varphi W = A\varphi_1 W. \tag{4.6}$$

Putting $X = \xi_2$ and $X = \xi_3$ into (4.2), we get, respectively,

$$\begin{cases} A\xi_2 - \alpha A^2\xi_2 = \alpha\xi_2 - \alpha^2 A\xi_2, \\ A\xi_3 - \alpha A^2\xi_3 = \alpha\xi_3 - \alpha^2 A\xi_3. \end{cases}$$

For $\alpha = 0$, clearly $\mathcal{Q}^\perp$ is invariant under the shape operator, i.e. $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$. Thus, let us consider $\alpha \neq 0$. Then the previous equations imply that

$$\begin{cases} A^2\xi_2 = \dfrac{\alpha^2+1}{\alpha} A\xi_2 + \xi_2, \\ A^2\xi_3 = \dfrac{\alpha^2+1}{\alpha} A\xi_3 + \xi_3. \end{cases} \tag{4.7}$$

Moreover, restricting $X = \xi_2$, $Y = \xi_3$ and putting $Z = W \in \mathcal{Q}$, Equation (4.2) becomes

$$\begin{aligned} &4\eta_3(AW)\xi_2 - 4\eta_2(AW)\xi_3 + 2\varphi AW - 2\varphi_1 AW + \eta_3(A^2W)A\xi_2 - \eta_2(A^2W)A\xi_3 \\ &= 2A\varphi W - 2A\varphi_1 W - 2\eta_3(AW)A^2\xi_2 + 2\eta_2(AW)A^2\xi_3. \end{aligned}$$

Applying (3.5), (4.6) and (4.7) to this equation, it follows $\eta_3(AW)\xi_2 = \eta_2(AW)\xi_3$. This means $\eta_3(AW) = \eta_2(AW) = 0$ for any tangent $W \in \mathcal{Q}$. It completes the proof. □

By Theorem 1.1 in the Introduction, we assert that a real hypersurface M with the assumptions given in Lemma 4.2 is locally congruent to one of the following real hypersurfaces:

($\mathcal{T}_A$) a tube over a totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$,

($\mathcal{T}_{A_0}$) a tube over the complex and totally complex totally geodesic $\mathbb{C}H^m$ in $SU_{2,m}/S(U_2 \cdot U_m)$;

or

($\mathcal{H}_A$) a horosphere in $SU_{2,m}/S(U_2 \cdot U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.

Therefore, by virtue of Lemma 4.2 we conclude that if $\xi \in \mathcal{Q}^\perp$, then M is locally congruent to either $\mathcal{T}_A$, $\mathcal{T}_{A_0}$ or $\mathcal{H}_A$. Such real hypersurfaces of type $\mathcal{T}_A$, $\mathcal{T}_{A_0}$ and $\mathcal{H}_A$ in $SU_{2,m}/S(U_2 \cdot U_m)$ are denoted by M_A. In [2, 3], Berndt and Suh gave some information related to the shape operator A of $\mathcal{T}_A$, $\mathcal{T}_{A_0}$ and $\mathcal{H}_A$ as follows.

Proposition 4.3 ([2, 3]) *Let M_A be a connected real hypersurface of type $\mathcal{T}_A$, $\mathcal{T}_{A_0}$ or $\mathcal{H}_A$ in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then one of the following statements holds:*

($\mathcal{T}_A$) *M_A has exactly four distinct constant principal curvatures*

$$\alpha = 2\coth(2r),\ \beta = \coth(r),\ \lambda_1 = \tanh(r),\ \lambda_2 = 0,$$

and the corresponding principal curvature spaces are

$$T_\alpha = span\{\xi\},\ T_\beta = span\{\xi_2, \xi_3\},\ T_{\lambda_1} = E_{-1},\ T_{\lambda_2} = E_{+1}.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are complex (with respect to J) and totally complex (with respect to $\mathfrak{J}$).

($\mathcal{T}_{A_0}$) *M_A has exactly four distinct constant principal curvatures*

$$\alpha = 2\sqrt{2}\coth(2\sqrt{2}r), \beta = \sqrt{2}\coth(\sqrt{2}r), \lambda = \sqrt{2}\tanh(\sqrt{2}r), \mu = 0$$

and the corresponding principal curvature spaces are

$$T_\alpha = \mathbb{R}\xi$$
$$T_\lambda = \mathcal{Q}^\perp \ominus \mathbb{R}\xi = \mathbb{H}N \ominus \mathbb{C}N$$
$$T_\beta = \{v \in \mathcal{Q} : \varphi v = \varphi_1 v\}$$
$$T_\mu = \{v \in \mathcal{Q} : \varphi v = -\varphi_1 v\}.$$

The corresponding multiplicities of the principal curvatures are

$$m_\alpha = 1, m_\lambda = 2, m_\beta = 2(m-1) = m_\mu.$$

($\mathcal{H}_A$) *M_A has exactly three distinct constant principal curvatures*

$$\alpha = 2,\ \beta = 1,\ \lambda = 0$$

with corresponding principal curvature spaces

$$T_\alpha = span\{\xi\},\ T_\beta = span\{\xi_2, \xi_3\} \oplus E_{-1},\ T_\lambda = E_{+1}.$$

Here, E_{+1} and E_{-1} are the eigenbundles of $\varphi\varphi_1|_{\mathcal{Q}}$ with respect to the eigenvalues $+1$ and -1, respectively.

In the remaining part of this section, by using Proposition 4.3, we will check whether the shape operator A on a real hypersurface M_A of type $\mathcal{T}_A$ ($\mathcal{T}_{A_0}$ or $\mathcal{H}_A$, resp.) satisfies semisymmetry condition. In order to do this, we assume that the shape operator A of M_A is semisymmetric.

Case I. $\mathcal{T}_A$.
From (4.2), [1, Proposition 3], and $\xi \in \mathcal{Q}^\perp$, we have

$$(\lambda_1 - \alpha)(\alpha\lambda_1 - 1)X = 0$$

for any tangent vector $X \in T_{\lambda_1} = \{X \in T_xM|\, X \perp \xi_\nu,\ \varphi X = \varphi_1 X,\ x \in M\}$. Since $\alpha = 2\coth(2r)$ and $\lambda = \tanh(r)$, it implies that every $X \in T_{\lambda_1}$ is a zero vector. This gives rise to a contradiction. In fact, the dimension of the eigenspace T_{λ_1} is $2m-2$ where $m \geq 3$.

Case II. $\mathcal{T}_{A_0}$.
From (4.2), [1, Proposition 3], and $\xi \in \mathcal{Q}^\perp$, we have

$$(\beta - \alpha)(\alpha\beta - 1)X = 0$$

for any tangent vector $X \in T_\beta = \{X \in T_xM|\, X \perp \xi_\nu,\ \varphi X = \varphi_1 X,\ x \in M\}$. Since $\alpha = 2\sqrt{2}\coth(2\sqrt{2}r)$ and $\beta = \sqrt{2}\coth(\sqrt{2}r)$, it becomes

$$\big(\sqrt{2}\tanh(\sqrt{2}r)\big)\big(2\coth^2(\sqrt{2}r) + 1\big)X = 0$$

which implies that every $X \in T_\beta$ is a zero vector. This gives rise to a contradiction. In fact, the dimension of the eigenspace T_β is $2m-2$ where $m \geq 3$.

Case III. $\mathcal{H}_A$**.**

From (4.2), [1, Proposition 3], and $\xi \in \mathcal{Q}^\perp$, we have

$$(\lambda-\alpha)(\alpha\lambda-1)X=0$$

for any tangent vector $X \in T_\lambda = \{X \in T_xM \,|\, X \perp \xi_\nu,\ \varphi X = \varphi_1 X,\ x \in M\}$. Since $\alpha = 2$ and $\lambda = 0$, it implies that every $X \in T_\lambda$ is a zero vector. This gives rise to a contradiction. In fact, the dimension of the eigenspace T_λ is $2m-2$ where $m \geq 3$.

Summing up these observations, we assert that the shape operator of real hypersurfaces M_A of three kinds of model spaces $\mathcal{T}_A$, $\mathcal{T}_{A_0}$ and $\mathcal{H}_A$ in $SU_{2,m}/S(U_2 \cdot U_m)$ does not satisfy the property of semisymmetry.

Due to Lemma 4.1, let us suppose that $\xi \in \mathcal{Q}$ (i.e. $JN \perp \mathfrak{J}N$) in this section. Related to this condition, Suh proved:

Theorem 4.4 ([16]) *Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with the Reeb vector field belonging to the maximal quaternionic subbundle $\mathcal{Q}$. Then one of the following statements holds*

($\mathcal{T}_B$) *M is an open part of a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$,*

($\mathcal{H}_B$) *M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$, or*

($\mathcal{E}$) *The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$.*

By virtue of this result, we assert that a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$ satisfying the hypotheses in our Theorem 1.6 is locally congruent to an open part of one of the model spaces mentioned in above Theorem 1.3. Hereafter, unless otherwise stated, such real hypersurfaces of type of $\mathcal{T}_B$, $\mathcal{H}_B$, and $\mathcal{E}$ in $SU_{2,m}/S(U_2 \cdot U_m)$ are denoted by M_B.

Now, let us check whether the shape operator A of a real hypersurfaces $\mathcal{T}_B$, $\mathcal{H}_B$, and $\mathcal{E}$ satisfies our conditions. In order to do this, let us introduce the following proposition given by Berndt and Suh [3].

Proposition 4.5 ([3]) *Let M_B be a real hypersurface of type $\mathcal{T}_B$ (resp. $\mathcal{H}_B$ or $\mathcal{E}$) in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then M_B has distinct principal curvatures as follows.*

($\mathcal{T}_B$) *M_B has five (four for $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$ in which case $\alpha = \lambda_2$) distinct constant principal curvatures*

$$\alpha = \sqrt{2}\tanh(\sqrt{2}r),\ \beta = \sqrt{2}\coth(\sqrt{2}r),\ \gamma = 0,$$

$$\lambda_1 = \frac{1}{\sqrt{2}}\tanh(\frac{1}{\sqrt{2}}r),\ \lambda_2 = \frac{1}{\sqrt{2}}\coth(\frac{1}{\sqrt{2}}r),$$

and the corresponding principal curvature spaces are

$$T_\alpha = \mathit{span}\{\xi\}, \quad T_\beta = \mathit{span}\{\xi_1, \xi_2, \xi_3\},$$

$$T_\gamma = \mathit{span}\{\varphi\xi_1, \varphi\xi_2, \varphi\xi_3\}.$$

The principal curvature spaces T_{λ_1} *and* T_{λ_2} *are invariant under* $\mathfrak{J}$ *and are mapped onto each other by* J*. In particular, the quaternionic dimension of* $SU_{2,m}/S(U_2U_m)$ *must be even.*

($\mathcal{H}_B$) M_B *has exactly three distinct constant principal curvatures*

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$\begin{aligned} &T_\alpha = span\{\xi, \xi_1, \xi_2, \xi_3\}, \quad T_\gamma = span\{\varphi\xi_1, \varphi\xi_2, \varphi\xi_3\}, \\ &T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}. \end{aligned}$$

($\mathcal{E}$) M_B *has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$\begin{aligned} &T_\alpha = span\{\xi, \xi_1, \xi_2, \xi_3\}, \quad T_\gamma = span\{\varphi\xi_1, \varphi\xi_2, \varphi\xi_3\}, \\ &T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}. \end{aligned}$$

If μ *is another (possibly nonconstant) principal curvature function, then* $JT_\mu \subset T_\lambda$ *and* $\mathfrak{J}T_\mu \subset T_\lambda$*. Thus, the corresponding multiplicities are*

$$m(\alpha) = 4, \quad m(\gamma) = 3, \quad m(\lambda), \quad m(\mu).$$

To check our converse problem we suppose that the structure Jacobi operator R_ξ of M_B is semisymmetric. By virtue of Proposition 4.5, we see that the structure vector field ξ of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ belongs to the distribution $\mathcal{Q}$.

Case I. $\mathcal{T}_B$**.**

If we put X as a unit vector field $\xi_1 \in T_\beta$ into (4.2), then we obtain $\alpha\beta(\alpha - \beta)\xi_1 = 0$. As we know $\alpha = \sqrt{2}\tanh(\sqrt{2}r)$, $\beta = \sqrt{2}\coth(\sqrt{2}r)$ on M_B, we get a contradiction.

Case II. $\mathcal{H}_B$ **or** $\mathcal{E}$**.**

If we put X as a unit vector field $\varphi\xi_1 \in T_\gamma$ into (4.2), then we obtain $4\alpha\varphi\xi_1 = 0$. As we know $\alpha = \sqrt{2}$ on $\mathcal{H}_B$ or $\mathcal{E}$, we get $\varphi\xi_1 = 0$ which is a contradiction.

Therefore we assert that the shape operator A of a model space neither of type (A) nor type (B) in $SU_{2,m}/S(U_2 \cdot U_m)$ does not satisfy the semisymmetry condition.

Summing up these discussions, we complete the proof of our Theorem 1.4 given in Introduction.

5. Semisymmetric structure Jacobi operator

In this section, we give a complete proof of our Theorem 1.6. Suppose the structure Jacobi operator of M has semisymmetry, that is, M satisfies the condition $(R(X,Y)R_\xi)Z = 0$. Besides, from the relation between (†) and (‡) we see that the given condition is equivalent to

$$R(X,Y)(R_\xi Z) = R_\xi(R(X,Y)Z). \tag{5.1}$$

Let $SU_{2,m}/S(U_2 \cdot U_m)$ and M be a complex hyperbolic two-plane Grassmannian and a Hopf real hypersurface. Hereafter, unless otherwise stated, we consider that X and Y are any tangent vector fields on M. The structure Jacobi operator R_ξ of M in $SU_{2,m}/S(U_2 \cdot U_m)$ is given by

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= -\frac{1}{2}\Big[X - \eta(X)\xi - \sum\nolimits_{\nu=1}^{3}\Big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu\Big\} \\ &\qquad - \sum\nolimits_{\nu=1}^{3}\Big\{3\eta_\nu(\varphi X)\varphi_\nu\xi + \eta_\nu(\xi)\varphi_\nu\varphi X\Big\}\Big] + \alpha AX - \eta(AX)A\xi, \end{aligned} \tag{5.2}$$

where α is the Reeb curvature defined by $\alpha = g(A\xi,\xi)$ on M (see [17]).

Put $Y = Z = \xi$ into (5.1), due to $R_\xi\xi = 0$, we get:

$$R_\xi(R_\xi X) = 0. \tag{5.3}$$

Using these observation from now on we show that ξ belongs to either $\mathcal{Q}$ or its orthogonal complement $\mathcal{Q}^\perp$.

Lemma 5.1 *M is a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with semisymmetric structure Jacobi operator, then ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.*

Proof Suppose that ξ satisfies (*) for some unit vectors $X_0 \in \mathcal{Q}$ and $\xi_1 \in \mathcal{Q}^\perp$.

In this case also we use [8, 9] again, we have

$$AX_0 = \alpha X_0, \quad \text{and} \quad A\xi_1 = \alpha\xi_1. \tag{5.4}$$

Substituting $X = \xi_1$ in (5.2), we have $R_\xi(\xi_1) = \alpha^2\xi_1 - \alpha^2\eta(\xi_1)\xi$. This gives that

$$\begin{aligned} R_\xi(R_\xi\xi_1) &= R_\xi\big(\alpha^2\xi_1 - \alpha^2\eta(\xi_1)\xi\big) \\ &= \alpha^2 R_\xi\xi_1 - \alpha^2\eta(\xi_1)R_\xi\xi \\ &= \alpha^4\xi_1 - \alpha^4\eta(\xi_1)\xi. \end{aligned}$$

So, the condition of semisymmetric structure Jacobi operator implies

$$\alpha^4\xi_1 - \alpha^4\eta(\xi_1)\xi = 0.$$

From this, taking the inner product with $X_0 \in \mathcal{Q}$, it gives $\alpha^4\eta(\xi_1)\eta(X_0) = 0$. So we obtain the following three cases: $\alpha = 0$, $\eta(X_0) = 0$ or $\eta(\xi_1) = 0$. When α identically vanishes, by virtue of (3.9) we conclude that ξ

belongs to either $\mathcal{Q}$ or $\mathcal{Q}^{\perp}$. For $\eta(\xi_1) = 0$, then ξ belongs to $\mathcal{Q}$ because of our notation (*). Moreover, ξ belongs to $\mathcal{Q}^{\perp}$ if $\eta(X_0) = 0$. Accordingly, it completes the proof of our Lemma. □

According to Lemma 5.1, we consider the case $\xi \in \mathcal{Q}^{\perp}$.

Lemma 5.2 *Let M be a Hopf real hypersurface with semisymmetric structure Jacobi operator in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^{\perp}$, then the shape operator of M satisfies $A\mathcal{Q} \subset \mathcal{Q}$.*

Proof We may put $\xi = \xi_1$, because $\xi \in \mathcal{Q}^{\perp}$. Differentiating $\xi = \xi_1$ for any direction X on M, we obtain

$$\begin{cases} q_2(X) = 2g(AX, \xi_2),\ q_3(X) = 2g(AX, \xi_3) \text{ and} \\ AX = \eta(AX)\xi + 2g(AX, \xi_2)\xi_2 + 2g(AX, \xi_3)\xi_3 - \varphi\varphi_1 AX. \end{cases} \tag{5.5}$$

Putting $X = \xi_2$ into (5.2), it follows that $R_\xi(\xi_2) = 2\xi_2 + \alpha A\xi_2$. If the smooth function α vanishes, it makes a contradiction. In fact, from (5.3) we see that $R_\xi(R_\xi\xi_2) = \xi_2 = 0$. Thus we may consider that the smooth function α is nonvanishing.

On the other hand, it follows that for any $W \in \mathcal{Q}$ Equation (5.2) becomes

$$R_\xi(W) = -\frac{1}{2}W - \frac{1}{2}\varphi_1\varphi W + \alpha AW. \tag{5.6}$$

From this and (5.3), it follows that

$$\begin{aligned} 0 &= R_\xi(R_\xi W) \\ &= \frac{1}{2}W - \frac{1}{2}\varphi_1\varphi W - \alpha^2 A^2 W. \end{aligned} \tag{5.7}$$

Taking the inner product with ξ_2 and ξ_3, respectively, using $\alpha \neq 0$, it becomes

$$\eta_2(A^2W) = 0, \quad \eta_3(A^2W) = 0. \tag{5.8}$$

According to (5.2), we also have $R_\xi(A\xi_2) = 2A\xi_2 + \alpha A^2\xi_2$. By virtue of (5.3), we get

$$\begin{aligned} 0 = R_\xi(R_\xi\xi_2) &= R_\xi(-\xi_2 + \alpha A\xi_2) \\ &= -R_\xi(\xi_2) + \alpha R_\xi(A\xi_2) \\ &= \xi_2 - 2\alpha A\xi_2 + \alpha^2 A^2\xi_2. \end{aligned}$$

Again taking the inner product with $W \in \mathcal{Q}$ and using the fact $\alpha \neq 0$, we have

$$-2\alpha g(A\xi_2, W) + \alpha^2 g(A^2\xi_2, W) = 0. \tag{5.9}$$

From this and (5.8), we obtain $\eta_2(AW) = 0$ for any tangent vector field $W \in \mathcal{Q}$.

Similarly, putting $X = \xi_3$ into (5.2), using the same method, we also obtain $\eta_3(AW) = 0$ for any tangent vector field $W \in \mathcal{Q}$.

Until now, we proved that if M satisfies our assumptions, then the distribution $\mathcal{Q}$ (resp., $\mathcal{Q}^{\perp}$) is invariant under the shape operator, that is, $A\mathcal{Q} \subset \mathcal{Q}$. This gives a complete proof of our lemma. □

From such a point of view, let us consider the converse problem. By using Proposition 4.3, let us check whether the shape operator A on a real hypersurface M_A of type $\mathcal{T}_A$, $\mathcal{T}_{A_0}$ or $\mathcal{H}_A$, resp. satisfies semisymmetry condition. In order to do this, we assume that the structure Jacobi operator R_ξ of M_A is semisymmetric.

In order to check our problem for a model space M_A, we suppose that M_A has semisymmetric structure Jacobi operator.

Case I. $\mathcal{T}_A$**.**
By virtue of Proposition 4.3, we see that $\xi = \xi_1 \in T_\alpha$ and $\xi_j \in T_\beta$ for $j = 2, 3$. From this, the semisymmetric condition for R_ξ becomes

$$\begin{aligned} R_\xi(R_\xi \xi_2) &= \xi_2 - 2\alpha\beta\xi_2 + \alpha^2\beta^2\xi_2 \\ &= (\alpha\beta - 1)^2\xi_2 = 0. \end{aligned}$$

But since $\alpha = 2\coth(2r)$ and $\beta = \coth(r)$, we obtain $(\alpha\beta - 1)^2 = \coth^2(r) \neq 0$. This gives $\xi_2 = 0$ which is a contradiction.

Case II. $\mathcal{T}_{A_0}$**.**
From (4.2), Proposition 4.3, and $\xi \in \mathcal{Q}^{\perp}$, we have

$$\begin{aligned} R_\xi(R_\xi \xi_2) &= \xi_2 - 2\alpha\lambda\xi_2 + \alpha^2\lambda^2\xi_2 \\ &= (\alpha\lambda - 1)^2\xi_2 = 0. \end{aligned}$$

Since $\alpha = 2\sqrt{2}\coth(2\sqrt{2}r)$ and $\lambda = \sqrt{2}\tanh(\sqrt{2}r)$, it becomes

$$\big(2\tanh^2(\sqrt{2}r) + 1\big)\xi_2 = 0$$

which implies that ξ_2 is a zero vector. This gives rise to a contradiction.

Case III. $\mathcal{H}_A$**.**

By virtue of Proposition 4.3, we see that $\xi = \xi_1 \in T_\alpha$ and $\xi_j \in T_\beta$ for $j = 2, 3$. From this, the semisymmetry condition for R_ξ becomes

$$\begin{aligned} R_\xi(R_\xi \xi_2) &= \xi_2 - 2\alpha\beta\xi_2 + \alpha^2\beta^2\xi_2 \\ &= (\alpha\beta - 1)^2\xi_2 = 0. \end{aligned}$$

But since $\alpha = 2$ and $\beta = 1$, this gives $\xi_2 = 0$ which is a contradiction.

Summing up these observations, we assert that the structure Jacobi operator of real hypersurfaces M_A of three kinds of model spaces $\mathcal{T}_A$, $\mathcal{T}_{A_0}$ and $\mathcal{H}_A$ in $SU_{2,m}/S(U_2 \cdot U_m)$ does not satisfy the property of semisymmetry.

In the sequel, we check whether R_ξ of a model space M_B of type (B) is semisymmetric. To do this, we assume that R_ξ of M_B satisfies the condition (5.1). On a tangent vector space T_xM_B at any point $x \in M_B$, the Reeb vector ξ belongs to $\mathcal{Q}$. The condition of (5.1) implies that for $X=\xi_2\in T_\beta$ and $Y=Z=\xi$,

$$R_\xi(R_\xi\xi_2)=(\alpha\beta)^2\xi_2=0.$$

On the other hand, from Proposition 4.5, since $\alpha=\sqrt{2}\tan(\sqrt{2}r)$ and $\beta=\sqrt{2}\cot(\sqrt{2}r)$ on $\mathcal{T}_B$ or $\alpha=\sqrt{2}$ and $\beta=\sqrt{2}$ on the other cases. In all cases, we get $(\alpha\beta)^2=4$. So, we consequently see that the tangent vector ξ_2 must be zero, which gives a contradiction.

Therefore we assert that the structure Jacobi operator R_ξ of a model space of type (A) or type (B) in $SU_{2,m}/S(U_2\cdot U_m)$ does not satisfy the semisymmetry condition. Summing up these discussions, we complete the proof of our Theorem 1.6 given in the Introduction.

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