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# $(p,q)$-Chebyshev polynomials for the families of biunivalent function associating a new integral operator with $(p,q)$-Hurwitz zeta function

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**Abstract:** In the present article, making use of the $(p,q)$-Hurwitz zeta function, we provide and investigate a new integral operator. Also, we define two families $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$ and $\mathcal{SC}_{p,q}(\lambda,\zeta,\vartheta,u,\tau)$ of biunivalent and holomorphic functions in the unit disc connected with $(p,q)$-Chebyshev Polynomials. Then we find coefficient estimates $|a_2|$ and $|a_3|$. Finally, we obtain Fekete-Szegö inequalities for these families.

**Key words:** Biunivalent function, $(p,q)$-Chebyshev polynomial, $(p,q)$-Hurwitz zeta function, a new integral operator, coefficient estimates, and Fekete-Szegö inequality

## 1. Introduction

Let $\Omega$ be the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

where the class $\Omega$ is analytic in the open unit disk $\Sigma = \{z : |z| < 1\}$. We also define to $\mathcal{P}$ as the class of all $\Omega$ functions that are univalent in $\Sigma$.

By the *Koebe One Quarter Theorem* [17] it is clear that the function $f \in \mathcal{P}$ has an inverse function $f^{-1}$, which is defined by

$f^{-1}(f(z)) = z$ and $f\left(f^{-1}(\widetilde{\omega})\right) = \eta$, $\left(|\widetilde{\omega}| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$,

where

$$f^{-1}(\widetilde{\omega}) = \widetilde{\omega} + \sum_{n=2}^{\infty} g_n \widetilde{\omega}^n.$$

It is clear that

$$\widetilde{\omega} = f\left(f^{-1}(\widetilde{\omega})\right) = \widetilde{\omega} + (g_2 + a_2)\widetilde{\omega}^2 + \left(g_3 - 2a_2^2 + a_3\right)\widetilde{\omega}^3 + \left(g_4 + 5a_2^3 - 5a_2a_3 + a_4\right)\widetilde{\omega}^4 + \cdots.$$

Assume that $g_2 = -a_2, g_3 = 2a_2^2 - a_3$ and $g_4 = -5a_2^3 + 5a_2a_3 - a_4$, we find that

$$f^{-1}(\widetilde{\omega}) = \widetilde{\omega} - a_2\widetilde{\omega}^2 + \left(2a_2^2 - a_3\right)\widetilde{\omega}^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\widetilde{\omega}^4 + \cdots. \tag{1.2}$$

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If both $f$ and $f^{-1}$ are univalent in $\Omega$, a function $f \in \Omega$ is said to be biunivalent in $\Omega$. We denote the biunivalent function by $\Xi$. After that, many mathematicians have been interested in studying biunivalent functions and they have provided and investigated the bounds for the coefficients $|a_n|$ (see, [10, 11], [20, 21], [25], and [32]).
The Fekete-Szegö inequality (problem), related to the Bieberbach conjecture, is an inequality for the coefficients of univalent analytic functions $f(z)$ in (1.1) discovered by Fekete and Szegö [19]. The Fekete-Szegö problem is the search for the estimate of the maximum value of the coefficient functional $\left|a_3 - \gamma a_2^2\right|$. Many authors have presented many subclasses of biunivalent analytic functions and examined applications of Fekete-Szegö inequality, such as Amourah et al. [4], Arikan et al. [6], Darus and Thomas [13], Deniz and Orhan [16], Yousef et al. ([43] and [44]), and Srivastava et al. ([34]–[38]).
The function $f(z)$ is called subordinate to $\hbar(z)$, denoted by $f(z) \prec \hbar(z)$; if $f(z)$ and $\hbar(z)$ are two analytic functions in $\Omega$ and there is a Schwarz function $\varphi$ with $\varphi(z) = 0$, $|\varphi(z)| < 1$ and $f(z) = \hbar(\varphi(z))$. In addition, we get the following equivalence if the function $\hbar$ is univalent in $\Omega$

$$f(z) \prec \hbar(z) \Leftrightarrow f(0) = \hbar(0) \text{ and } f(\Omega) \subset \hbar(\Omega).$$

For any $j \geq 2$ and $0 < q < p \leq 1$, The second type of $(p,q)$-Chebyshev polynomial is known as the following relationship:

$$\mathcal{F}_j(x,\alpha,p,q) = \left(p^j + q^j\right) x\mathcal{F}_{j-1}(x,\alpha,p,q) + (pq)^{j-1}\alpha\mathcal{F}_{j-2}(x,\alpha,p,q) \tag{1.3}$$

with $\alpha$ is a variable and the initial values $\mathcal{F}_0(x,\alpha,p,q) = 1$ and $\mathcal{F}_1(x,\alpha,p,q) = (p+q)x$.
Depending on the values of $p, q, s,$ and $x$, we get special polynomials from $(p,q)$-Chebyshev polynomial such as (Fibonacci polynomials, Pell polynomials, Jacobsthal polynomials, and Chebyshev polynomial of the second type), it was presented by many researchers (see, [1], [3, 4], [10, 11], and [18]).
In 2019, Kizilates et al. [27] provided and investigated the first and second types of $(p,q)$- Chebyshev polynomial and obtained derived formulas, generating functions, and some important results of this polynomial.
The generating function of the second type is defined as follows.

$$\mathcal{C}_{p,q}(z) = \frac{1}{1 - xpz\eta_p - xqz\eta_q - \alpha pqz^2\eta_{p,q}} = \sum_{j=0}^{\infty} \mathcal{F}_j(x,\alpha,p,q)\, z^j \,(z \in \Sigma). \tag{1.4}$$

Let $\aleph(z)$ and $\varphi(\widetilde{\omega})$ be two analytic functions in the unit disk $\Omega$ with $\aleph(0) = \varphi(0) = 0$, $|\aleph(z)| < 1, |\varphi(\widetilde{\omega})| < 1$, and

$$\aleph(z) = \sigma_1 z + \sigma_2 z^2 + \sigma_3 z^3 + \cdots, \text{ and } \varphi(\widetilde{\omega}) = l_1\widetilde{\omega} + l_2\widetilde{\omega}^2 + l_3\widetilde{\omega}^3 + \cdots \quad (z, \widetilde{\omega} \in \Sigma). \tag{1.5}$$

The $(p,q)$-calculus denotes the possibility of extending the $q$-calculus to postquantum calculus. Chakrabarti and Jagannathan [12] proposed the $(p,q)$-calculus in quantum algebras to generalize the $q$-series, which has numerous applications in science and engineering. After that, many articles provided and investigated the application of $(p,q)$-calculus (see; [2], [7], [22], [23], [29], and [33]).
The $(p,q)$-derivative operator is a kind of derivative operator defined by

$$d_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z}\,(z \neq 0,\ 0 < q < p < 1,\ and\ p \neq q).$$

It is clear that for $n \in \mathbb{N}=\{1,2,3,\dots\}$ and $z \in \Sigma$

$$d_{pq}\left\{\sum_{j=1}^{\infty} a_j z^j\right\} = \sum_{j=1}^{\infty} [j]_{pq} z^{j-1},$$

where $[j]_{p,q} = \frac{p^j - q^j}{p-q} = p^{j-1} + p^{j-2}q + p^{j-3}q^2 + \cdots + pq^{j-2} + q^{j-1}$ and $[0]_{p,q} = 0$.
This is a natural extension of the $q$-number ([24]),

$$\lim_{p\to 1} [j]_{p,q} = [j]_q = \frac{1-q^j}{1-q}, q \neq 1.$$

We introduce the $p,q$-Hurwitz zeta function, which is a generalization of $q$–Hurwitz zeta function (see, [30], [31], and [40]), motivated by the work cited above.
In the beginning, we define the $(p,q)$-Hurwitz zeta function $\zeta_{p,q}(u,\tau;z)$ by the following form

$$\zeta_{p,q}(u,\tau;z) = \sum_{j=0}^{\infty} \frac{z^j}{[j+u]_{p,q}^{\tau}},$$

where $u \in \mathbb{C}\backslash\mathbb{Z}_0^-, \tau \in \mathbb{C}$, when $|z|<1$, and $\mathrm{R}(\tau)>1$ when $|z|=1$.
Now, by the functions $f(z)$ in (1.1), we present the $(p,q)$-Srivastava-Attiya operator $\mathcal{J}_{p,q,\tau}^{u} f(z) : \Omega \to \Omega$ as follows

$$\mathcal{J}_{p,q,\tau}^{u} f(z) = \left(\Phi_{\tau}^{u}(p,q;z) * f(z)\right)\left(z \in \Sigma, u \in \mathbb{C}\backslash\mathbb{Z}_0^-; \tau \in \mathbb{C}\right), \tag{1.6}$$

where

$$\Phi_{\tau}^{u}(p,q;z) = [1+u]_{pq}^{\tau}\left[\zeta_{p,q}(u,\tau;z) - (u)_{pq}^{-\tau}\right]. \tag{1.7}$$

From (1.6) and (1.7), we note that

$$\mathcal{J}_{p,q,\tau}^{u} f(z) = z + \sum_{j=2}^{\infty}\left(\frac{[1+u]_{pq}}{[j+u]_{pq}}\right)^{\tau} a_j z^j. \tag{1.8}$$

From (1.8), we observe that
1. When $p=1$, we get a $q$-Srivastava-Attiya operator [39].

$$\mathcal{J}_{q,\tau}^{b} f(z) = z + \sum_{n=2}^{\infty}\left(\frac{[1+u]_q}{[n+u]_q}\right)^{\tau} a_n z^n.$$

2. When $p=1$ and $q \to 1$, we get a Srivastava-Attiya operator [34].

$$\mathcal{J}_{1s,\tau}^{u} f(z) = z + \sum_{j=2}^{\infty}\left(\frac{1+u}{j+u}\right)^{\tau} a_j z^j.$$

The aim of this work is to define a $(p,q)$-integral operator by using $(p,q)$-Hurwitz zeta function, which is a generalization of $q$-Srivastava-Attiya integral operator. After that, we determine initial coefficient bounds for some classes of analytic functions defined by subordination. Finally, our results deal with the Fekete-Szegö problem for $(p,q)$-Chebyshev polynomials.

**2. New concepts and results**

At he beginning, we introduce the concept of new families $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$ and $\mathcal{SC}_{p,q}\left(\lambda,\zeta,\vartheta,u,\tau\right)$.

**Definition 2.1** *For $0 \le \xi \le 1, \zeta \ge 0$ and $0 \le \delta \le 1$. A function $f \in \Xi$ is said to be in the family $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,\ u,\tau)$ if the following subordinate conditions are satisfied:*

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right)'}{\left[\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right]^{1-\zeta}}+\xi\left(\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right)'\left(\frac{\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)}{z}\right)^{\delta}\prec\mathcal{C}_{p,q}\left(z\right) \tag{2.1}$$

*and*

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right)'}{\left[\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right]^{1-\zeta}}+\xi\left(\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right)'\left(\frac{\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)}{\widetilde{\omega}}\right)^{\delta}\prec\mathcal{C}_{p,q}\left(\widetilde{\omega}\right). \tag{2.2}$$

**Remark 2.2** *It is worth noting that the family $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$ contains many subfamilies, we mention them as follows:*

*1- We get the subfamily $\mathcal{SM}_{p,q}(\xi,\zeta,u,\tau)$, when $\delta=0$; define as below*

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right)'}{\left[\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right]^{1-\zeta}}+\xi\left(\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right)'\prec\mathcal{C}_{p,q}\left(z\right),$$

*and*

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right)'}{\left[\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right]^{1-\zeta}}+\xi\left(\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right)'\prec\mathcal{C}_{p,q}\left(\widetilde{\omega}\right).$$

*2- We get the subfamily $\mathcal{SM}_{p,q}(\xi,\delta,u,\tau)$, when $\zeta=1$; define as below*

$$(1-\xi)\left(\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right)'+\xi\left(\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right)'\left(\frac{\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)}{z}\right)^{\delta}\prec\mathcal{C}_{p,q}\left(z\right),$$

*and*

$$(1-\xi)\left(\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right)'+\xi\left(\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right)'\left(\frac{\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)}{\widetilde{\omega}}\right)^{\delta}\prec\mathcal{C}_{p,q}\left(\widetilde{\omega}\right).$$

*3- We get the subfamily $\mathcal{SM}_{p,q}(\xi,\zeta,u,\tau)$, when $\xi=0$; define as below*

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right)'}{\left[\mathcal{J}^{u}_{p,q,\tau}f\left(z\right)\right]^{1-\zeta}}\prec\mathcal{C}_{p,q}\left(z\right),$$

*and*

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right)'}{\left[\mathcal{J}^{u}_{p,q,\tau}f^{-1}\left(\widetilde{\omega}\right)\right]^{1-\zeta}}\prec\mathcal{C}_{p,q}\left(\widetilde{\omega}\right).$$

**Definition 2.3** *For $\lambda\in\mathbb{C}\backslash\{0\}$, $\zeta\ge 0$ and $\vartheta\ge 0$. A function $f\in\Xi$ is said to be in the family $\mathcal{SC}_{p,q}\left(\lambda,\zeta,\vartheta,u,\tau\right)$ if satisfies the following subordinate conditions:*

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f\left(z\right)+\vartheta z^{2}(\mathcal{J}_{p,q,\tau}^{u}f\left(z\right))''}{z}+(\zeta-2\vartheta)\big(\mathcal{J}_{p,q,\tau}^{u}f\left(z\right)\big)'-1\right)\prec\mathcal{C}_{p,q}\left(z\right), \tag{2.3}$$

and

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\,\mathcal{J}_{p,q,\tau}^{u}f^{-1}\left(\widetilde{\omega}\right)+\vartheta\widetilde{\omega}^{2}\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}\left(\widetilde{\omega}\right)\big)''}{\widetilde{\omega}}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}\left(\widetilde{\omega}\right)\right)'-1\right)\prec\mathcal{C}_{p,q}\left(\widetilde{\omega}\right). \tag{2.4}$$

**Remark 2.4** *It is worth noting that the family $\mathcal{SC}_{p,q}\left(\lambda,\zeta,\vartheta,u,\tau\right)$ contains many subfamilies, we mention them as follows:*

*1- We get the subfamily $\mathcal{SC}_{p,q}\left(\lambda,\vartheta,u,\tau\right)$, when $\zeta=1+2\vartheta$;*

$$\frac{1}{\lambda}\left(\vartheta z(\mathcal{J}_{p,q,\tau}^{u}f\left(z\right))''+\big(\mathcal{J}_{p,q,\tau}^{u}f\left(z\right)\big)'-1\right)\prec\mathcal{C}_{p,q}\left(z\right),$$

*and*

$$\frac{1}{\lambda}\left(\vartheta\widetilde{\omega}\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}\left(\widetilde{\omega}\right)\big)''+\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}\left(\widetilde{\omega}\right)\big)'-1\right)\prec\mathcal{C}_{p,q}\left(\widetilde{\omega}\right).$$

*2- We get the subfamily $\mathcal{SC}_{p,q}(\lambda,\zeta,u,\tau)$, when $\vartheta=0$;*

$$\frac{1}{\lambda}\left((1-\zeta)\,\frac{\mathcal{J}_{p,q,\tau}^{u}f\left(z\right)}{z}+\zeta\big(\mathcal{J}_{p,q,\tau}^{u}f\left(z\right)\big)'-1\right)\prec\mathcal{C}_{p,q}\left(z\right),$$

*and*

$$\frac{1}{\lambda}\left(\frac{(1-\zeta)\,\mathcal{J}_{p,q,\tau}^{u}f^{-1}\left(\widetilde{\omega}\right)}{\widetilde{\omega}}+\zeta\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}\left(\widetilde{\omega}\right)\big)'-1\right)\prec\mathcal{C}_{p,q}\left(\widetilde{\omega}\right).$$

*3- We get the subfamily $\mathcal{SC}_{p,q}(\lambda,u,\tau)$, when $\zeta=1$ and $\vartheta=0$;*

$$\frac{1}{\lambda}\left(\big(\mathcal{J}_{p,q,\tau}^{u}f\left(z\right)\big)'-1\right)\prec\mathcal{C}_{p,q}\left(z\right),$$

*and*

$$\frac{1}{\lambda}\left(\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}\left(\widetilde{\omega}\right)\big)'-1\right)\prec\mathcal{C}_{p,q}\left(\widetilde{\omega}\right).$$

**Theorem 2.5** *Let $f(z)\in\Omega$ be in the family $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$, then*

$$|a_2|\leq\frac{|(p+q)\,x|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\mathcal{H}_{p,q,x}\left(\xi,\zeta,\delta,u,\tau\right)+\mathcal{N}_{p,q,x}\left(\xi,\zeta,\delta,u,\tau\right)-\mathcal{Q}_{p,q,x}\left(\xi,\zeta,\delta,u,\tau\right)\right]}}, \tag{2.5}$$

*and*

$$|a_3|\leq\frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{\left[(1-\xi)\,(\zeta+1)+\xi\delta+2\right](1+u)_{pq}^{\tau}}+\frac{(p+q)^{2}x^{2}(2+u)_{pq}^{2\tau}}{\left[(1-\xi)\,(\zeta+1)+\xi\delta+1\right]^{2}(1+u)_{pq}^{2\tau}}, \tag{2.6}$$

*where* $\mathcal{H}_{p,q,x}\left(\xi,\zeta,\delta,u,\tau\right)=(p+q)^{2}x^{2}\left((1-\xi)\,(\zeta+1)+\xi\delta+2\right)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,

$\mathcal{N}_{p,q,x}\left(\xi,\zeta,\delta,u,\tau\right)=(p+q)^{2}x^{2}\left(\frac{1}{2}\left(\zeta-\xi\zeta\right)(\zeta+3)+\xi(\delta\left(\delta+1\right)-2)\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$,

*and* $\mathcal{Q}_{p,q,x}\left(\xi,\zeta,\delta,u,\tau\right)=\left(p^{2}+q^{2}\right)(p+q)\,x^{2}\left[(1-\xi)\,(\zeta+1)+\xi\delta+1\right]^{2}(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

**Proof** Suppose that $f(z) \in \mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$, there are two analytic functions $\aleph, \varphi$ as defined in (1.5), such that

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'}{\left[\mathcal{J}_{p,q,\tau}^{u}f(z)\right]^{1-\zeta}}+\xi\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'\left(\frac{\mathcal{J}_{p,q,\tau}^{u}f(z)}{z}\right)^{\delta}\prec\mathcal{C}_{p,q}(\aleph(z)), \tag{2.7}$$

and

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'}{\left[\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right]^{1-\zeta}}+\xi\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'\left(\frac{\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})}{\widetilde{\omega}}\right)^{\delta}\prec\mathcal{C}_{p,q}(\varphi(\widetilde{\omega})). \tag{2.8}$$

From (2.7) and (2.8), we have

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'}{\left[\mathcal{J}_{p,q,\tau}^{u}f(z)\right]^{1-\zeta}}+\xi\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'\left(\frac{\mathcal{J}_{p,q,\tau}^{u}f(z)}{z}\right)^{\delta}=1+\mathcal{F}_1(x,\alpha,p,q)\,\aleph(z)+\ \mathcal{F}_2(x,\alpha,p,q)\,(\aleph(z))^2+\dots \tag{2.9}$$

and

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'}{\left[\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right]^{1-\zeta}}+\xi\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'\left(\frac{\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})}{\widetilde{\omega}}\right)^{\delta}$$

$$=1+\mathcal{F}_1(x,\alpha,p,q)\,\varphi(\widetilde{\omega})+\mathcal{F}_2(x,\alpha,p,q)\,(\varphi(\widetilde{\omega}))^2+\cdots. \tag{2.10}$$

Comparing Equations (1.5) and (2.7)–(2.10), we get the following relationship

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'}{\left[\mathcal{J}_{p,q,\tau}^{u}f(z)\right]^{1-\zeta}}+\xi\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'\left(\frac{\mathcal{J}_{p,q,\tau}^{u}f(z)}{z}\right)^{\delta}$$

$$=1+\mathcal{F}_1(x,\alpha,p,q)\,\sigma_1 z+\left[\mathcal{F}_1(x,s,p,q)\,\sigma_2+\ \mathcal{F}_2(x,s,p,q)\sigma_1^2\right]z^2+\cdots. \tag{2.11}$$

and

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'}{\left[\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right]^{1-\zeta}}+\xi\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'\left(\frac{\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})}{\widetilde{\omega}}\right)^{\delta}$$

$$=1+\mathcal{F}_1(x,\alpha,p,q)\,l_1\widetilde{\omega}+\left[\mathcal{F}_1(x,s,p,q)l_2+\mathcal{F}_2(x,s,p,q)l_1^2\right]\widetilde{\omega}^2+\cdots. \tag{2.12}$$

Since $|\aleph(z)|<1, |\varphi(\widetilde{\omega})|<1,$ then

$$|\sigma_j|<1 \ and \ |l_j|<1 \ for \ all \ j\in\mathbb{N}. \tag{2.13}$$

By calculating the right-hand side of two conditions in (2.11) and (2.12), and comparing the coefficients, we get

$$[(1-\xi)(\zeta+1)+\xi\delta+1]\frac{(1+u)_{pq}^{\tau}}{(2+u)_{pq}^{\tau}}a_2=\mathcal{F}_1(x,\alpha,p,q)\,\sigma_1 \tag{2.14}$$

$$[(1-\xi)(\zeta+1)+\xi\delta+2]\,\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}a_3+\frac{[(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2)](1+u)_{pq}^{2\tau}}{2(2+u)_{pq}^{2\tau}}a_2^2$$

$$=\mathcal{F}_1(x,s,p,q)\,\sigma_2+\mathcal{F}_2(x,s,p,q)\,\sigma_1^2 \tag{2.15}$$

$$-[(1-\xi)(\zeta+1)+\xi\delta+1]\frac{(1+u)_{pq}^{\tau}}{(2+u)_{pq}^{\tau}}a_2=\mathcal{F}_1(x,\alpha,p,q)\,l_1 \tag{2.16}$$

and

$$[(1-\xi)(\zeta+1)+\xi\delta+2]\,\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}(2a_2^2-a_3)+\frac{[(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2)](1+u)_{pq}^{2\tau}}{2(2+u)_{pq}^{2\tau}}a_2^2$$

$$=\mathcal{F}_1(x,s,p,q)l_2+\mathcal{F}_2(x,s,p,q)l_1^2. \tag{2.17}$$

From (2.14) and (2.16), we have

$$\sigma_1=-l_1 \tag{2.18}$$

and

$$\frac{2[(1-\xi)(\zeta+1)+\xi\delta+1]^2(1+u)_{pq}^{2\tau}}{(2+u)_{pq}^{2\tau}}a_2^2=\mathcal{F}_1^2(x,\alpha,p,q)\left(\sigma_1^2+l_1^2\right). \tag{2.19}$$

Now, by adding (2.15) to (2.17), we obtain

$$[2((1-\xi)(\zeta+1)+\xi\delta+2)\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}+\frac{2[\frac{1}{2}(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2)](1+u)_{pq}^{2\tau}}{(2+u)_{pq}^{2\tau}}]a_2^2$$

$$=\mathcal{F}_1(x,s,p,q)(\sigma_2+l_2)+\mathcal{F}_2(x,s,p,q)(\sigma_1^2+l_1^2). \tag{2.20}$$

By (2.19) we substitute the value of $\left(\sigma_1^2+l_1^2\right)$ into Equation (2.20), we get

$$\begin{aligned}&[2\mathcal{F}_1^2(x,\alpha,p,q)((1-\xi)(\zeta+1)+\xi\delta+2)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}\\&+2\mathcal{F}_1^2(x,\alpha,p,q)\left(\tfrac{1}{2}(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2)\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}\\&-2\mathcal{F}_2(x,s,p,q)[(1-\xi)(\zeta+1)+\xi\delta+1]^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}]a_2^2\end{aligned}$$

$$=\mathcal{F}_1^3(x,\alpha,p,q)(2k+u)^{2\tau}(3k+u)^{\tau}(\sigma_2+l_2). \tag{2.21}$$

From (1.3), (2.13), and (2.21), we have

$$|a_2|\leq\frac{|(p+q)x|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\mathcal{H}_{p,q,x}(\xi,\zeta,\delta,u,\tau)+\mathcal{N}_{p,q,x}(\xi,\zeta,\delta,u,\tau)-\mathcal{Q}_{p,q,x}(\xi,\zeta,\delta,u,\tau)\right]}}.$$

Now, to get the bound on $|a_3|$, using (2.18) and subtracting (2.17) from (2.15)

$$2\left[(1-\xi)(\zeta+1)+\xi\delta+2\right]\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}(a_3-a_2^2)=\mathcal{F}_1(x,s,p,q)(\sigma_2-l_2)+\mathcal{F}_2(x,s,p,q)(\sigma_1^2-l_1^2). \tag{2.22}$$

According to (2.18) and (2.22), we get

$$a_3=\frac{\mathcal{F}_1(x,s,p,q)(3+u)_{pq}^{\tau}(\sigma_2-l_2)}{2\left[(1-\xi)(\zeta+1)+\xi\delta+2\right](1+u)_{pq}^{\tau}}+\frac{\mathcal{F}_1^2(x,\alpha,p,q)(2+u)_{pq}^{2\tau}\left(\sigma_1^2+l_1^2\right)}{2[(1-\xi)(\zeta+1)+\xi\delta+1]^2(1+u)_{pq}^{2\tau}}.$$

Then

$$|a_3| \leq \frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{[(1-\xi)\,(\zeta+1)+\xi\delta+2]\,(1+u)_{pq}^{\tau}} + \frac{(p+q)^2 x^2 (2+u)_{pq}^{2\tau}}{[(1-\xi)\,(\zeta+1)+\xi\delta+1]^2 (1+u)_{pq}^{2\tau}}.$$

The proof is complete. □

For the special case $\delta = 0$, Theorem 2.5 becomes:

**Corollary 2.6** *Let $f(z) \in \Omega$ be in the family $\mathcal{SM}_{p,q}(\xi, \zeta, u, \tau)$, then*

$$|a_2| \leq \frac{|(p+q)\,x|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\tilde{N}_{p,q,x}\,(\xi,\zeta,u,\tau)+\tilde{K}_{p,q,x}\,(\xi,\zeta,u,\tau)-\tilde{L}_{p,q,x}\,(\xi,\zeta,u,\tau)\right]}},$$

$$|a_3| \leq \frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{[\zeta\,(1-\xi)+\xi+3]\,(1+u)_{pq}^{\tau}} + \frac{(p+q)^2 x^2 (2+u)_{pq}^{2\tau}}{[\zeta\,(1-\xi)+\xi+2]^2 (1+u)_{pq}^{2\tau}},$$

*where* $\tilde{N}_{p,q,x}\,(\xi,\zeta,u,\tau) = (p+q)^2 x^2\,(\zeta\,(1-\xi)+\xi+3)\,(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,
$\tilde{K}_{p,q,x}\,(\xi,\zeta,u,\tau) = (p+q)^2 x^2\left(\frac{1}{2}\,(\zeta-\xi\zeta)\,(\zeta+3)+4\xi\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$, *and*
$\tilde{L}_{p,q,x}\,(\xi,\zeta,u,\tau) = \left(p^2+q^2\right)(p+q)\,x^2(\zeta\,(1-\xi)+\xi+2)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

For the special case $\zeta = 1$, Theorem 2.5 becomes:

**Corollary 2.7** *Let $f(z) \in \Omega$ be in the family $\mathcal{SM}_{p,q}(\xi, \delta, u, \tau)$, then*

$$|a_2| \leq \frac{|(p+q)\,x|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\tilde{V}_{p,q,x}\,(\xi,\delta,u,\tau)+\tilde{U}_{p,q,x}\,(\xi,\delta,u,\tau)-\tilde{O}_{p,q,x}\,(\xi,\delta,u,\tau)\right]}},$$

$$|a_3| \leq \frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{(\xi(1-\delta)+3)\,(1+u)_{pq}^{\tau}} + \frac{(p+q)^2 x^2 (2+u)_{pq}^{2\tau}}{(\xi(\delta-1)+2)^2 (1+u)_{pq}^{2\tau}},$$

*where* $\tilde{V}_{p,q,x}\,(\xi,\delta,u,\tau) = (p+q)^2 x^2\,(\xi(1-\delta)+3)\,(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,
$\tilde{U}_{p,q,x}\,(\xi,\delta,u,\tau) = (p+q)^2 x^2\left(\frac{1}{2}(\xi(\delta\,(\delta+1)-2)\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$, *and*
$\tilde{O}_{p,q,x}\,(\xi,\delta,u,\tau) = \left(p^2+q^2\right)(p+q)\,x^2(\xi(\delta-1)+2)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

**Theorem 2.8** *Let $f(z) \in \Omega$ be in the family $\mathcal{SC}_{p,q}\,(\lambda, \zeta, \vartheta, u, \tau)$, then*

$$|a_2| \leq \frac{|(p+q)\,x\lambda|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\,[Q_{p,q,x}\,(\lambda,\zeta,\vartheta,k,u,\tau)-M_{p,q,x}\,(\lambda,\zeta,\vartheta,k,u,\tau)]}},$$

*and*

$$|a_3| \leq |a_2|^2 + \frac{(p+q)\,x(3+u)_{pq}^{\tau}}{\lambda\,(1+2\zeta+2\vartheta)\,(1+u)_{pq}^{\tau}},$$

*where* $Q_{p,q,x}(\lambda,\zeta,\vartheta,u,\tau)=\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(p+q)^{2}x^{2}$
*and* $M_{p,q,x}(\lambda,\zeta,\vartheta,u,\tau)=\left(p^{2}+q^{2}\right)(p+q)\,x^{2}(1+\zeta)^{2}(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

**Proof** Suppose that $f(z)\in\mathcal{SC}_{p,q}(\lambda,\zeta,\vartheta,u,\tau)$. There are two analytic functions $\aleph,\varphi$ as defined in (1.5), such that

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f(z)+\vartheta z^{2}(\mathcal{J}_{p,q,\tau}^{u}f(z))''}{z}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'-1\right)\prec\mathcal{C}_{p,q}(\aleph(z)),\tag{2.23}$$

and

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})+\vartheta\widetilde{\omega}^{2}(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega}))''}{\widetilde{\omega}}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'-1\right)\prec\mathcal{C}_{p,q}(\varphi(\widetilde{\omega})).\tag{2.24}$$

From (2.23) and (2.24), we have

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f(z)+\vartheta z^{2}(\mathcal{J}_{p,q,\tau}^{u}f(z))''}{z}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'-1\right)$$
$$=1+\mathcal{F}_{1}(x,\alpha,p,q)\,\aleph(z)+\mathcal{F}_{2}(x,\alpha,p,q)\,(\aleph(z))^{2}+\cdots.\tag{2.25}$$

and

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})+\vartheta\widetilde{\omega}^{2}(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega}))''}{\widetilde{\omega}}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'-1\right)$$
$$=1+\mathcal{F}_{1}(x,\alpha,p,q)\,\varphi(\widetilde{\omega})+\mathcal{F}_{2}(x,\alpha,p,q)\,(\varphi(\widetilde{\omega}))^{2}+\cdots.\tag{2.26}$$

Comparing Equations (1.5) and (2.23)–(2.26), we get the following relationship

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f(z)+\vartheta z^{2}(\mathcal{J}_{p,q,\tau}^{u}f(z))''}{z}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'-1\right)$$
$$=1+\mathcal{F}_{1}(x,\alpha,p,q)\,\sigma_{1}z+\left[\mathcal{F}_{1}(x,s,p,q)\sigma_{2}+\mathcal{F}_{2}(x,s,p,q)\sigma_{1}^{2}\right]z^{2}+\cdots\tag{2.27}$$

and

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})+\vartheta\widetilde{\omega}^{2}(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega}))''}{\widetilde{\omega}}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'-1\right)$$
$$=1+\mathcal{F}_{1}(x,\alpha,p,q)\,l_{1}\widetilde{\omega}+\left[\mathcal{F}_{1}(x,s,p,q)l_{2}+\mathcal{F}_{2}(x,s,p,q)l_{1}^{2}\right]\widetilde{\omega}^{2}+\cdots.\tag{2.28}$$

Since $|\aleph(z)|<1,|\varphi(\widetilde{\omega})|<1,$ then

$$|\sigma_{j}|<1\ and\ |l_{j}|<1\ for\ all\ j\in\mathbb{N}.\tag{2.29}$$

By calculating the right-hand side of two conditions in (2.27) and (2.27), and comparing the coefficients, we get

$$\frac{1}{\lambda}(1+\zeta)\frac{(1+u)_{pq}^{\tau}}{(2+u)_{pq}^{\tau}}a_{2}=\mathcal{F}_{1}(x,\alpha,p,q)\,\sigma_{1},\tag{2.30}$$

$$\frac{1}{\lambda}\left(1+2\zeta+2\vartheta\right)\frac{\left(1+u\right)_{pq}^{\tau}}{\left(3+u\right)_{pq}^{\tau}}a_3=\mathcal{F}_1\left(x,s,p,q\right)\sigma_2+\mathcal{F}_2\left(x,s,p,q\right)\sigma_1^2, \tag{2.31}$$

$$-\frac{1}{\lambda}\left(1+\zeta\right)\frac{\left(1+u\right)_{pq}^{\tau}}{\left(2+u\right)_{pq}^{\tau}}a_2=\mathcal{F}_1\left(x,\alpha,p,q\right)l_1, \tag{2.32}$$

and

$$\frac{1}{\lambda}\left(1+2\zeta+2\vartheta\right)\frac{\left(1+u\right)_{pq}^{\tau}}{\left(3+u\right)_{pq}^{\tau}}(2a_2^2-a_3)=\mathcal{F}_1(x,s,p,q)l_2+\mathcal{F}_2(x,s,p,q)l_1^2. \tag{2.33}$$

From (2.30) and (2.32), we have

$$\sigma_1=-l_1 \tag{2.34}$$

and

$$\frac{2}{\lambda^2}(1+\zeta)^2\frac{\left(1+u\right)_{pq}^{2\tau}}{\left(2+u\right)_{pq}^{2\tau}}a_2^2=\mathcal{F}_1^2\left(x,\alpha,p,q\right)\left(\sigma_1^2+l_1^2\right). \tag{2.35}$$

Now, by adding (2.31) to (2.33), we obtain

$$\frac{2}{\lambda}\left(1+2\zeta+2\vartheta\right)\frac{\left(1+u\right)_{pq}^{\tau}}{\left(3+u\right)_{pq}^{\tau}}a_2^2=\mathcal{F}_1\left(x,s,p,q\right)\left(\sigma_2+l_2\right)+\mathcal{F}_2(x,s,p,q)(\sigma_1^2+l_1^2). \tag{2.36}$$

By (2.35) we substitute the value of $\left(\sigma_1^2+l_1^2\right)$ into Equation (2.36), we get

$$2\left[\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}(2+u)_{pq}^{2\tau}\mathcal{F}_1^2\left(x,\alpha,p,q\right)-\mathcal{F}_2\left(x,s,p,q\right)\left(1+\zeta\right)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}\right]a_2^2$$
$$=\mathcal{F}_1^3\left(x,\alpha,p,q\right)\lambda^2(2k+u)^{2\tau}(3k+u)^{\tau}(\sigma_2+l_2). \tag{2.37}$$

From (1.3), (2.29) and (2.37), we have

$$|a_2|\leq\frac{\left|\left(p+q\right)x\lambda\right|\left(2+u\right)_{pq}^{\tau}\sqrt{\left(p+q\right)x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[Q_{p,q,x}\left(\lambda,\zeta,\vartheta,u,\tau\right)-M_{p,q,x}\left(\lambda,\zeta,\vartheta,u,\tau\right)\right]}}.$$

Now, to get the bound on $|a_3|$, using (2.34) and subtracting (2.33) from (2.31)

$$\frac{2}{\lambda}\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}a_3=\frac{2}{\lambda}\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}a_2^2+(3+u)_{pq}^{\tau}\mathcal{F}_1\left(x,s,p,q\right)\left(\sigma_2-l_2\right). \tag{2.38}$$

According to (2.34) and (2.38), we get

$$\frac{2}{\lambda}\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}|a_3|\leq\frac{2}{\lambda}\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}|a_2|^2+2(3+u)_{pq}^{\tau}\mathcal{F}_1\left(x,s,p,q\right).$$

Then

$$|a_3|\leq|a_2|^2+\frac{\left|\left(p+q\right)x\right|\left(3+u\right)_{pq}^{\tau}}{\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}}.$$

Here, the proof is complete. □

For the special case $\zeta=1+2\vartheta$, Theorem 2.8 becomes:

**Corollary 2.9** *Let* $f(z) \in \Omega$ *be in the family* $\mathcal{SC}_{p,q}(\lambda, \vartheta, u, \tau)$, *then*

$$|a_2| \leq \frac{|(p+q)\,x\lambda|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\tilde{Q}_{p,q,x}(\lambda,\vartheta,u,\tau)-\tilde{M}_{p,q,x}(\lambda,\vartheta,u,\tau)\right]}},$$

$$|a_3| \leq |a_2|^2 + \frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{\lambda(1+2\zeta)(1+u)_{pq}^{\tau}},$$

*where* $\tilde{Q}_{p,q,x}(\lambda,\vartheta,u,\tau) = 3\lambda(1+2\vartheta)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(p+q)^2x^2$
*and* $\tilde{M}_{p,q,x}(\lambda,\vartheta,u,\tau) = 2\left(p^2+q^2\right)(p+q)\,x^2(1+\zeta)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

For the special case $\vartheta = 0$, Theorem 2.8 becomes:

**Corollary 2.10** *Let* $f(z) \in \Omega$ *be in the family* $\mathcal{SC}_{p,q}(\lambda, \zeta, u, \tau)$, *then*

$$|a_2| \leq \frac{|(p+q)\,x\lambda|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\tilde{D}_{p,q,x}(\lambda,\zeta,u,\tau)-\tilde{F}_{p,q,x}(\lambda,\zeta,u,\tau)\right]}},$$

$$|a_3| \leq |a_2|^2 + \frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{\xi(1+2\zeta)(1+u)_{pq}^{\tau}},$$

*where* $\tilde{D}_{p,q,x}(\lambda,\zeta,u,\tau) = \lambda(1+2\zeta)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(p+q)^2x^2$
*and* $\tilde{F}_{p,q,x}(\lambda,\zeta,u,\tau) = \left(p^2+q^2\right)(p+q)\,x^2(1+\zeta)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

## 3. Fekete-Szegö inequality

**Theorem 3.1** *Let* $f(z)$ *be in the family* $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$ *and* $\gamma \in \mathbb{R}$, *then*

$$\left|a_3-\gamma a_2^2\right| \leq \begin{cases} \dfrac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{[(1-\xi)(\zeta+1)+\xi\delta+2]\,(1+u)_{pq}^{\tau}}, & \text{if} \quad |\gamma-1| \leq \chi_1, \\ \dfrac{\left|(p+q)^3x^3\right|(2+u)_{pq}^{2\tau}(3+u)_{pq}^{\tau}\,|1-\gamma|}{|[(p+q)\left(\mathcal{A}(\xi,\zeta,\delta,u,\tau)+\mathcal{B}(\xi,\zeta,\delta,u,\tau)\right)-(p^2+q^2)\,\mathcal{G}(\xi,\zeta,\delta,u,\tau)]\,(p+q)\,x^2-pq\alpha\mathcal{G}(\xi,\zeta,\delta,u,\tau)|}, \\ \text{if} \quad |\gamma-1| \geq \chi_1 \end{cases}$$

*where* $\mathcal{A}(\xi,\zeta,\delta,u,\tau) = ((1-\xi)(\zeta+1)+\xi\delta+2)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,
$\mathcal{B}(\xi,\zeta,\delta,u,\tau) = \left(\frac{1}{2}(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$,
$\mathcal{G}(\xi,\zeta,\delta,u,\tau) = [(1-\xi)(\zeta+1)+\xi\delta+1]^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$,
*and* $\chi_1 = \frac{[(p+q)(\mathcal{A}(\xi,\zeta,\delta,u,\tau)+\mathcal{B}(\xi,\zeta,\delta,u,\tau))-\left(p^2+q^2\right)\mathcal{G}(\xi,\zeta,\delta,u,\tau)](p+q)x^2-pq\alpha\mathcal{G}(\xi,\zeta,\delta,u,\tau)}{(p+q)^2x^2[(1-\xi)(\zeta+1)+\xi\delta+2](2+u)_{pq}^{2\tau}(1+u)_{pq}^{\tau}}$.

**Proof** By Equations (2.21) and (2.22), it follows that

$$a_3-\gamma a_2^2=\frac{\mathcal{F}_1\left(x,s,p,q\right)\left(3+u\right)_{pq}^{\tau}\left(\sigma_2-l_2\right)}{2\left[\left(1-\xi\right)\left(\zeta+1\right)+\xi\delta+2\right]\left(1+u\right)_{pq}^{\tau}}+\left(1-\gamma\right)a_2^2$$

$$=\frac{\mathcal{F}_1\left(x,s,p,q\right)\left(3+u\right)_{pq}^{\tau}\left(\sigma_2-l_2\right)}{2\left[\left(1-\xi\right)\left(\zeta+1\right)+\xi\delta+2\right]\left(1+u\right)_{pq}^{\tau}}+$$

$$\frac{\mathcal{F}_1^3\left(x,\alpha,p,q\right)\left(3+u\right)_{pq}^{\tau}\left(2+u\right)_{pq}^{2\tau}\left(\sigma_2+l_2\right)\left(1-\gamma\right)}{\mathcal{F}_1^2\left(x,\alpha,p,q\right)\left(\mathcal{A}\left(\xi,\zeta,\delta,u,\tau\right)+\mathcal{B}\left(\xi,\zeta,\delta,u,\tau\right)\right)-\mathcal{F}_2\left(x,s,p,q\right)\mathcal{G}\left(\xi,\zeta,\delta,u,\tau\right)}$$

$$=\mathcal{F}_1\left(x,\alpha,p,q\right)[\left(\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)+\frac{(3+u)_{pq}^{\tau}}{2[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}\right)\sigma_2+\left(\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)-\frac{(3+u)_{pq}^{\tau}}{2[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}\right)l_2],$$

where $\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)=\frac{\mathcal{F}_1^2(x,\alpha,p,q)(3+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(1-\gamma)}{\mathcal{F}_1^2(x,\alpha,p,q)(\mathcal{A}(\xi,\zeta,\delta,u,\tau)+\mathcal{B}(\xi,\zeta,\delta,u,\tau))-\mathcal{F}_2(x,s,p,q)\mathcal{G}(\xi,\zeta,\delta,u,\tau)}$.

Thus, according to $(p,q)$-Chebyshev polynomials in (1.3), we conclude that

$$\left|a_3-\gamma a_2^2\right|\leq\begin{cases}\frac{(p+q)x(3+u)_{pq}^{\tau}}{[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}, & 0\leq|\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)|\leq\frac{(3k+u)^{\tau}}{2[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}\\ 2\left|(p+q)x\right|\left|\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta\right)\right|, & for\ |\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)|\geq\frac{(3+u)_{pq}^{\tau}}{2[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}\end{cases}$$

After making several simplifications, the proof of Theorem 3.1 is complete. □

**Theorem 3.2** *Let $f(z)$ be in the family $\mathcal{SC}_{p,q}\left(\lambda,\zeta,\vartheta,u,\tau\right)$ and $\gamma\in\mathbb{R}$, then*

$$\left|a_3-\gamma a_2^2\right|\leq\begin{cases}\dfrac{\left|(p+q)x\right|\left(3+u\right)_{pq}^{\tau}}{\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}} \quad if \quad |\gamma-1|\leq\chi_2,\\ \dfrac{\left|(p+q)^3x^3\right|\lambda^2(3+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}\left|1-\gamma\right|}{\left|[(p+q)\widetilde{\mathcal{A}}\left(\lambda,\zeta,\delta,u,\tau\right)-(p^2+q^2)\widetilde{\mathcal{B}}\left(\lambda,\zeta,\delta,u,\tau\right)](p+q)x^2-pq\alpha\widetilde{\mathcal{B}}\left(\lambda,\zeta,\delta,u,\tau\right)\right|}, \\ if \quad |\gamma-1|\geq\chi_2\end{cases}$$

*where* $\widetilde{\mathcal{A}}\left(\lambda,\zeta,\delta,u,\tau\right)=\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,

$\widetilde{\mathcal{B}}\left(\lambda,\zeta,\delta,u,\tau\right)=\left(1+\zeta\right)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$,

*and* $\chi_2=\frac{[(p+q)\widetilde{\mathcal{A}}(\lambda,\zeta,\delta,u,\tau)-\left(p^2+q^2\right)\widetilde{\mathcal{B}}(\lambda,\zeta,\delta,u,\tau)](p+q)x^2-pq\alpha\widetilde{\mathcal{B}}(\lambda,\zeta,\delta,u,\tau)}{(p+q)^2x^2\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}}$.

**Proof** By Equations (2.37) and (2.38), it follows that

$$a_3-\gamma a_2^2=\frac{(3+u)_{pq}^{\tau}\mathcal{F}_1\left(x,s,p,q\right)\left(\sigma_2-l_2\right)}{2\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}}+(1-\gamma)a_2^2$$

$$=\frac{\mathcal{F}_1\left(x,\alpha,p,q\right)}{2}\left[\left(\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)+\frac{\left(3+u\right)_{pq}^{\tau}}{\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}}\right)\sigma_2\right.$$

$$+\left(\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)-\frac{(3+u)_{pq}^{\tau}}{\lambda\left(1+2\zeta+2\vartheta\right)(1+u)_{pq}^{\tau}}\right)l_2]$$

where

$$\Upsilon\left(\gamma,\zeta,\xi,\vartheta\right)=\frac{\mathcal{F}_1^2\left(x,\alpha,p,q\right)\xi^2(3+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(1-\gamma)}{\mathcal{F}_1^2\left(x,\alpha,p,q\right)\widetilde{\mathcal{A}}\left(\xi,\zeta,\delta,u,\tau\right)-\mathcal{F}_2(x,s,p,q)\widetilde{\mathcal{B}}\left(\xi,\zeta,\delta,u,\tau\right)}.$$

Thus, according to $(p,q)$-Chebyshev polynomials in (1.3), we conclude that

$$\left|a_3-\gamma a_2^2\right|\leq\begin{cases}\frac{|(p+q)x|(3+u)_{pq}^{\tau}}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}}, & 0\leq\left|\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)\right|\leq\frac{(3+u)_{pq}^{\tau}}{\xi(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}},\\ 2\left(p+q\right)x\left|\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)\right|, & for\ \ \left|\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)\right|\geq\frac{(3+u)_{pq}^{\tau}}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}}\end{cases}$$

After making several simplifications, the proof of Theorem 3.2 is complete. □

## 4. Conclusion

In this article, we introduced a new integral operator defined by $(p,q)$-Hurwitz zeta function, which is a generalization of the $q$-Srivastava-Attiya operator. We also provided two families $\mathcal{SM}_{p,q}\left(\xi,\zeta,\delta,u,\tau\right)$ and $\mathcal{SC}_{p,q}\left(\lambda,\zeta,\vartheta,u,\tau\right)$ of biunivalent and holomorphic functions in the unit disk, which is defined by $(p,q)$-Chebyshev polynomials, and we obtained Fekete-Szegö inequalities for these families. We also think that this construction has many applications.

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