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Equicontinuity and sensitivity on countable amenable semigroup

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Abstract: In this paper, we obtain the classification of topological dynamical systems with a discrete action. The equicontinuity and sensitivity for amenable discrete countable semigroup action are shown by the left Følner sequence. We consider the notion of uniquely ergodic and mean equicontinuous on amenable discrete countable semigroup action and develop the notion of density with respect to the Følner sequence on equicontinuous and sensitivity.

Key words: Measure-preserving dynamical system, semigroup action, equicontinuity, sensitivity, amenable semigroup, Følner sequence, density

1. Introduction

Throughout the paper, the topological dynamical system (t.d.s. for short) is denoted by (X, S) or $(X, \langle T_s \rangle_{s \in S})$, where S is a discrete (infinite) countable semigroup, (X, d) is a compact metric space and $T_s : X \to X$ for every $s \in S$ is a continuous map such that for all $s, t \in S$, $T_s \circ T_t = T_{st}$. When $S = \mathbb{Z}_+$ the action is generated by a continuous evolution map T. For simplicity, we write t.d.s., instead of (X, T). The C*-algebra of all complex-valued continuous functions equipped by supermom norm denoted by $C(X)$. Therefore, by adjoint-map $U_{T_s} : C(X) \to C(X)$ is well defined by $\varphi \circ T_s = U_{T_s} \circ \varphi$ for every $\varphi \in C(X)$ and $s \in S$. For simplicity, we define ${}_s\varphi := \varphi \circ T_s = U_{T_s}(\varphi)$. So, ${}_s\varphi(x) = \varphi(sx)$ for every $s \in S$, $x \in X$, and $\varphi \in C(X)$.

Let $(X, \langle T_s \rangle_{s \in S})$ be a t.d.s. In this paper, $(X \times X, \{T_s \times T_s\}_{s \in S})$ denotes the topological dynamical system $(x, y) \mapsto (T_s x, T_s y) : X \times X \to X \times X$ for $s \in S$. We will write it $(X \times X, S \times S)$ for simplicity.

Let $(X, \mathcal{B}, \mu)$ be a probability space. A self-map $T : X \to X$ is called to be measurable if $T^{-1}B \in \mathcal{B}$ for all $B \in \mathcal{B}$. We say that μ is a T-invariant measure, if $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}$, and we say that $(X, T, \mathcal{B}, \mu)$ is measure preserving dynamical system (m.p.s for short). Let $M(X, T)$ denote the collection of all $T-$invariant measures on X.

Lemma 1.1 *[14] Assume that X is a compact metric space and $\{\mu_n\}$ is a sequence of measures in $M(X, T)$ and $\mu \in M(X, T)$; Then, we have the following equivalent properties:*

1. *$\mu_n \to \mu$ in the weak*-topology;*
2. *For every closed subset F of X, $\mu(F) \geq \limsup_{n \to +\infty} \mu_n(F)$;*
3. *For every open subset U of X, $\mu(U) \leq \liminf_{n \to +\infty} \mu_n(U)$.*

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By Krylov–Bogolioubov Theorem [6], $M(X,T) \neq \emptyset$. The topological dynamical system (X,T) is uniquely ergodic if $M(X,T)$ is a one-point set. In 1952, Oxtoby proved equivalent conditions for uniquely ergodic of the dynamical system. The following theorem can be seen in Section 5.3 of [11]. For a dynamical system (X,T), $f \in C(X)$ and $n \in \mathbb{Z}^+$, we define mean average of f by $M_f^n(x) = (1/n)\sum_{i=0}^{n-1} f(T^i x)$.

Theorem 1.2 *Let (X,T) be a dynamical system. Then the following statements are equivalent:*

1. *(X,T) is uniquely ergodic;*
2. *For each $f \in C(X)$, $\{M_f^n(x)\}_{n=1}^{\infty}$ converges uniformly on X to a constant;*
3. *For each $f \in C(X)$, there is a subsequence $\{M_f^{n_k}\}_{k=1}^{\infty}$ which converges pointwise on X to a constant;*
4. *(X,T) contains only one minimal set, and for each $f \in C(X)$, $\{M_f^n(x)\}_{n=1}^{\infty}$ converges uniformly on X.*

We prove the above theorem for a countable amenable semigroup action S, (see Theorem 2.5). A dynamical system (X,T) is called equicontinuous if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x, y \in X$ with $d(x,y) < \delta$, $d(T^n x, T^n y) < \epsilon$ for $n \in \mathbb{Z}^+$. In fact, the collection of maps $\{T^n | n \in \mathbb{Z}^+\}$ is uniformly equicontinuous.

Let (X,T) be a dynamical system, a point $x \in X$ is called mean equicontinuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $y \in X$ with $d(x,y) < \delta$,

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n-1} d(T^i x, T^i y) < \epsilon.$$

A transitive system (see subsection 2.2) is called almost mean equicontinuous if there is at least one mean equicontinuous point. A dynamical system (X,T) is called mean sensitive, if there exists a $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there exists $y \in U$ such that

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n-1} d(T^i x, T^i y) > \delta.$$

LI, TU, and YE in [9] expressed an important notion of the mean equicontinuity and mean sensitively. In [8], the authors showed that a t.d.s., (X,T) is called density-one-equicontinuous (resp. Banach density-one-equicontinuous), if for any $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in X$ with $d(x,y) < \delta$, then $d(T^n x, T^n y) \leq \epsilon$ for all $n \in \mathbb{Z}^+$ except a set of zero upper density (resp. zero upper Banach density).

In this paper, we generalize the results in [8, 9], for dynamical system $(X, \langle T_s \rangle_{s\in S})$ where S is a left amenable countable semigroup. Furthermore, Theorem 3.8 is an extension of the following theorem.

Theorem 1.3 *[9] Let (X,T) be a dynamical system. Then the following statements are equivalent:*

1. *(X,T) is mean equicontinuous;*
2. *For each $f \in C(X \times X)$, the sequence $\{M_f^{n_k}\}_{k=1}^{\infty}$ is uniformly equicontinuous;*

3. *For each* $f \in C(X \times X)$, *the sequence* $\{M_f^{n_k}\}_{k=1}^{\infty}$ *is uniformly convergent to a* $T \times T-$ *invariant continuous function* $f^* \in C(X \times X)$.

The topological dynamical system (X, T) is called sensitive, if there is a $\delta > 0$ such that for any nonempty open subset $U \subseteq X$, there exist $x, y \in U$ such that

$$\limsup_{k \to +\infty} d(T^k x, T^k y) > \delta.$$

In other words, (X, T) is sensitive, if $N_T(U, \delta)$ is nonempty for some $\delta > 0$, where

$$N_T(U, \delta) := \{n \in \mathbb{Z}^+ : \text{there exist } x, y \in U \text{with } d(T^n(x), T^n(y)) > \delta\},$$

here δ will be referred to as a constant of sensitivity. We will introduce the concept of sensitivity to the dynamical system $(X, \langle T_s \rangle_{s \in S})$, where S is a left amenable countable semigroup.

This paper is arranged as follows: In Section 2 we give some basic notions in semigroups and topological dynamics. Section 3 is devoted to the basic properties of $\mathcal{F}$-mean equicontinuous and $\mathcal{F}$-density equicontinuous to the dynamical system $(X, \langle T_s \rangle_{s \in S})$ in which S is an amenable semigroup. In the last section, the structure of $\mathcal{F}$-density-sensitivity is studied by notion of density and Følner sequence $\mathcal{F}$ on S.

2. Preliminaries

In this section, we introduce some basic facts and notions in semigroups and topological dynamics which will be used later.

2.1. Notions of size in semigroups

Let S be a semigroup. For $a \in S$ and $A, B \subseteq S$ define $a^{-1}A = \{s \in S : as \in A\}$, $B^{-1}A = \bigcup_{b \in B} b^{-1}A$, $AB = \{ab : a \in A, b \in B\}$ and $AA^{-1} = \{x \in S : \text{there exists } y \in A \text{ such that } xy \in A\}$.

Definition 2.1 *Let* S *be a semigroup and* $P_f(S)$ *be the set of all finite and nonempty subsets of* S.

i) A subset B *on* S *is (left) syndetic, if there is a* $F \in P_f(S)$ *such that* $BF^{-1} = \bigcup_{f \in F} Bf^{-1} = S.$

ii) A subset B *on* S *is (left) thick, if for every* $F \in P_f(S)$ *there is a* $t \in S$ *such that* $tF \subseteq B$.

iii) A subset B *on* S *is called (left) piecewise syndetic, if there is a* $F \in P_f(S)$ *such that for some* $H \in P_f(S)$, BH^{-1} *is (left) thick.*

iv) A subset B *on* S *is called (left) thickly syndetic, if for every* $A \in P_f(S)$ *there is a syndetic set* $Q_A \subseteq S$ *such that* $Q_A A \subseteq B$ *(cf.[13]).*

Similarly, one can define a right syndetic set, a right piecewise syndetic set, a right thick set and a right thickly syndetic set, respectively. But we remark that left syndetic and right syndetic sets are different. For more details, see [7, Lemma 13.39]. Every thickly syndetic set is thick and syndetic. In other words, let B be a thickly syndetic and $t \in S$. Then, there is a syndetic set $Q_t \subseteq S$ such that $tQ_t \subseteq B$. So, B is syndetic. One can check that a thickly syndetic set is thick.

The cardinality of A denotes by $|A|$. The upper density subset A of $\mathbb{N}$ is defined by

$$\overline{D}(A) = \limsup_{n\to+\infty} \frac{|A \cap \{1,2,..,n\}|}{n}.$$

Similarly, the lower density of A is defined by

$$\underline{D}(A) = \liminf_{n\to+\infty} \frac{|A \cap \{1,2,..,n\}|}{n}.$$

The density subset A of $\mathbb{N}$ is $D(A) = \overline{D}(A) = \underline{D}(A)$ whenever the limits are equal.

The upper Banach density subset A of $\mathbb{N}$ is defined by

$$\overline{BD}(A) = \limsup_{N-M\to+\infty} \frac{|A \cap \{M, M+1,.., M+N\}|}{N-M},$$

and the lower Banach density subset A of $\mathbb{N}$ is defined by

$$\underline{BD}(A) = \liminf_{n\to+\infty} \frac{|A \cap \{M, M+1,.., M+N\}|}{N-M}.$$

The Banach density subset A of $\mathbb{N}$ is $BD(A) = \overline{BD}(A) = \underline{BD}(A)$ whenever the limits are equal. It should be noted that for any $A \subseteq \mathbb{N}$, one has $\underline{BD}(A) \le \underline{D}(A) \le \overline{D}(A) \le \overline{BD}(A)$ and $D(A) = 1 - D(A^c)$. For more details, see [1] .

Definition 2.2 (See [5]) *A discrete infinite countable semigroup S will be called left amenable if it admits a left Følner sequence, i.e. a sequence $\{F_n\}_{n=1}^{\infty}$ of nonempty finite subsets on S such that for all $g \in S$*

$$\lim_{n\to+\infty} \frac{|gF_n \bigtriangleup F_n|}{|F_n|} = 0,$$

where $\bigtriangleup$ stands for the symmetric difference of sets.
Let $A \subseteq S$ and $\mathcal{F} := \{F_n\}_{n=1}^{\infty}$ be a left Følner sequence. We define the upper density of A with respect to $\mathcal{F}$ by

$$\overline{D}_{\mathcal{F}}(A) = \limsup_{n\to+\infty} \frac{|A \cap F_n|}{|F_n|}.$$

Definition 2.3 (See [4]) *For $A \subseteq S$, the Banach density of A with respect to left Følner sequence $\mathcal{F} := \{F_n\}_{n=1}^{\infty}$ is denoted by $BD(A)$ and defined by*

$$BD(A) := \sup\left\{\overline{D}_{F_n}(A) : (F_n) \text{ is a Følner sequence on } S\right\}.$$

2.2. The dynamical system

For a dynamical system $(X, \langle T_s \rangle_{s\in S})$, the orbit of x is the set $orb(x,S) := \{T_s(x) : s \in S\}$. For $U \subset X$ and $s \in S$ define $s^{-1}U := \{x \in X : sx \in U\}$. Let V and U be two nonempty subsets of X and $x \in X$. We have $N(x,U) := \{s \in S : sx \in U\}$, and $N(U,V) := \{s \in S : U \cap s^{-1}V \neq \emptyset\}$. We say $(X, \langle T_s \rangle_{s\in S})$ is transitive if $N(U,V) \neq \emptyset$ for every nonempty open subset U and V of X and it is weakly mixing if $X \times X$ is transitive and it is also said that $x \in X$ is a transitive point if $orb(x,S) = X$.

In a complete metric space, the countable union of nowhere dense sets are said to be meager; the meager complement set is called a residual set. The set of all transitive points in X is a dense G_δ-set for any transitive topological dynamical system. We say that $(X, \langle T_s \rangle_{s\in S})$ is minimal if every point of X is a transitive point. For a dynamical system $(X, \langle T_s \rangle_{s\in S})$, a point $x \in X$ is minimal point if and only if $N(x, U)$ is a syndetic subset on S for every nonempty open subset U of X.

Definition 2.4 *Suppose that S is a discrete infinite countable left amenable semigroup and $(X, \langle T_s \rangle_{s\in S})$ is a dynamical system. If $x, y \in X$, then pair $(x, y) \in X \times X$ is called Banach proximal, if for any $\epsilon > 0$, the set $\{s \in S : d(T_s x, T_s y) < \epsilon\}$ has Banach density one, i.e. $BD(\{s \in S : d(T_s x, T_s y) < \epsilon\}) = 1$.*

Let $(X, \mathcal{B}, \mu)$ be a probability space and $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s. We say that a measure μ on X is $S-$invariant if $\mu(T_s^{-1}A) = \mu(A)$ for every $A \in \mathcal{B}$ and $s \in S$, also $(X, \langle T_s \rangle_{s\in S})$ is a measure preserving dynamical system, (m.p.s for short). The collection of all $S-$invariant measures on X is denoted by $M(X, S)$. For $\mu \in M(X, S)$, the support of μ is defined by $supp(\mu) = \{x \in X : \mu(U) > 0 \text{ for every neighbourhood } U \text{ of } x\}$.

The following proof process is taken from the [6, Theorem 8.2] and [10, Theorem 3.11], but it has its own details. We have the following theorem for uniquely ergodic dynamical system on amenable semigroup.

Theorem 2.5 *Suppose that $(X, \mathcal{B}, \mu)$ is a probability space and $(X, \langle T_s \rangle_{s\in S})$ is an m.p.s such that for some $s \in S$, T_s is injective and $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ is a left Følner sequence on S. For every $\varphi \in C(X)$ define $\varphi_F(x) := \frac{1}{|F|}\sum_{g\in F} \varphi(T_g x) = \frac{1}{|F|}\sum_{g\in F} \varphi(gx)$, where F is a nonempty finite subset on S. Then the following conditions are equivalent:*

1. *$(X, \langle T_s \rangle_{s\in S})$ is uniquely ergodic;*
2. *For every $\varphi \in C(X)$, $\{\varphi_{F_n}\}_{n=1}^{\infty}$ converges uniformly on X to a constant;*
3. *For every $\varphi \in C(X)$, there is a sub-family $\{F_{n_i}\} \subseteq \{F_n\}$ such that $\{\varphi_{F_{n_i}}\}_{i=1}^{\infty}$ converges pointwise on X to a constant.*

Proof $(1) \Rightarrow (2)$. Assuming μ to is the uniquely ergodic measure of the dynamical system $(X, \langle T_s \rangle_{s\in S})$, we must show that for any $\varphi \in C(X)$, the sequence $\{\varphi_{F_n}\}_{n=1}^{\infty}$ converges uniformly on X to $\int_X \varphi d\mu$.

Suppose, contrary to our claim that there exists a function $\psi \in C(X)$, $\epsilon_0 > 0$, sequence $n_i \geq i$ and a sequence of points $\{x_i\} \subseteq X$ such that

$$|\psi_{F_{n_i}}(x_i) - \int_X \psi d\mu| \geq \epsilon_0 > 0.$$

The sequence $\psi_{F_{n_i}}$ is bounded by $||\psi||_\infty$; so, by the Arzela–Ascoli Theorem [12, pp. 394], it has a convergent subsequence. Without restriction of generality, we can assume

$$\lim_{i\to+\infty} \psi_{F_{n_i}}(x_i) = \lim_{i\to+\infty} \frac{1}{|F_{n_i}|} \sum_{s\in F_{n_i}} \psi(sx_i) \longrightarrow \alpha \neq \int_X \psi d\mu. \tag{2.1}$$

Also, since X is a compact metric, $C(X)$ is a separable; So, we can take $W := \{\varphi_1 = \psi, \varphi_2 = 1, \varphi_3, \varphi_4, \ldots, \varphi_n, \ldots\}$ as a countable dense set of $C(X)$ and define $Z := \text{span}\{\varphi_1 = \psi, \varphi_2 = 1, \varphi_3, \varphi_4, \ldots\}$.

For $n_\ell \in \mathbb{N}$ define $L_{n_\ell} : W \longrightarrow \mathbb{R}$ by

$$L_{n_\ell}(\varphi) := \frac{1}{|F_{n_\ell}|} \sum_{s \in F_{n_\ell}} \varphi(s\tilde{x_\ell}),$$

such that $\tilde{x_\ell} \in X$ and $n, \ell \in \mathbb{N}$. Clearly, L_{n_ℓ} is linear, positive and $L_{n_\ell}(1) = 1$ for all $n_\ell \in \mathbb{N}$.

Thus, according to Section 3 of [2], L_{n_ℓ} is a mean, and the sequence $\{L_{n_\ell}\}$ has a w*-limit point L. Therefore, we can define the linear operator $L : W \longrightarrow \mathbb{R}$ by

$$L(\varphi) := \lim_{\ell \to +\infty} \frac{1}{|F_{n_\ell}|} \sum_{s \in F_{n_\ell}} \varphi(s\tilde{x_\ell}),$$

for every $\varphi \in W$.

It is obvious that L is a mean on W. Since $L : W \longrightarrow \mathbb{R}$ is a uniform continuous function on W and W is a dense subset of $C(X)$, L has a unique extension $L : C(X) \longrightarrow \mathbb{C}$. We define $SZ := \{s\varphi : s \in S, \varphi \in Z\}$. Therefore, we have

$$\begin{aligned}
|L(s\varphi) - L(\varphi)| &= \lim_{\ell \to +\infty} \left| \frac{1}{|F_{n_\ell}|} \sum_{t \in F_{n_\ell}} s\varphi\left(t\widetilde{x}_\ell\right) - \frac{1}{|F_{n_\ell}|} \sum_{t \in F_{n_\ell}} \varphi\left(t\widetilde{x}_\ell\right) \right| \\
&= \lim_{\ell \to +\infty} \left| \frac{1}{|F_{n_\ell}|} \sum_{t \in F_{n_\ell}} \varphi\left(st\widetilde{x}_\ell\right) - \frac{1}{|F_{n_\ell}|} \sum_{t \in F_{n_\ell}} \varphi\left(t\widetilde{x}_\ell\right) \right| \\
&= \lim_{\ell \to +\infty} \left| \frac{1}{|F_{n_\ell}|} \sum_{t \in F_{n_\ell}} \left(\varphi\left(st\widetilde{x}_\ell\right) - \varphi\left(t\widetilde{x}_\ell\right)\right) \right| \\
&\le \lim_{\ell \to +\infty} \frac{\|\varphi\|_\infty \left|sF_{n_\ell} \Delta F_{n_\ell}\right|}{|F_{n_\ell}|} \\
&= 0.
\end{aligned}$$

As a result, if $\varphi, s\varphi \in W$ for $s \in S$, then $L(s\varphi) = L(\varphi)$. If $s\varphi = p\psi$ for $\varphi, \psi \in W$ and $s, p \in S$ then

$$\begin{aligned}
|L(s\varphi) - L(p\varphi)| &= \lim_{\ell\to+\infty}\left|\frac{1}{|F_{n_\ell}|}\sum_{t\in F_{n_\ell}} s\varphi\left(t\widetilde{x}_\ell\right) - \frac{1}{|F_{n_\ell}|}\sum_{t\in F_{n_\ell}} p\varphi\left(t\widetilde{x}_\ell\right)\right| \\
&= \lim_{\ell\to+\infty}\left|\frac{1}{|F_{n_\ell}|}\sum_{t\in F_{n_\ell}} \varphi\left(st\widetilde{x}_\ell\right) - \frac{1}{|F_{n_\ell}|}\sum_{t\in F_{n_l}} \varphi\left(pt\widetilde{x}_\ell\right)\right| \\
&= \lim_{\ell\to+\infty}\left|\frac{1}{|F_{n_\ell}|}\sum_{t\in F_{n_\ell}} \left(\varphi\left(st\widetilde{x}_\ell\right) - \varphi\left(pt\widetilde{x}_\ell\right)\right)\right| \\
&\le \lim_{\ell\to+\infty}\frac{\|\varphi\|_\infty\,|sF_{n_\ell}\Delta pF_{n_\ell}|}{|F_{n_\ell}|} \\
&\le \lim_{\ell\to+\infty}\|\varphi\|_\infty\left(\frac{|sF_{n_\ell}\Delta F_{n_\ell}|}{|F_{n_\ell}|} + \frac{|F_{n_\ell}\Delta pF_{n_\ell}|}{|F_{n_\ell}|}\right) \\
&= 0.
\end{aligned}$$

So, the linear operator L is well defined on SZ.

T_s is injective for some $s\in S$; so, $U_{T_s}: C(X)\to C(X)$ is surjective by [3, Exercises 9 pp. 284]. Therefore,

$$C(X) = U_{T_s}(C(X)) = U_{T_s}(\overline{Z}) \subseteq \overline{U_{T_s}(Z)} = \overline{sZ} \subseteq \overline{SZ}.$$

SZ is a dense subset of $C(X)$. Thus, for any $\varphi\in C(X)$ and every $\{\tilde\varphi_j\}\subseteq SZ$ with

$$\lim_{j\to\infty}\|\tilde\varphi_j - \varphi\|_\infty = 0,$$

we have

$$L(\varphi) := \lim_{j\to\infty} L\left(\tilde\varphi_j\right).$$

For some Borel probability measures ν on X, we have $L(\varphi) = \int_X \varphi\ \mathrm{d}\nu$ by the Riesz representation theorem. In the next step, we must show that ν is an $S-$invariant measure.

Considering $s\in S$ and $\varphi\in C(X)$, there exists $\{\varphi_j\}\subseteq Z$ with the property that $\lim_{j\to+\infty}\|\varphi_j-\varphi\| = 0$. Then $\{{}_s\varphi_j\}\subseteq SZ$ and $\lim_{j\to+\infty}\|{}_s\varphi_j -{}_s\varphi\| = 0$; therefore, we have

$$L\left({}_s\varphi\right) = \lim_{j\to\infty} L\left({}_s\varphi_j\right) = \lim_{j\to\infty} L\left(\varphi_j\right) = L(\varphi).$$

So, ν is $S-$invariant. Since $(X, \langle T_s\rangle_{s\in S})$ is uniquely ergodic, $\nu = \mu$. Considering $\psi\in W$ and equality (2.1), one has

$$\int_X \psi\ \mathrm{d}\mu = \int_X \psi\ \mathrm{d}\nu = L(\psi) = \lim_{\ell\to\infty}\frac{1}{|F_{n_\ell}|}\sum_{s\in F_{n_\ell}}\psi\left(s\tilde x_\ell\right) = \lim_{\ell\to\infty}\psi_{F_{n_\ell}}(\tilde x_\ell) = \alpha.$$

Therefore, $\alpha = \int_X \psi\ \mathrm{d}\mu$, which contradicts our assumption.

(2) $\Rightarrow$(3). It is trivial.

(3) $\Rightarrow$(1). Similarly, we can extend the operator L to $C(X)$ such that $L(1)=1$ and L is a positive linear operator from $C(X)$ into $\mathbb{C}$. By the Riesz representation theorem, $L(\varphi)=\int_X \varphi\ \mathrm{d}\nu$ for some Borel probability measures ν such that ν is $S-$invariant. Let $\mu \in M(X,S)$ be an arbitrary $S-$invariant measure. For all $\varphi_j \in W$, and $x \in X$, we have

$$\lim_{\ell\to+\infty} \frac{1}{|F_{n_\ell}|} \sum_{s\in F_{n_\ell}} \varphi_j(sx) = L(\varphi_j) = \int_X \varphi_j\ \mathrm{d}\nu.$$

Since μ is $S-$invariant, one has $\int_X \varphi_j(sx)\mathrm{d}\mu = \int_X \varphi_j(x)\mathrm{d}\mu$; therefore,

$$\int_X \frac{1}{|F_{n_\ell}|} \sum_{s\in F_{n_\ell}} \varphi_j(sx)\mathrm{d}\mu = \frac{1}{|F_{n_\ell}|} \sum_{s\in F_{n_\ell}} \int_X \varphi_j(sx)\mathrm{d}\mu = \int_X \varphi_j\ \mathrm{d}\mu.$$

By the dominated convergence theorem, we get

$$\int_X \varphi_j\ \mathrm{d}\mu = \lim_{\ell\to\infty} \int_X \frac{1}{|F_{n_\ell}|} \sum_{s\in F_{n_\ell}} \varphi_j(sx)\mathrm{d}\mu = \int_X \left(\int_X \varphi_j\ \mathrm{d}\nu\right) \mathrm{d}\mu = \int_X \varphi_j\ \mathrm{d}\nu.$$

Since W is dense in $C(X)$, we have $\int_X \varphi\ \mathrm{d}\mu = \int_X \varphi\ \mathrm{d}\nu$ for all $\varphi \in C(X)$; so, $\nu=\mu$.
Therefore, $(X, \langle T_s\rangle_{s\in S})$ is uniquely ergodic. □

3. Mean equicontinuous and density equicontinuous

In this section, we mainly discuss the properties of mean equicontinuous and density equicontinuous for dynamical system on left amenable countable semigroups.

Definition 3.1 *Let (X,d) be a compact metric space, S be a discrete infinite countable semigroup and $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system.*

i) A subset A of S acts equicontinuously at $x_0 \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for each $x \in X$, $d(x_0, x) < \delta$ implies $d(T_a(x_0), T_a(x)) < \epsilon$ for every $a \in A$.

ii) A point $x_0 \in X$ is called an equicontinuity point if $A := S$ acts equicontinuously at x_0. Denoted by $Eq(X)$, the collection of all equicontinuity point $x_0 \in X$. If $Eq(X) = X$, then $(X, \langle T_s\rangle_{s\in S})$ is equicontinuous.

Definition 3.2 *Let (X,d) be a compact metric space, $K \subseteq X$, S be a discrete infinite countable semigroup, $\mathcal{F}$ be a left Følner sequence on S, and $(X, \langle T_s\rangle_{s\in S})$ be a t.d.s. We call K is $S_{\mathcal{F}}-$equicontinuous sets, if for any $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that whenever $x, y \in K$ with $d(x,y) < \delta$, then*

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s x, T_s y) \geq \epsilon\}) = 0.$$

If X itself is $S_{\mathcal{F}}-$equicontinuous, then $(X, \langle T_s\rangle_{s\in S})$ is called equicontinuous.

Lemma 3.3 *Let $(X, \langle T_s\rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. If $K \subseteq X$ is $S_{\mathcal{F}}-$equicontinuous set, then* *(a) the closure $cl(K)$ is an $S_{\mathcal{F}}-$ equicontinuous set,*

(b) any subset of K is an $S_{\mathcal{F}}-$ equicontinuous set.

Proof It is obvious according to Definition 3.2. □

Proposition 3.4 *Let $(X, \langle T_s\rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. The union of finitely many closed $S_{\mathcal{F}}-$ equicontinuous sets is $S_{\mathcal{F}}-$ equicontinuous set.*

Proof Let A and B be two $S_{\mathcal{F}}-$equicontinuous subsets of X such that $A\cup B$ is not a $S_{\mathcal{F}}-$equicontinuous set. Then there exists $\epsilon > 0$ such that, for each $i \in \mathbb{N}$, there exist $x_i, y_i \in A \cup B$ such that $d(x_i, y_i) < 1/i$ and

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s x_i, T_s y_i) \geq \epsilon\}) > 0.$$

Without loss of generality, since $d(x_i, y_i) < 1/i$ and X is compact, we have $\lim_{i\to+\infty} x_i = \lim_{i\to+\infty} y_i = z$ for some $z \in X$. Also, A and B are closed; So, $z \in A \cap B$.

Furthermore, A and B are $S_{\mathcal{F}}-$equicontinuous sets; so, for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in A$ or $x, y \in B$ with $d(x, y) < \delta$,

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s x, T_s y) \geq \epsilon\}) = 0.$$

Now, for $\epsilon/2 > 0$ choose $x_i \in A$ and $y_i \in B$ and $z \in A \cap B$ such that $d(x_i, z) < \delta$ and $d(y_i, z) < \delta$. According to the definition of $S_{\mathcal{F}}-$equicontinuous sets for A and B, we have

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s x_i, T_s z) \geq \frac{\epsilon}{2}\}) = 0,$$

and

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s y_i, T_s z) \geq \frac{\epsilon}{2}\}) = 0.$$

We know that

$$d(T_s x_i, T_s y_i) \leq d(T_s x_i, T_s z) + d(T_s y_i, T_s z).$$

Therefore,

$$\{s \in S : d(T_s x, T_s y) \geq \epsilon\} \subseteq \{s \in S : d(T_s x_i, T_s z) \geq \frac{\epsilon}{2}\} \cap \{s \in S : d(T_s z, T_s y_i) \geq \frac{\epsilon}{2}\}.$$

On the other hand,

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s x_i, T_s y_i) \geq \epsilon\}) = 0$$

is a contradiction. □

3.1. $\mathcal{F}$-mean equicontinuous

We present the notion of mean in equicontinuous system on a left amenable semigroup.

Definition 3.5 *Let (X, S) be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. We say $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}-$mean equicontinuous, if for any $\epsilon > 0$, there is $\delta > 0$ such that*

$$\limsup_{n\to+\infty} \frac{1}{|F_n|} \sum_{g\in F_n} d(T_g(x), T_g(y)) < \epsilon,$$

whenever $x, y \in X$ and $d(x, y) < \delta$.

We notice that if $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous, then so is $(X \times X, S \times S)$.

Definition 3.6 *Let $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s., and $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ be a left Følner sequence on S. We call $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}-$mean-L-stable, if for every $\epsilon > 0$, there is a $\delta > 0$ with the property that $d(x, y) < \delta$ implies $d(T_s x, T_s y) < \epsilon$ for all $s \in S$ except a set of upper density less than ϵ. i.e. there exists $\delta > 0$ such that for each $\epsilon > 0$, we have*

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s(x), T_s(y)) \geq \epsilon\}) \leq \epsilon.$$

We need the following lemma for the next theorem.

Lemma 3.7 *Let $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s., and $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ be a left Følner sequence on S. Then $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous if and only if $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}-$mean-L-stable.*

Proof Sufficiency. Suppose that $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous. There exists a $\delta > 0$ with the property that for every $\epsilon > 0$ and $x, y \in X$ with $d(x, y) < \delta$, we have

$$\limsup_{n\to+\infty} \frac{1}{|F_n|} \sum_{g\in F_n} d(T_g(x), T_g(y)) < \epsilon^2.$$

Let $E_\epsilon = \{s \in S : d(T_s(x), T_s(y)) \geq \epsilon\}$; so, for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $d(x, y) < \delta(\epsilon)$, then

$$\epsilon^2 > \limsup_{n\to+\infty} \frac{1}{|F_n|} \sum_{g\in F_n} d(T_g(x), T_g(y)) \geq \limsup_{n\to+\infty} \frac{\epsilon}{|F_n|} |E_\epsilon \cap F_n|,$$

so,

$$\overline{D}_{\mathcal{F}}(E_\epsilon) \leq \epsilon.$$

Necessity. Let $(X, \langle T_s \rangle_{s\in S})$ be $\mathcal{F}-$mean-L-stable. Fix $\epsilon > 0$; so, there is a $\delta > 0$ such that for each $x, y \in X$, $d(x, y) < \delta$ implies that $\overline{D}_{\mathcal{F}}(E_\epsilon) \leq \epsilon$. Pick $M := \text{diam}X = \sup\{d(x, y) \ : \ x, y \in X\}$. Therefore,

$$\limsup_{n\to+\infty}\frac{1}{|F_n|}\sum_{g\in F_n}d(T_g(x),T_g(y))\leq \limsup_{n\to+\infty}\frac{1}{|F_n|}\left(\sum_{g\in F_n\cap E_\epsilon}d(T_g(x),T_g(y))+\sum_{g\in F_n\setminus E_\epsilon}d(T_g(x),T_g(y))\right)$$
$$\leq \limsup_{n\to+\infty}\frac{1}{|F_n|}\left(\sum_{g\in F_n\cap E_\epsilon}d(T_g(x),T_g(y))+\epsilon|F_n\cap E_\epsilon^c|\right)$$
$$=\limsup_{n\to+\infty}\frac{1}{|F_n|}\sum_{g\in F_n\cap E_\epsilon}d(T_g(x),T_g(y))+\limsup_{n\to+\infty}\frac{\epsilon}{|F_n|}|F_n\cap E_\epsilon^c|$$
$$\leq \limsup_{n\to+\infty}\left(\frac{M|F_n\cap E_\epsilon|}{|F_n|}+\frac{\epsilon|F_n\cap E_\epsilon^c|}{|F_n|}\right)$$
$$\leq M\overline{D}_{\mathcal{F}}(E_\epsilon)+\epsilon\overline{D}_{\mathcal{F}}(E_\epsilon)$$
$$\leq M\epsilon+\epsilon\leq\epsilon.$$

So, $(X,\langle T_s\rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous. □

We have the following characterization of uniquely ergodic of the dynamical system and $\mathcal{F}$-mean equicontinuous systems.

Theorem 3.8 *Assume that $(X,\mathcal{B},\mu)$ is a probability space and $(X,\langle T_s\rangle_{s\in S})$ is an m.p.s with the property that for some $s\in S$, T_s is injective and $\mathcal{F}=\{F_n\}_{n=1}^{\infty}$ is a left Følner sequence on S. For every $\varphi\in C(X)$ define $\varphi_F(x):=\frac{1}{|F|}\sum_{g\in F}\varphi(T_gx)=\frac{1}{|F|}\sum_{g\in F}\varphi(gx)$, where F is a nonempty finite subset on S. Then the following conditions are equivalent:*

1. *$(X,\langle T_s\rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous;*
2. *For each $\varphi\in C(X\times X)$, the sequence $\{\varphi_{F_n}\}_{n=1}^{\infty}$ is uniformly equicontinuous;*
3. *For each $\varphi\in C(X\times X)$, the sequence $\{\varphi_{F_n}\}_{n=1}^{\infty}$ is uniformly convergent to a $S\times S-$invariant continuous function $\varphi^*\in C(X\times X)$.*

Proof The proof is similar to Theorem 2.5. We suppose that $\varphi\in C(X)$ instead of $\varphi\in C(X\times X)$ because $(X,\langle T_s\rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous, then so is $(X\times X,S\times S)$.

(1) $\Rightarrow$ (2). Pick $\varphi\in C(X)$. Therefore, it is shown that for any $\epsilon>0$, there exists $\delta>0$ such that for every $x,y\in X$ with $d(x,y)<\delta$, we have

$$|\varphi_{F_n}(x)-\varphi_{F_n}(y)|<\epsilon,\quad \forall n\in\mathbb{N}.$$

Fix $\epsilon>0$. Since φ is uniformly continuous, there exists $\delta_1>0$ with the property that if $x,y\in X$ with $d(x,y)<\delta_1$, then

$$|\varphi(x)-\varphi(y)|<\frac{\epsilon}{2}. \tag{3.1}$$

Also, $(X,\langle T_s\rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous, by Lemma 3.7, $(X,\langle T_s\rangle_{s\in S})$ is $\mathcal{F}-$mean-L-stable. Now, let $E_\epsilon(x,y)=\{s\in S: d(T_s(x),T_s(y))\geq\epsilon\}$, then we have $\overline{D}_{\mathcal{F}}(E_\epsilon(x,y))\leq\epsilon$.

Suppose that $0 < \zeta < \min\{\delta_1, \epsilon/2||\varphi||_\infty\}$. There is $\delta_2 \in (0, \delta_1)$ such that for every $x, y \in X$, if $d(x,y) < \delta_2$ then

$$\overline{D}_{\mathcal{F}}(E_\epsilon(x,y)) < \zeta. \tag{3.2}$$

For every $x, y \in X$ with $d(x,y) < \delta_2$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\begin{aligned} |\varphi_{F_n}(x) - \varphi_{F_n}(y)| &= \left| \frac{1}{|F_n|} \sum_{g \in F_n} \varphi(gx) - \frac{1}{|F_n|} \sum_{g \in F_n} \varphi(gy) \right| \\ &\leq \frac{1}{|F_n|} \sum_{g \in F_n} |\varphi(gx) - \varphi(gy)|. \end{aligned}$$

According to the formulae (3.1) and (3.2), we have

$$\begin{aligned} \frac{1}{|F_n|} \sum_{g \in F_n} |\varphi(gx) - \varphi(gy)| &\leq \frac{1}{|F_n|} \left(\sum_{g \in F_n \cap E_\epsilon} |\varphi(gx) - \varphi(gy)| + \sum_{g \in F_n \setminus E_\epsilon} |\varphi(gx) - \varphi(gy)| \right) \\ &\leq \frac{1}{|F_n|} \left(\sum_{g \in F_n \cap E_\epsilon} |\varphi(gx - gy)| + \sum_{g \in F_n \setminus E_\epsilon} |\varphi(gx) - \varphi(gy)| \right) \\ &\leq \zeta . ||\varphi||_\infty + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Since (X, d) is a compact metric space, there exists $\delta_3 > 0$ with the property that for every $x, y \in X$ with $d(x,y) < \delta_3$, we have

$$|\varphi_{F_n}(x) - \varphi_{F_n}(y)| < \epsilon, \quad \forall n = 1, 2, \ldots, N.$$

In the next step, we take $0 < \delta < \min\{\delta_2, \delta_3\}$. Then for every $x, y \in X$ with $d(x,y) < \delta$, we have

$$|\varphi_{F_n}(x) - \varphi_{F_n}(y)| < \epsilon, \quad \forall n = 1, 2, \ldots, N.$$

As a result, we have $\{\varphi_{F_n}\}_{n=1}^{\infty}$ is uniformly equicontinuous.

(2) $\Rightarrow$ (3). Since the sequence $\{\varphi_{F_n}\}_{n=1}^{\infty}$ is uniformly equicontinuous, it is uniformly bounded, too. In this way, there exists a subsequence $\{\varphi_{F_{n_k}}\}_{k=1}^{\infty}$ which is uniformly convergent to $\varphi^* \in C(X)$ by the Arzela–Ascoli Theorem. For every $x \in X$ and for every $s \in S$,

$$
\begin{aligned}
|\varphi^*(T_s x)-\varphi^*(x)| &= \lim_{k\to+\infty}\frac{1}{|F_{n_k}|}\left|\left(\sum_{g\in F_{n_k}}\varphi(T_g(T_s(x)))-\sum_{g\in F_{n_k}}\varphi(T_g(x))\right)\right| \\
&= \lim_{k\to+\infty}\frac{1}{|F_{n_k}|}\left|\left(\sum_{g\in F_{n_k}}(\varphi(T_{gs}(x))-\varphi(T_g(x)))\right)\right| \\
&\leq \lim_{k\to+\infty}\frac{||\varphi||_\infty\,|sF_{n_k}\Delta F_{n_k}|}{|F_{n_k}|} \\
&= 0.
\end{aligned}
$$

On the other hand, $\varphi^*|_{\overline{orb(x,S)}}$ is constant for every $x\in X$, because φ^* is continuous. By Theorem 2.5, $(\overline{orb(x,S)},\langle T_s\rangle_{s\in S})$ is a uniquely ergodic dynamical system. Therefore, $\lim_{n\to+\infty}\varphi_{F_n}=\varphi^*$ uniformly on $(\overline{orb(x,S)},\langle T_s\rangle_{s\in S})$.

Thus, for every $x\in X$, $\lim_{n\to+\infty}\varphi_{F_n}(x)=\varphi^*(x)$. Since $\{\varphi_{F_n}\}_{n=1}^{\infty}$ is uniformly equicontinuous, we notice that $\lim_{n\to+\infty}\varphi_{F_n}(x)=\varphi^*(x)$ uniformly on X.

(3) $\Rightarrow$ (1). We notice that $d(\cdot,\cdot)$ is a continuous function on $X\times X$; so, the sequence $\{d_{F_n}\}_{n=1}^{\infty}$ is uniformly equicontinuous where $d_{F_n}(x,y)=\frac{1}{|F_n|}\sum_{g\in F_n}d(T_g(x),T_g(y))$. Thus, for every $\epsilon>0$, there exists $\delta>0$ such that $d_{F_n}(x,y)<\epsilon/2$ for every $x,y\in X$ with $d(x,y)<\delta$. Then for every $x,y\in X$ with $d(x,y)<\delta$, we have

$$
\limsup_{n\to+\infty}\frac{1}{|F_n|}\sum_{g\in F_n}d(T_g(x),T_g(y))\leq\sup_n d_{F_n}(x,y)<\epsilon.
$$

Therefore, $(X,\langle T_s\rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous. □

It is shown in [11], if (X,T) is a transitive dynamical system, then (X,T) is uniquely ergodic. Similarly, by Lemma 3.7, Theorem 2.5, and Theorem 3.8, we obtain that the following proposition for $(X,\langle T_s\rangle_{s\in S})$, where S is a left amenable discrete infinite countable semigroup.

Proposition 3.9 *Let $(X,\mathcal{B},\mu)$ be a probability space and $(X,\langle T_s\rangle_{s\in S})$ be an m.p.s with the property that for some $s\in S$, T_s be injective and $\mathcal{F}$ be a left Følner sequence on S.*

1. *If $(X,\langle T_s\rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous, then for every $x\in X$, $(\overline{orb(x,S)},\langle T_s\rangle_{s\in S})$ is uniquely ergodic. Especially, if $(X,\langle T_s\rangle_{s\in S})$ is also transitive dynamical system, then $(X,\langle T_s\rangle_{s\in S})$ is uniquely ergodic.*

2. *If the minimal dynamical system $(X,\langle T_s\rangle_{s\in S})$ is $\mathcal{F}-$mean-L-stable, then it is uniquely ergodic.*

Definition 3.10 *Let $(X,\langle T_s\rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. We say $(X,\langle T_s\rangle_{s\in S})$ is $\mathcal{F}-$Banach mean equicontinuous if and only if for any $\epsilon>0$, there is a $\delta>0$ such that for every $y\in B(x,\delta)$, then $BD(\{s\in S:d(T_s(x),T_s(y))>\epsilon\})\leq\epsilon$.*

3.2. $\mathcal{F}$-density equicontinuous

We express the notion of density in equicontinuous system with respect to Følner sequence.

Definition 3.11 *Let $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. For $t \in [0,1]$, it is said that $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-density-t-equicontinuous, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in X$ with $d(x,y) < \delta$, implies that*

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s(x), T_s(y)) > \epsilon\}) \leq 1 - t.$$

It is evident that for $t = 1$, we have the notion of $\mathcal{F}$-density-equicontinuity.

Now, we examine the relationship between $\mathcal{F}$-mean equicontinuous and $\mathcal{F}$-density- equicontinuous in the following statements. In particular conditions, these concepts will be equivalent.

Proposition 3.12 *Let $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. The $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous if and only if $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-density-t-equicontinuous for every $t \in [0,1)$.*

Proof Sufficiency. Suppose that $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous and $E_\epsilon(x,y) := \{s \in S : d(T_s(x), T_s(y)) \geq \epsilon\}$. By Lemma 3.7, we have $\overline{D}_{\mathcal{F}}(E_\epsilon(x,y)) \leq \epsilon$. For $\epsilon \in (0,t]$, take $\delta_1 := \delta(\epsilon)$. Then $d(x,y) < \delta_1$ such that $\overline{D}_{\mathcal{F}}(E_\epsilon(x,y)) \leq \epsilon \leq t$. Also, for $\epsilon \in (t,1]$, take $\delta_2 := \delta(t)$. Then $d(x,y) < \delta_2$ such that

$$\overline{D}_{\mathcal{F}}(E_\epsilon(x,y)) \leq \overline{D}_{\mathcal{F}}(E_t(x,y)) \leq t.$$

Therefore, for every $\epsilon > 0$, $\delta := \min\{\delta_1, \delta_2\}$, we have $\overline{D}_{\mathcal{F}}(E_\epsilon(x,y)) \leq t$, whenever $d(x,y) < \delta$.

It follows that $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-density-(1-t)-equicontinuous; so, $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-density-t-equicontinuous for every $t \in [0,1)$ because t is arbitrary.

Necessity. Now, let $(X, \langle T_s \rangle_{s\in S})$ be a $\mathcal{F}$-density-t-equicontinuous for every $t \in [0,1)$ and pick $\epsilon \in (0,1)$ and $t \in (1-\epsilon, 1)$. Therefore, there is a $\delta(\epsilon, t) > 0$ such that if $x, y \in X$ with $d(x,y) < \delta(\epsilon, t)$, then

$$\overline{D}_{\mathcal{F}}(E_\epsilon(x,y)) \leq 1 - t \leq \epsilon.$$

Hence, $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-mean equicontinuous, and the proof is completed. □

Definition 3.13 *Let $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. For any $\epsilon > 0$, we call a point $x \in X$ is $\mathcal{F}$-Banach density-equicontinuous point if there is a $\delta > 0$ such that for every $y \in B(x, \delta)$*

$$BD(\{s \in S : d(T_s(x), T_s(y)) > \epsilon\}) = 0.$$

If every point of X is $\mathcal{F}$-Banach density-equicontinuous point, then a t.d.s., $(X, \langle T_s \rangle_{s\in S})$ is called an $\mathcal{F}$-Banach density-equicontinuous.

Theorem 3.14 *Suppose that $(X, \langle T_s \rangle_{s\in S})$ is a t.d.s., and $\mathcal{F}$ is a left Følner sequence on S. $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-Banach density-equicontinuous if and only if it is $\mathcal{F}$-density-equicontinuous.*

Proof Sufficiency. We assume that t.d.s., $(X, \langle T_s\rangle_{s\in S})$ is $\mathcal{F}$-density-quicontinuous and $x \in X$ is a $\mathcal{F}$-density-equicontinuous point but not $\mathcal{F}$-Banach density-equicontinuous point. Therefore, for any $\delta > 0$ there is $\epsilon > 0$ and $x_\delta \in B(x, \delta)$ such that

$$\overline{BD}(\{s \in S : d(T_s(x), T_s(x_\delta)) > \epsilon\}) > 0.$$

Since $x_\delta \in B(x, \delta)$, we consider a sequence x_n such that $\lim_{n\to+\infty} x_n = x$ and

$$\overline{BD}(\{s \in S : d(T_s(x), T_s(x_n)) > \epsilon\}) > 0.$$

Let $E_n = \{s \in S : d(T_s(x), T_s(x_n)) > \epsilon\}$, then for some subsequence $\{F_{n_k}\}_{n=1}^{\infty}$ we have

$$\lim_{k\to+\infty} \frac{|E_n \cap F_{n_k}|}{|F_{n_k}|} > 0,$$

for every $n \in \mathbb{N}$. So, we take that

$$\lim_{k\to+\infty} \frac{1}{|F_{n_k}|} \sum_{g\in F_{n_k}} \delta_{(T_g(x),T_g(x_n))} = \mu_n,$$

under the weak* topology and $\mu_n \in M(X, S)$.

Also, for $s \in S$ we have $\mu_n \in M(X \times X, S \times S)$, and $\operatorname{supp}(\mu_n) \subseteq (\overline{\operatorname{orb}(x, x_n)}, T_s \times T_s)$. Since d is a continuous function on $X \times X$, the set $U_\epsilon = \{(y_1, y_2) \in \ X \times X : d(y_1, y_2) > \epsilon\}$ is a nonempty open subset of $X \times X$. Then by Lemma 1.1, we have

$$0 < \overline{BD}_{\mathcal{F}}(E_n) \leq \limsup_{k\to+\infty} \frac{1}{|F_{n_k}|} \sum_{g\in F_{n_k}} \delta_{(T_g(x),T_g(x_n))}\left(\overline{U_\epsilon}\right) \leq \mu_n\left(\overline{U_\epsilon}\right).$$

On the other hand, x is a $\mathcal{F}-$density-equicontinuous point; so, for some large enough number $n \in \mathbb{N}$, the set $H_n = \{s \in S : d(T_s(x), T_s(x_n)) > \epsilon/2\}$ has zero upper density. i.e. $\overline{D}_{\mathcal{F}}(H_n) = 0$.

Also, X is $\mathcal{F}-$density-equicontinuous, then so is $X \times X$ and so $(\overline{\operatorname{orb}(x, x_n)}, S \times S)$ is uniquely ergodic, by Proposition 3.9. Therefore,

$$0 = \overline{D}_{\mathcal{F}}(H_n) \geq \liminf_{k\to+\infty} \frac{1}{|F_{n_k}|} \sum_{g\in F_{n_k}} \delta_{(T_g(x),T_g(x_n))}\left(U_{\epsilon/2}\right) \geq \mu_n\left(U_{\epsilon/2}\right),$$

because $\mu_n\left(U_{\epsilon/2}\right) \geq \mu_n\left(\overline{U_\epsilon}\right)$, it is a contradiction. Therefore, $(X, \langle T_s\rangle_{s\in S})$ is $\mathcal{F}-$Banach density-equicontinuous.

Necessity. It is obvious. □

Definition 3.15 *Let $(X, \langle T_s\rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. We call*

i) a point $x \in X$ is an $\mathcal{F}-$density-equicontinuous point (resp. $\mathcal{F}-$Banach density-equicontinuous point) if, for any $\epsilon > 0$, there is a $\delta > 0$ with the property that if $y \in B(x, \delta)$, then $d(T_s(x), T_s(y)) < \epsilon$ for all $s \in S$ except a set of zero upper density with respect to $\mathcal{F}$ (resp. zero upper $\mathcal{F}-$Banach density).

ii) $(X, \langle T_s\rangle_{s\in S})$ is almost $\mathcal{F}-$density-equicontinuous if there exists a transitive and $\mathcal{F}-$density-equicontinuous point.

Denote by $D(X,S)$, the set of all $\mathcal{F}-$density-equicontinuous points in X. For every $\epsilon > 0$ define

$$D_\epsilon(X,S) = \{x \in X : \ \forall y, z \in B(x,\delta)\ ,\ \exists \delta > 0\ ,\ \overline{D}_{\mathcal{F}}\left(\{s \in S : d(T_s(y), T_s(z)) > \epsilon\}\right) = 0\}.$$

It is obvious that if $0 < \epsilon_1 < \epsilon_2$ then $D_{\epsilon_1}(X,S) \supseteq D_{\epsilon_2}(X,S)$.

Proposition 3.16 *Let $(X, \langle T_s\rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S.*

(1) For each $\epsilon > 0$, $D_\epsilon(X,S)$ is open, $T_s^{-1}(D_\epsilon(X,S)) \subset D_\epsilon(X,S)$ for every $s \in S$ and

$$D(X,S) = \lim_{k\to+\infty} \bigcap_{n=1}^{k} D_{\frac{1}{|F_n|}}(X,S)$$

is a G_δ subset of X.

(2) If $(X, \langle T_s\rangle_{s\in S})$ is transitive dynamical system, then $D(X,S)$ is either residual or empty. If additionally $(X, \langle T_s\rangle_{s\in S})$ is almost $\mathcal{F}-$ density-equicontinuous, then every transitive point belongs to $D(X,S)$.

(3) If $(X, \langle T_s\rangle_{s\in S})$ is minimal dynamical system and almost $\mathcal{F}-$ density-equicontinuous, then it is $\mathcal{F}-$ density-equicontinuous.

Proof Suppose that $x \in D_\epsilon(X,S)$. We consider $\delta > 0$ satisfying the properties of the definition of $D_\epsilon(X,S)$ for $x \in X$. Now, fix $y \in B(x,\delta/2)$. If $z, w \in B(y,\delta/2)$, then $z, w \in B(x,\delta)$; so, $\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s(z), T_s(w)) > \epsilon\}) = 0$. Therefore, $B(x,\delta/2) \subset D_\epsilon(X,S)$ and $D_\epsilon(X,S)$ are open.

Let $x \in X$ with $T_s^{-1}(x) \in D_\epsilon(X,S)$, for every $s \in S$. We can choose $\delta > 0$ satisfying the properties of the definition of $D_\epsilon(X,S)$ for $T_s^{-1}(x) \in X$. There exists $\zeta > 0$ and $g, g' \in S$ with the property that $d(T_g(y), T_g(z)) < \epsilon$ for any $y, z \in B(x,\zeta)$. If $y, z \in B(x,\zeta)$, then $T_g(y), T_g(z) \in B(T_g(x),\delta)$. Therefore,

$$d(T_g(y), T_g(z)) = d(T_g(g'y), T_g(g'z)) < \epsilon.$$

So, $x \in D_\epsilon(X,S)$. On the other hand $T_s^{-1}(D_\epsilon(X,S)) \subset D_\epsilon(X,S)$ for every $s \in S$.
If $x \in X$ belongs to all intersections of $D_{\frac{1}{|F_n|}}(X,S)$, then obviously, $x \in D(X,S)$. Conversely, if $x \in D(X,S)$ then there exists $\delta > 0$ for all $y \in B(x,\delta)$ such that

$$d(T_g(x), T_g(y)) < \frac{1}{2|F_n|}.$$

If $y, z \in B(x,\delta)$, then

$$\begin{aligned} d(T_g(y), T_g(z)) &\le (d(T_g(x), T_g(y)) + d(T_g(x), T_g(z)) \\ &\le \frac{1}{2|F_n|} + \frac{1}{2|F_n|} \\ &= \frac{1}{|F_n|}. \end{aligned}$$

Thus, $x \in D_{\frac{1}{|F_n|}}(X,S)$.

By the transitivity of $(X, \langle T_s \rangle_{s\in S})$, every $D_\epsilon(X,S)$ is either dense or empty; since $D_\epsilon(X,S)$ is inversely invariant and open; therefore, $D(X,S)$ is either residual or empty by the Baire category theorem. If $D(X,S)$ is residual, then every $D_\epsilon(X,S)$ is dense and open. If $\epsilon > 0$ and $x \in X$ is a transitive point, then there exists some $g \in S$ such that $gx \in D_\epsilon(X,S)$. So, $x \in D_\epsilon(X,S)$; since $D_\epsilon(X,S)$ is inversely invariant. Thus, $x \in D(X,S)$. □

4. $\mathcal{F}$-density-sensitivity

In this section, we show the notion of sensitivity. Let $(X, \langle T_s \rangle_{s\in S})$ be a dynamical system where S is a discrete (infinite) left amenable countable semigroup and (X,d) is a compact metric space and $U \subseteq X$. We define

$$N_T(U,\delta) = \{s \in S : \text{there exists } x, y \in U \text{ such that } d(T_s(x), T_s(y)) > \delta\}.$$

Here δ will be referred to as a constant of sensitivity.

Definition 4.1 *A dynamical system $(X, \langle T_s \rangle_{s\in S})$ is sensitive, (thickly syndetically sensitive, thickly sensitive, respectively) if the set $N_T(U,\delta)$ is a nonempty set (thickly syndetic set, thick set, respectively) for every nonempty open subset U of X.*

By Definitions 4.1, we have syndetically sensitive from thickly sensitive.

Definition 4.2 *A dynamical system $(X, \langle T_s \rangle_{s\in S})$ is multisensitive, if the collection $\{N_T(U,\delta) :\ U$ is an open and nonempty subset of $X\}$ has the finite intersection property. Here δ will be referred to as a constant of sensitivity.*

Proposition 4.3 *Let $(X, \langle T_s \rangle_{s\in S})$ be a dynamical system and T_s be a homeomorphism for each $s \in S$. If $(X, \langle T_s \rangle_{s\in S})$ is multisensitive, then $(X, \langle T_s \rangle_{s\in S})$ is thickly sensitive.*

Proof Let $(X, \langle T_s \rangle_{s\in S})$ be a multisensitive with a constant of sensitivity δ. Let $F := \{s_1, s_2, \dots, s_k\}$ and U be an open subset of X. Pick $U_i := s_i U$. For any finite collection of nonempty open subsets $U_1, U_2, \dots, U_k$ of X, we have $\bigcap_{i=1}^k N_T(U_i,\delta) \neq \emptyset$. Then for every $1 \leq i \leq k$ there are $x_i, y_i \in U$ such that $d(T_{s_i}x_i, T_{s_i}y_i) > \delta$. Let $x = T_{s_i}x_i$ and $y = T_{s_i}x_j$. Thus $d(T_s x, T_s y) = d(T_s T_{s_i} x_i, T_s T_{s_i} y_i) > \delta$, for some $s \in \bigcap_{i=1}^k N_T(U_i,\delta)$. Therefore, there exists $s \in S$ such that $ss_i \subseteq N_T(U,\delta)$ for $i = 1, 2, \dots, k$. The set $N_T(U,\delta)$ is thick, and $(X, \langle T_s \rangle_{s\in S})$ is thickly sensitive. □

Especially the thickly sensitive is equivalent to the multisensitive. In the following statement, we prove it.

Proposition 4.4 *Suppose that $(X, \langle T_s \rangle_{s\in S})$ is a dynamical system with the point transitive system and T_s is a homeomorphism for every $s \in S$. If $(X, \langle T_s \rangle_{s\in S})$ is thickly sensitive, then $(X, \langle T_s \rangle_{s\in S})$ is multisensitive.*

Proof Assume that $(X, \langle T_s \rangle_{s\in S})$ is a thickly sensitive with a constant of sensitivity δ. Let $U_1, U_2, \dots, U_n$ be nonempty open subsets of X. Let $z \in X$ be a transitive point of $(X, \langle T_s \rangle_{s\in S})$. Then there is $s_i \in S$ such that

$s_i z \in U_i$ where $1 \le i \le n$. i.e. $N(z, U_i) \neq \emptyset$. Choose a nonempty open subset $U \subseteq X$ such that $s_i U \subseteq U_i$ for all $i \in \{1, 2, \ldots, n\}$. Since $(X, \langle T_s \rangle_{s\in S})$ is thickly sensitive, $N_T(U_i, \delta)$ is a thick set.

So, for $F := \{s_1, s_2, \ldots, s_n\} \subseteq S$ there exists $s \in S$ such that $sF \subseteq N_T(U, \delta)$. Since $\bigcap_{i=1}^{k} N_T(s_i U, \delta) \subseteq \bigcap_{i=1}^{k} N_T(U_i, \delta)$, we have $s \in \bigcap_{i=1}^{k} N_T(s_i U, \delta)$ and hence $s \in \bigcap_{i=1}^{k} N_T(U_i, \delta)$. This implies that $(X, \langle T_s \rangle_{s\in S})$ is multisensitive. □

Definition 4.5 *Let $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. We say that $\mathcal{F}$-mean sensitive if there exists $\delta > 0$ such that for every $x \in X$ and every $\epsilon > 0$ there is $y \in B(x, \epsilon)$ satisfying*

$$\limsup_{n\to+\infty} \frac{1}{|F_n|} \sum_{g\in F_n} d(T_g(x), T_g(y)) > \delta.$$

Definition 4.6 *Let $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. We call $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-density-n-sensitive, if there is $\delta > 0$ with the property that for any nonempty open subset U of X there are $x_1, \ldots, x_n \in U$ for $2 \le n \in \mathbb{N}$ such that the set*

$$\overline{D}_{\mathcal{F}}(\{s \in S : \min_{1\le i\neq j\le n} d(T_s x_i, T_s x_j) > \delta\}) > 0.$$

Here δ will be referred to as a constant of sensitivity. When $n = 2$, for simplicity, we call that $(X, \langle T_s \rangle_{s\in S})$ is $\mathcal{F}$-density-sensitive.

Definition 4.7 *Let $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s., and $\mathcal{F}$ be a left Følner sequence on S. We call the nondiagonal tuple $(x_1, x_2, \ldots, x_n) \in X^{(n)}$ for $2 \le n \in \mathbb{N}$, is a $\mathcal{F}$-density-n-sensitive tuple point (or say $\mathcal{F}$-density-sensitive-n-tuple point) if for any $\delta > 0$ and nonempty open subset U of X, there exist $y_1, y_2, \ldots, y_n \in U$ such that*

$$\overline{D}_{\mathcal{F}}(\{s \in S : T_s y_i \in B(x_i, \delta), i = 1, 2, \ldots, n\}) > 0.$$

Denote by $DS_n(X, S)$ all $\mathcal{F}$-density-sensitive n-tuples points. A tuple $(x_1, x_2, \ldots, x_n) \in DS_n(X, S)$ is referred to be essential if $x_i \neq x_j$ for any $1 \le i < j \le n$. We show the collection of all such n-tuples points as $DS_n^e(X, S)$.

Proposition 4.8 *Let $(X, \langle T_s \rangle_{s\in S})$ be a t.d.s., and let $\mathcal{F}$ be a left Følner sequence on S. We have:*

1. *If $(X, \langle T_s \rangle_{s\in S})$ is a transitive dynamical system, then it is either $\mathcal{F}$-density-sensitive or almost $\mathcal{F}$-density-equicontinuous;*
2. *If $(X, \langle T_s \rangle_{s\in S})$ is a minimal dynamical system, then it is either $\mathcal{F}$-density-sensitive or $\mathcal{F}$-density-equicontinuous.*

Proof Suppose that $(X, \langle T_s \rangle_{s\in S})$ is not almost $\mathcal{F}$-density-equicontinuous, see Definition 3.15. There is a transitive point $x \in X$ which is not an $\mathcal{F}$-density-equicontinuous point, by Proposition 3.16. Let U be a nonempty open subset of X. So, there are $s \in S$ and $\epsilon_0 > 0$ such that $T_s B(x, \epsilon_0) \subset U$. Since x is not an $\mathcal{F}$-density-equicontinuous point, there is a $\delta > 0$ such that for any $\ell \in \mathbb{N}$ with $0 < 1/\epsilon_0 < \ell$, there are $n_\ell \in \mathbb{N}$ and $y_\ell \in X$ with $d(x, y_\ell) < 1/\ell$ such that

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s x, T_s y_\ell) > \delta\}) > 0.$$

Pick $u = T_s x$ and $v_\ell = T_s y_\ell$. Thus, $u, v_\ell \in U$ and

$$\overline{D}_{\mathcal{F}}(\{s \in S : d(T_s u, T_s v_\ell) > \delta\}) = \overline{D}_{\mathcal{F}}(\{s \in S : d(T_s x, T_s y_\ell) > \delta\}) > 0.$$

It follows that $(X, \langle T_s \rangle_{s \in S})$ is $\mathcal{F}$-density-sensitive.
The the proof of part (2) follows from Proposition 3.16 and part (1). □

The following proposition states that $\mathcal{F}$-density-n-sensitivity can be characterized by $\mathcal{F}$-density-sensitive-n-tuples.

Proposition 4.9 *Suppose that $(X, \langle T_s \rangle_{s \in S})$ is a transitive t.d.s., and $2 \leq n \in \mathbb{N}$ and $\mathcal{F}$ is a left Følner sequence on S. Then $(X, \langle T_s \rangle_{s \in S})$ is $\mathcal{F}$-density-n-sensitive if and only if $DS_n^e(X, S) \neq \emptyset$.*

Proof Sufficiency. Assume that $(x_1, \ldots, x_n) \in DS_n^e(X, S)$. Set $\delta = \frac{1}{2}\min\{d(x_i, x_j) : 1 \leq i \neq j \leq n\}$. Since a tuple $(x_1, x_2, \ldots, x_n) \in DS_n(X, S)$ is mentioned to be essential if $x_i \neq x_j$ for each $1 \leq i < j \leq n$. Let $U_1, \ldots, U_n$ be open neighbourhoods of $x_1, \ldots, x_n$ for $1 \leq i \neq j \leq n$, respectively. Since $(x_1, \ldots, x_n) \in DS_n^e(X, S)$, then for every nonempty open subset U of X there are $y_1, y_2, \ldots, y_n \in U$ by

$$\overline{D}_{\mathcal{F}}(\{s \in S : T_s y_i \in U_i, i = 1, 2, \ldots, n\}) > 0.$$

Hence,

$$\overline{D}_{\mathcal{F}}(\{s \in S : \min_{1 \leq i \neq j \leq n} d(T_s y_i, T_s y_j) > \delta\}) > 0.$$

So, $(X, \langle T_s \rangle_{s \in S})$ is $\mathcal{F}$-density-n-sensitive.

Necessity. First, suppose that $(X, \langle T_s \rangle_{s \in S})$ is $\mathcal{F}$-density-n-sensitive. Assume that

$$X_\delta^n = \{(x_1, x_2, \ldots, x_n) \in X^{(n)} : \min_{1 \leq i < j \leq n} d(x_i, x_j) \geq \delta\}.$$

It is obvious that X_δ^n is a closed subset of $X^{(n)}$. Let $x \in X$ be a transitive point. There are $x_m^1, x_m^2, \ldots, x_m^n \in B(x, 1/m)$ for every $m \in \mathbb{N}$ such that

$$\overline{D}_{\mathcal{F}}(\{s \in S : (T_s x_m^1, \ldots, T_s x_m^n) \in X_\delta^n\}) > 0,$$

because $(X, \langle T_s \rangle_{s \in S})$ is $\mathcal{F}$-density-n-sensitive. Now, we can take an open cover $\{A_1^1, \ldots, A_1^{N_1}\}$ of X_δ^n for some $N_1 \in \mathbb{N}$ with the property that $\max\{diam(A_1^i) : i = 1, \ldots, n\} < 1$. So, there is $1 \leq N_1^m \leq N_1$ for every $m \in \mathbb{N}$ such that

$$\overline{D}_{\mathcal{F}}(\{s \in S : (T_s x_m^1, \ldots, T_s x_m^n) \in \overline{A_1^{N_1^m}} \cap X_\delta^n\}) > 0.$$

Without restriction of generality, we suppose that $N_1^m = 1$ for all $m \in \mathbb{N}$ and

$$\overline{D}_{\mathcal{F}}(\{s \in S : (T_s x_m^1, \ldots, T_s x_m^n) \in \overline{A_1^1} \cap X_\delta^n\}) > 0.$$

By repeating the above method, for every $\ell \geq 1$ we have cover $\overline{A_\ell^1} \cap X_\delta^n$ by finite nonempty open subsets with diameters less than $1/(\ell+1)$, i.e. $\overline{A_\ell^1} \cap X_\delta^n \subset \bigcup_{i=1}^{N_{\ell+1}} A_i^{\ell+1}$ and $diam(A_i^{\ell+1}) < 1/(\ell+1)$. Then for every $m \in \mathbb{N}$, there is $1 \leq N_{\ell+1}^m \leq N_{\ell+1}$ such that

$$\overline{D}_{\mathcal{F}}(\{s \in S : (T_s x_m^1, \ldots, T_s x_m^n) \in \overline{A_{\ell+1}^{N_{\ell+1}^m}} \cap X_\delta^n\}) > 0.$$

There is no loss of generality in assuming $N_{\ell+1}^m = 1$ for all $m \in \mathbb{N}$ and

$$\overline{D}_{\mathcal{F}}(\{s \in S : (T_s x_m^1, \ldots, T_s x_m^n) \in \overline{A_{\ell+1}^1} \cap X_\delta^n\}) > 0.$$

Obviously, there is a unique point $(z_1, \ldots, z_n) \in \bigcap_{l=1}^{\infty} \overline{A_{\ell+1}^1} \cap X_\delta^n$. Now, we must show that $(z_1, \ldots, z_n) \in DS_n^e(X, S)$. For any $\epsilon > 0$ there is $\ell \in \mathbb{N}$ such that $\overline{A_{\ell+1}^1} \cap X_\delta^n \subset V_1 \times \cdots \times V_n$, where $V_i = B(z_i, \epsilon)$ for $i = 1, \ldots, n$.

By the above construction, for every $B(x, 1/m)$, there are $x_m^1, \ldots, x_m^n \in B(x, 1/m)$ such that

$$\overline{D}_{\mathcal{F}}(\{s \in S : (T_s x_m^1, \ldots, T_s x_m^n) \in \overline{A_\ell^1} \cap X_\delta^n\}) > 0.$$

Furthermore, for all $m \in \mathbb{N}$,

$$\overline{D}_{\mathcal{F}}(\{s \in S : (T_s x_m^1, \ldots, T_s x_m^n) \in V_1 \times \cdots \times V_n\}) > 0.$$

For every nonempty open subset U of X, since x is a transitive point, there is $t \in S$ such that $T_k x \in U$ and so for some $m_0 \in \mathbb{N}$, we have $T_t B(x, 1/m_0) \subset U$. It means that $T_t x_{m_0}^1, \ldots, T_t x_{m_0}^n \in U$ and

$$\overline{D}_{\mathcal{F}}(\{s \in S : (T_t(T_s x_m^1), \ldots, T_t(T_s x_m^n)) \in V_1 \times \cdots \times V_n\}) > 0.$$

Thus, $(z_1, \ldots, z_n) \in DS_n^e(X, S)$ and the proof is completed. □

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