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Formulas for special numbers and polynomials derived from functional equations of their generating functions

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Abstract: The main purpose of this paper is to investigate various formulas, identities and relations involving Apostol type numbers and parametric type polynomials. By using generating functions and their functional equations, we give many relations among the certain family of combinatorial numbers, the Vieta polynomials, the two-parametric types of the Apostol-Euler polynomials, the Apostol-Bernoulli polynomials, the Apostol-Genocchi polynomials, the Fibonacci and Lucas numbers, the Chebyshev polynomials, and other special numbers and polynomials. Moreover, we give some formulas related to trigonometric functions, special numbers and special polynomials. Finally, some remarks and observations on the results of this paper are given.

Key words: Apostol type numbers and polynomials, Vieta polynomials, Fibonacci and Lucas numbers, parametric type polynomials, special numbers and polynomials, generating functions

1. Introduction

Special functions involving generating functions, trigonometric functions, special numbers and polynomials have been investigated by many researchers. There are many different applications of these functions in theory special functions, applied mathematics, mathematical physics, and other areas. Furthermore, special polynomials and special numbers with their generating functions can also be used to solve many different real-world problems and mathematical problems.

The motivation of this paper is to give various kind novel computational formulas, relations and identities with the aid of generating functions, their functional equations, and trigonometric functions. These formulas, relations and identities include many special numbers and special polynomials, as well as some special combinatorial sums. Some of these are the two-parametric types of the Apostol-Euler polynomials, the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials, the combinatorial numbers, the Apostol type numbers, the Fibonacci numbers, the Lucas numbers, the Vieta-Fibonacci polynomials, the Vieta-Lucas polynomials and the Chebyshev polynomials.

The notations and definitions involving special numbers and polynomials with their generating functions are given as follows:

Let $\mathbb{N}=\{1,2,3,\cdots\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex

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numbers and $i^2 = -1$. Furthermore,

$$0^u = \begin{cases} 1, & (u = 0) \\ 0, & (u \in \mathbb{N}) \end{cases}$$

and

$$\binom{r}{u} = \frac{(r)_u}{u!} = \frac{r(r-1)(r-2)\cdots(r-u+1)}{u!} \qquad (u \in \mathbb{N};\ r \in \mathbb{C}),$$

with $\binom{r}{0} = 1$ and $(r)_0 = 1$ (*cf.* [1–23]).

The Apostol-Bernoulli polynomials of order α are defined by

$$R_B(w, x; \alpha, \gamma) = \left(\frac{w}{\gamma e^w - 1}\right)^\alpha e^{xw} = \sum_{m=0}^{\infty} \mathcal{B}_m^{(\alpha)}(x; \gamma) \frac{w^m}{m!}, \tag{1.1}$$

where $|w| < 2\pi$ when $\gamma = 1$; $|w| < |\log(\gamma)|$ when $\gamma \neq 1$; $1^\alpha := 1$ (*cf.* [16, 18, 20–23]).

Substituting $\alpha = 0$ into (1.1), we have

$$\mathcal{B}_m^{(0)}(x; \gamma) = x^m.$$

When $x = 0$ in (1.1), we have the Apostol-Bernoulli numbers $\mathcal{B}_m^{(\alpha)}(\gamma)$ of order α:

$$\mathcal{B}_m^{(\alpha)}(\gamma) = \mathcal{B}_m^{(\alpha)}(0; \gamma).$$

Setting $\gamma = 1$ and $x = 0$ in (1.1), we have the Bernoulli polynomials and numbers of order α:

$$B_m^{(\alpha)}(x) = \mathcal{B}_m^{(\alpha)}(x; 1) \qquad \text{and} \qquad B_m^{(\alpha)} = \mathcal{B}_m^{(\alpha)}(0; 1)$$

(*cf.* [3, 11, 13, 18, 20–23]).

The Apostol-Euler polynomials of order α are defined by

$$\left(\frac{2}{\gamma e^w + 1}\right)^\alpha e^{xw} = \sum_{m=0}^{\infty} \mathcal{E}_m^{(\alpha)}(x; \gamma) \frac{w^m}{m!}, \tag{1.2}$$

where $|w| < \pi$ when $\gamma = 1$; $|t| < |\log(-\gamma)|$ when $\gamma \neq 1$; $1^\alpha := 1$ (*cf.* [16, 18, 19, 22, 23]).

Substituting $\alpha = 0$ into (1.2), we have

$$\mathcal{E}_m^{(0)}(x; \gamma) = x^m.$$

When $x = 0$ in (1.2), we have the Apostol-Euler numbers of order α:

$$\mathcal{E}_m^{(\alpha)}(\gamma) = \mathcal{E}_m^{(\alpha)}(0; \gamma).$$

Substituting $\gamma = 1$ and $x = 0$ into (1.2), we have the Euler polynomials and numbers of order α:

$$E_m^{(\alpha)}(x) = \mathcal{E}_m^{(\alpha)}(x; 1) \qquad \text{and} \qquad E_m^{(\alpha)} = \mathcal{E}_m^{(\alpha)}(0; 1)$$

(*cf.* [3, 11, 13, 18–23]).

The Apostol-Genocchi polynomials of order α are defined by

$$\left(\frac{2w}{\gamma e^{w}+1}\right)^{\alpha} e^{xw}=\sum_{m=0}^{\infty} \mathcal{G}_{m}^{(\alpha)}(x;\gamma)\frac{w^{m}}{m!}, \tag{1.3}$$

where $|w|<\pi$ when $\gamma=1$; $|w|<|\log(-\gamma)|$ when $\gamma\neq 1$; $1^{\alpha}:=1$ (*cf.* [18, 22, 23]).

Substituting $\alpha=0$ into (1.3), we have

$$\mathcal{G}_{m}^{(0)}(x;\gamma)=x^{m}.$$

Setting $x=0$ in (1.3), we get the Apostol-Genocchi numbers of order α:

$$\mathcal{G}_{m}^{(\alpha)}(\gamma)=\mathcal{G}_{m}^{(\alpha)}(0;\gamma).$$

Substituting $\gamma=1$ and $x=0$ into (1.3), we have the Genocchi polynomials and numbers of order α:

$$G_{m}^{(\alpha)}(x)=\mathcal{G}_{m}^{(\alpha)}(x;1) \qquad \text{and} \qquad G_{m}^{(\alpha)}=\mathcal{G}_{m}^{(\alpha)}(0;1)$$

(*cf.* [18, 22, 23]).

The Chebyshev polynomials of the first kind $T_m(x)$ are defined by

$$\frac{1-xw}{1-2xw+w^{2}}=\sum_{m=0}^{\infty} T_{m}(x)w^{m} \tag{1.4}$$

(*cf.* [2–4]).

The Chebyshev polynomials of the second kind $U_m(x)$ are defined by

$$\frac{1}{1-2xw+w^{2}}=\sum_{m=0}^{\infty} U_{m}(x)w^{m} \tag{1.5}$$

(*cf.* [2–4]).

The Vieta-Lucas polynomials $v_m(x)$ are defined by

$$\frac{2-xw}{1-xw+w^{2}}=\sum_{m=0}^{\infty} v_{m}(x)\,w^{m} \tag{1.6}$$

with

$$v_{m}(x)=\sum_{j=0}^{\left[\frac{m}{2}\right]}(-1)^{j}\,\frac{m}{m-j}\binom{m-j}{j}x^{m-2j}, \tag{1.7}$$

where $v_0(x)=2$ and $[k]$ is the largest integer $\leq k$ (*cf.* [5, 12]).

The Vieta-Fibonacci polynomials $V_m(x)$ are defined by

$$\frac{1}{1-xw+w^{2}}=\sum_{m=1}^{\infty} V_{m}(x)\,w^{m-1} \tag{1.8}$$

with

$$V_m(x) = \sum_{j=0}^{\left[\frac{m-1}{2}\right]} (-1)^j \binom{m-1-j}{j} x^{m-2j-1}, \tag{1.9}$$

where $V_0(x) = 0$ (*cf.* [5, 12]).

Relations among the Vieta-Fibonacci polynomials, the Vieta-Lucas polynomials and the Chebyshev polynomials are given by

$$v_m(x) = 2T_m\left(\frac{x}{2}\right)$$

and

$$V_m(x) = U_{m-1}\left(\frac{x}{2}\right),$$

where $m \in \mathbb{N}$ (*cf.* [12, Eqs. (47.40) and (47.41)]; see also [5]).

The Fibonacci-type polynomials in two variables $\mathcal{G}_n(x, y; k, m, l)$ are defined by

$$\frac{1}{1 - x^k w - y^m w^{m+l}} = \sum_{n=0}^{\infty} \mathcal{G}_n(x, y; k, m, l) w^n, \tag{1.10}$$

where $k, m, l \in \mathbb{N}_0$ (*cf.* [17]). An explicit formula for the polynomials $\mathcal{G}_n(x, y; k, m, l)$ is given by

$$\mathcal{G}_n(x, y; k, m, l) = \sum_{s=0}^{\left[\frac{n}{m+l}\right]} \binom{n - s(m+l-1)}{s} y^{ms} x^{nk-msk-lsk},$$

(*cf.* [17, 18]).

Substituting $y = 1$ and $k = m = l = 1$ into (1.10), we have

$$F_n(x) = \mathcal{G}_{n-1}(x, 1; 1, 1, 1),$$

where $F_n(x)$ denotes the Fibonacci polynomials (*cf.* [1, 3, 12, 17, 18]).

By using the following functional equation, a relation between the $v_n(x)$ and the polynomials $\mathcal{G}_n(x, y; k, m, l)$ is easily given:

$$(2 - xw) \sum_{n=0}^{\infty} \mathcal{G}_n(x, -1; 1, 1, 1) w^n = \sum_{n=0}^{\infty} v_n(x) w^n.$$

Therefore

$$\sum_{n=0}^{\infty} 2\mathcal{G}_n(x, -1; 1, 1, 1) w^n - \sum_{n=1}^{\infty} x\mathcal{G}_{n-1}(x, -1; 1, 1, 1) w^n = \sum_{n=0}^{\infty} v_n(x) w^n.$$

Comparing the coefficients of w^n on both sides of the above equation, for $n \in \mathbb{N}$, we one has

$$v_n(x) = 2\mathcal{G}_n(x, -1; 1, 1, 1) - x\mathcal{G}_{n-1}(x, -1; 1, 1, 1),$$

(*cf.* [17]).

By performing similar operations of the previous formula, a relation between the polynomials $V_n(x)$ and the polynomials $\mathcal{G}_n(x, y; k, m, l)$ is also given as follows:

$$\sum_{n=0}^{\infty} \mathcal{G}_n(x, -1; 1, 1, 1)\, w^n = \sum_{n=0}^{\infty} V_{n+1}(x)\, w^n.$$

Comparing the coefficients of w^n on both sides of the above equation, we get the following relation:

$$V_{n+1}(x) = \mathcal{G}_n(x, -1; 1, 1, 1),$$

where $n \in \mathbb{N}_0$ (*cf.* [17]).

Let $k \in \mathbb{N}_0$ and $\gamma \in \mathbb{C}$. The numbers $y_1(m, k; \gamma)$ are defined by

$$R_y(w, k; \gamma) = \frac{(\gamma e^w + 1)^k}{k!} = \sum_{m=0}^{\infty} y_1(m, k; \gamma) \frac{w^m}{m!} \tag{1.11}$$

(*cf.* [19]).

By using (1.2) and (1.11), we have

$$\mathcal{E}_m^{(-k)}(\gamma) = k! 2^{-k} y_1(m, k; \gamma) \tag{1.12}$$

(*cf.* [19, Eq. (28)]).

Let $k \in \mathbb{N}_0$ and $\gamma \in \mathbb{C}$. The numbers $y_2(m, k; \gamma)$ are defined by

$$R_{y_2}(w, k; \gamma) = \frac{\left(\gamma e^w + \gamma^{-1} e^{-w} + 2\right)^k}{(2k)!} = \sum_{m=0}^{\infty} y_2(m, k; \gamma) \frac{w^m}{m!} \tag{1.13}$$

(*cf.* [19]).

The polynomials $C_m(x, y)$ and $S_m(x, y)$ are defined, respectively, as follows:

$$R_{\mathrm{C}}(w, x, y) = e^{xw} \cos(yw) = \sum_{m=0}^{\infty} C_m(x, y) \frac{w^m}{m!} \tag{1.14}$$

and

$$R_{\mathrm{S}}(w, x, y) = e^{xw} \sin(yw) = \sum_{m=0}^{\infty} S_m(x, y) \frac{w^m}{m!} \tag{1.15}$$

(*cf.* [6, 8–10, 14, 15, 23]).

Using (1.14) and (1.15), we have

$$C_m(x, y) = \sum_{s=0}^{\left[\frac{m}{2}\right]} (-1)^s \binom{m}{2s} x^{m-2s} y^{2s} \tag{1.16}$$

and

$$S_m(x, y) = \sum_{s=0}^{\left[\frac{m-1}{2}\right]} (-1)^s \binom{m}{2s+1} x^{m-2s-1} y^{2s+1} \tag{1.17}$$

(*cf.* [6, 8–10, 14, 15, 23]).

Substituting $x=y$ into (1.16) and (1.17), we have

$$C_m(x,x)=x^m\sqrt{2^m}\cos\left(\frac{m\pi}{4}\right) \tag{1.18}$$

and

$$S_m(x,x)=x^m\sqrt{2^m}\sin\left(\frac{m\pi}{4}\right) \tag{1.19}$$

(*cf.* [9, 23]).

The two-parametric types of the Apostol-Bernoulli polynomials of order α are defined by

$$R_{\mathrm{BC}}\left(w,x,y;\alpha,\gamma\right)=\left(\frac{w}{\gamma e^w-1}\right)^{\alpha}e^{xw}\cos\left(yw\right)=\sum_{m=0}^{\infty}\mathcal{B}_m^{(C,\alpha)}\left(x,y;\gamma\right)\frac{w^m}{m!} \tag{1.20}$$

and

$$R_{\mathrm{BS}}\left(w,x,y;\alpha,\gamma\right)=\left(\frac{w}{\gamma e^w-1}\right)^{\alpha}e^{xw}\sin\left(yw\right)=\sum_{m=0}^{\infty}\mathcal{B}_m^{(S,\alpha)}\left(x,y;\gamma\right)\frac{w^m}{m!} \tag{1.21}$$

(*cf.* [23]).

By using (1.20) and (1.21), we have

$$\mathcal{B}_m^{(C,\alpha)}\left(x,y;\gamma\right)=\sum_{j=0}^{m}\binom{m}{j}\mathcal{B}_{m-j}^{(\alpha)}\left(\gamma\right)C_j\left(x,y\right) \tag{1.22}$$

and

$$\mathcal{B}_m^{(S,\alpha)}\left(x,y;\gamma\right)=\sum_{j=0}^{m}\binom{m}{j}\mathcal{B}_{m-j}^{(\alpha)}\left(\gamma\right)S_j\left(x,y\right) \tag{1.23}$$

(*cf.* [23]).

The two-parametric types of the Apostol-Euler polynomials of order α are defined by

$$R_{\mathrm{EC}}\left(w,x,y;\alpha,\gamma\right)=\left(\frac{2}{\gamma e^w+1}\right)^{\alpha}e^{xw}\cos\left(yw\right)=\sum_{m=0}^{\infty}\mathcal{E}_m^{(C,\alpha)}\left(x,y;\gamma\right)\frac{w^m}{m!} \tag{1.24}$$

and

$$R_{\mathrm{ES}}\left(w,x,y;\alpha,\gamma\right)=\left(\frac{2}{\gamma e^w+1}\right)^{\alpha}e^{xw}\sin\left(yw\right)=\sum_{m=0}^{\infty}\mathcal{E}_m^{(S,\alpha)}\left(x,y;\gamma\right)\frac{w^m}{m!} \tag{1.25}$$

(*cf.* [23]).

Using (1.24) and (1.25), we have

$$\mathcal{E}_m^{(C,\alpha)}\left(x,y;\gamma\right)=\sum_{j=0}^{m}\binom{m}{j}\mathcal{E}_{m-j}^{(\alpha)}\left(\gamma\right)C_j\left(x,y\right) \tag{1.26}$$

and

$$\mathcal{E}_{m}^{(S,\alpha)}\left(x,y;\gamma\right)=\sum_{j=0}^{m}\binom{m}{j}\mathcal{E}_{m-j}^{(\alpha)}\left(\gamma\right)S_{j}\left(x,y\right) \tag{1.27}$$

(*cf.* [23]).

The two-parametric types of the Apostol-Genocchi polynomials of order α are defined by

$$R_{\mathrm{GC}}\left(w,x,y;\alpha,\gamma\right)=\left(\frac{2w}{\gamma e^{w}+1}\right)^{\alpha}e^{xw}\cos\left(yw\right)=\sum_{m=0}^{\infty}\mathcal{G}_{m}^{(C,\alpha)}\left(x,y;\gamma\right)\frac{w^{m}}{m!} \tag{1.28}$$

and

$$R_{\mathrm{GS}}\left(w,x,y;\alpha,\gamma\right)=\left(\frac{2w}{\gamma e^{w}+1}\right)^{\alpha}e^{xw}\sin\left(yw\right)=\sum_{m=0}^{\infty}\mathcal{G}_{m}^{(S,\alpha)}\left(x,y;\gamma\right)\frac{w^{m}}{m!} \tag{1.29}$$

(*cf.* [23]).

By (1.28) and (1.29), we have

$$\mathcal{G}_{m}^{(C,\alpha)}\left(x,y;\gamma\right)=\sum_{j=0}^{m}\binom{m}{j}\mathcal{G}_{m-j}^{(\alpha)}\left(\gamma\right)C_{j}\left(x,y\right) \tag{1.30}$$

and

$$\mathcal{G}_{m}^{(S,\alpha)}\left(x,y;\gamma\right)=\sum_{j=0}^{m}\binom{m}{j}\mathcal{G}_{m-j}^{(\alpha)}\left(\gamma\right)S_{j}\left(x,y\right) \tag{1.31}$$

(*cf.* [23]).

The rest of this paper is summarized as follows:

In Section 2, many formulas and identities including the combinatorial numbers, the Apostol type numbers and polynomials, the two-parametric types of the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and the Apostol-Genocchi polynomials, and the trigonometric functions are obtained.

In Section 3, some relations related to the Vieta-Fibonacci polynomials, the Vieta-Lucas polynomials, combinatorial numbers, the two-parametric types of the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and the Apostol-Genocchi polynomials, and special numbers are given.

In Section 4, some formulas involving the Fibonacci numbers, the Lucas numbers, the two-parametric types of the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials, the Apostol-Genocchi polynomials, and combinatorial numbers are presented.

2. Relations containing parametric type polynomials and special polynomials and numbers

In this section, using functional equation methods by the aid of the generating functions for the special polynomials, we obtain some relations and formulas related to the numbers $y_1\left(m,v;\gamma\right)$, the numbers $y_2\left(m,v;\gamma\right)$, the polynomials $C_m\left(x,y\right)$, the polynomials $S_m\left(x,y\right)$, the higher order of the Apostol-Euler numbers, the Apostol-Bernoulli numbers, the two-parametric types of the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and the Apostol-Genocchi polynomials, the sine and cosine functions.

Theorem 2.1 *Let $m, d \in \mathbb{N}_0$. Then we have*

$$C_m(x,y) = \frac{\gamma^d (2d)!}{2^{2d}} \sum_{j=0}^{m} \binom{m}{j} \sum_{s=0}^{j} \binom{j}{s} d^{m-j} y_2(s,d;\gamma) \mathcal{E}_{j-s}^{(C,2d)}(x,y;\gamma). \tag{2.1}$$

Proof Using (1.13), (1.14) and (1.24), we obtain the following functional equation

$$\frac{2^{2d}}{(2d)!} R_{\mathrm{C}}(w,x,y) = \gamma^d e^{wd} R_{y_2}(w,d;\gamma) R_{\mathrm{EC}}(w,x,y;2d,\gamma).$$

By using the above equation, we get

$$\sum_{m=0}^{\infty} C_m(x,y) \frac{w^m}{m!} = \gamma^d 2^{-2d} (2d)! \sum_{m=0}^{\infty} d^m \frac{w^m}{m!} \sum_{m=0}^{\infty} y_2(m,d;\gamma) \frac{w^m}{m!} \sum_{m=0}^{\infty} \mathcal{E}_m^{(C,2d)}(x,y;\gamma) \frac{w^m}{m!}.$$

Thus

$$\sum_{m=0}^{\infty} C_m(x,y) \frac{w^m}{m!} = \gamma^d 2^{-2d} (2d)! \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} d^{m-j} \sum_{s=0}^{j} \binom{j}{s} y_2(s,d;\gamma) \mathcal{E}_{j-s}^{(C,2d)}(x,y;\gamma) \frac{w^m}{m!}.$$

Comparing the coefficients of $\frac{w^m}{m!}$ on both sides of this last equation, we get Equation (2.1). □

Theorem 2.2 *Let $m, d \in \mathbb{N}_0$. Then we have*

$$\cos\left(\frac{m\pi}{4}\right) = \frac{\gamma^d (2d)!}{2^{2d} x^m \sqrt{2^m}} \sum_{j=0}^{m} \binom{m}{j} \sum_{s=0}^{j} \binom{j}{s} d^{m-j} y_2(s,d;\gamma) \sum_{k=0}^{j-s} \binom{j-s}{k} \sum_{r=0}^{\left[\frac{k}{2}\right]} (-1)^r \binom{k}{2r} x^k \mathcal{E}_{j-s-k}^{(2d)}(\gamma).$$

Proof Substituting $x = y$ into (2.1), and combining the final equation with (1.18), we get

$$\cos\left(\frac{m\pi}{4}\right) = \frac{\gamma^d (2d)!}{2^{2d} x^m \sqrt{2^m}} \sum_{j=0}^{m} \binom{m}{j} \sum_{s=0}^{j} \binom{j}{s} d^{m-j} y_2(s,d;\gamma) \mathcal{E}_{j-s}^{(C,2d)}(x,x;\gamma).$$

Combining the above equation with (1.16) and (1.26), we obtain

$$\cos\left(\frac{m\pi}{4}\right) = \frac{\gamma^d (2d)!}{2^{2d} x^m \sqrt{2^m}} \sum_{j=0}^{m} \binom{m}{j} \sum_{s=0}^{j} \binom{j}{s} d^{m-j} y_2(s,d;\gamma) \sum_{k=0}^{j-s} \binom{j-s}{k} \sum_{r=0}^{\left[\frac{k}{2}\right]} (-1)^r \binom{k}{2r} x^k \mathcal{E}_{j-s-k}^{(2d)}(\gamma).$$

Thus proof of the theorem is completed. □

Theorem 2.3 *Let $m, d \in \mathbb{N}_0$. Then we have*

$$S_m(x,y) = \frac{\gamma^d (2d)!}{2^{2d}} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{m}{k} \binom{k}{j} d^{m-k} y_2(j,d;\gamma) \mathcal{E}_{k-j}^{(S,2d)}(x,y;\gamma). \tag{2.2}$$

Proof By using (1.13), (1.15) and (1.25), we obtain the following functional equation

$$\frac{2^{2d}}{(2d)!}R_{\mathrm{S}}(w,x,y)=\gamma^{d}e^{wd}R_{y_2}(w,d;\gamma)R_{\mathrm{ES}}(w,x,y;2d,\gamma).$$

With the help of the above functional equation, the proof of (2.2) is completed by following exactly the same lines as the proof of the assertion of (2.1), and so we omit it. □

Theorem 2.4 *Let* $m,d\in\mathbb{N}_0$. *Then we have*

$$\sin\left(\frac{m\pi}{4}\right)=\frac{\gamma^{d}(2d)!}{2^{2d}x^{m}\sqrt{2^{m}}}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}d^{m-k}y_2(j,d;\gamma)\sum_{v=0}^{k-j}\binom{k-j}{v}\sum_{r=0}^{\left[\frac{v-1}{2}\right]}(-1)^{r}\binom{v}{2r+1}x^{v}\mathcal{E}_{k-j-v}^{(2d)}(\gamma).$$

Proof Substituting $x=y$ into (2.2), and combining the final equation with (1.19), we get

$$\sin\left(\frac{m\pi}{4}\right)=\frac{\gamma^{d}(2d)!}{2^{2d}x^{m}\sqrt{2^{m}}}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}d^{m-k}y_2(j,d;\gamma)\mathcal{E}_{k-j}^{(S,2d)}(x,x;\gamma).$$

Combining the above equation with (1.17) and (1.27), we have

$$\sin\left(\frac{m\pi}{4}\right)=\frac{\gamma^{d}(2d)!}{2^{2d}x^{m}\sqrt{2^{m}}}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}d^{m-k}y_2(j,d;\gamma)\sum_{v=0}^{k-j}\binom{k-j}{v}\sum_{r=0}^{\left[\frac{v-1}{2}\right]}(-1)^{r}\binom{v}{2r+1}x^{v}\mathcal{E}_{k-j-v}^{(2d)}(\gamma).$$

Thus proof of the theorem is completed. □

Replacing m by $4m$ in Theorem 2.4, we arrive at the following theorem:

Theorem 2.5 *Let* $m,d\in\mathbb{N}_0$. *Then we have*

$$\sum_{k=0}^{4m}\sum_{j=0}^{k}\binom{4m}{k}\binom{k}{j}d^{4m-k}y_2(j,d;\gamma)\sum_{v=0}^{k-j}\binom{k-j}{v}\sum_{r=0}^{\left[\frac{v-1}{2}\right]}(-1)^{r}\binom{v}{2r+1}x^{v}\mathcal{E}_{k-j-v}^{(2d)}(\gamma)=0.$$

By the aid of (1.11), (1.14) and (1.24), Kilar and Simsek [6] gave the following relation:

$$C_m(x,y)=2^{-d}d!\sum_{s=0}^{m}\binom{m}{s}y_1(s,d;\gamma)\mathcal{E}_{m-s}^{(C,d)}(x,y;\gamma). \tag{2.3}$$

Combining (2.3) with (1.12), we derive the following result:

Corollary 2.6 *Let* $m,d\in\mathbb{N}_0$. *Then we have*

$$C_m(x,y)=\sum_{s=0}^{m}\binom{m}{s}\mathcal{E}_{s}^{(-d)}(\gamma)\mathcal{E}_{m-s}^{(C,d)}(x,y;\gamma). \tag{2.4}$$

Substituting $x=y$ into (2.4), combining the final equation with (1.16) and (1.26), we arrive at the following theorem:

Theorem 2.7 *Let $m, d \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{\left[\frac{m}{2}\right]}(-1)^v\binom{m}{2r} = \sum_{s=0}^{m}\binom{m}{s}\mathcal{E}_s^{(-d)}(\gamma)\sum_{r=0}^{m-s}\binom{m-s}{r}\sum_{k=0}^{\left[\frac{r}{2}\right]}(-1)^k\binom{r}{2k}x^{r-m}\mathcal{E}_{m-s-r}^{(d)}(\gamma).$$

Theorem 2.8 *Let $m, d \in \mathbb{N}_0$ with $m \geq d$. Then we have*

$$(m)_d C_{m-d}(x,y) = (-1)^d d! \sum_{s=0}^{m}\binom{m}{s} y_1(s,d;-\gamma)\mathcal{B}_{m-s}^{(C,d)}(x,y;\gamma). \tag{2.5}$$

Proof By using (1.11), (1.14) and (1.20), we obtain the following functional equation:

$$w^d R_{\mathrm{C}}(w,x,y) = (-1)^d d! R_y(w,d;-\gamma) R_{\mathrm{BC}}(w,x,y;d,\gamma).$$

From the above equation, we get

$$\sum_{m=0}^{\infty}(m)_d C_{m-d}(x,y)\frac{w^m}{m!} = (-1)^d d! \sum_{m=0}^{\infty} y_1(m,d;-\gamma)\frac{w^m}{m!}\sum_{m=0}^{\infty}\mathcal{B}_m^{(C,d)}(x,y;\gamma)\frac{w^m}{m!}.$$

Therefore

$$\sum_{m=0}^{\infty}(m)_d C_{m-d}(x,y)\frac{w^m}{m!} = (-1)^d d! \sum_{m=0}^{\infty}\sum_{s=0}^{m}\binom{m}{s} y_1(s,d;-\gamma)\mathcal{B}_{m-s}^{(C,d)}(x,y;\gamma)\frac{w^m}{m!}.$$

Comparing the coefficients of $\frac{w^m}{m!}$ on both sides of the previous equation, we get the desired result. □

Theorem 2.9 *Let $m, d \in \mathbb{N}_0$. Then we have*

$$C_m(x,y) = \sum_{s=0}^{m}\binom{m}{s}\mathcal{B}_s^{(-d)}(\gamma)\mathcal{B}_{m-s}^{(C,d)}(x,y;\gamma). \tag{2.6}$$

Proof By (1.1), (1.14) and (1.20), we obtain

$$R_{\mathrm{C}}(w,x,y) = R_B(w,0;-d,\gamma) R_{\mathrm{BC}}(w,x,y;v,\gamma).$$

From the above equation, we get

$$\sum_{m=0}^{\infty} C_m(x,y)\frac{w^m}{m!} = \sum_{m=0}^{\infty}\mathcal{B}_m^{(-d)}(\gamma)\frac{w^m}{m!}\sum_{m=0}^{\infty}\mathcal{B}_m^{(C,d)}(x,y;\gamma)\frac{w^m}{m!}.$$

Therefore

$$\sum_{m=0}^{\infty} C_m(x,y)\frac{w^m}{m!} = \sum_{m=0}^{\infty}\sum_{s=0}^{m}\binom{m}{s}\mathcal{B}_s^{(-d)}(\gamma)\mathcal{B}_{m-s}^{(C,d)}(x,y;\gamma)\frac{w^m}{m!}.$$

Comparing the coefficients of $\frac{w^m}{m!}$ on both sides of the previous equation, we have the desired result. □

Theorem 2.10 *Let* $m, d \in \mathbb{N}_0$. *Then we have*

$$S_m(x,y) = \sum_{j=0}^{m} \binom{m}{j} \mathcal{B}_j^{(-d)}(\gamma)\, \mathcal{B}_{m-j}^{(S,d)}(x,y;\gamma). \tag{2.7}$$

Proof By using (1.1), (1.15) and (1.21), we get the following functional equation:

$$R_{\mathrm{S}}(w,x,y) = R_B(w,0;-d,\gamma)\, R_{\mathrm{BS}}(w,x,y;d,\gamma).$$

With the help of the above functional equation, the proof of (2.7) is completed by following exactly the same lines as the proof of the assertion of (2.6), and so we omit it. □

Theorem 2.11 *Let* $m, d \in \mathbb{N}_0$ *with* $m \geq 2d$. *Then we have*

$$C_{m-2d}(x,y) = \frac{(2d)!\gamma^d}{(m)_{2d}\, 2^{2d}} \sum_{k=0}^{m} \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} d^{m-k} y_2(j,d;\gamma)\, \mathcal{G}_{k-j}^{(C,2d)}(x,y;\gamma).$$

Proof By using (1.13), (1.14) and (1.28), we derive the following functional equation:

$$w^{2d} R_{\mathrm{C}}(w,x,y) = 2^{-2d}(2d)!\gamma^d e^{wd} R_{y_2}(w,d;\gamma) R_{\mathrm{GC}}(w,x,y;2d,\gamma).$$

From the above equation, we get

$$w^{2d} \sum_{m=0}^{\infty} C_m(x,y) \frac{w^m}{m!} = 2^{-2d}(2d)!\gamma^d \sum_{m=0}^{\infty} d^m \frac{w^m}{m!} \sum_{m=0}^{\infty} y_2(m,d;\gamma) \frac{w^m}{m!} \sum_{m=0}^{\infty} \mathcal{G}_m^{(C,2d)}(x,y;\gamma) \frac{w^m}{m!}.$$

Hence,

$$\sum_{m=0}^{\infty} (m)_{2d}\, C_{m-2d}(x,y) \frac{w^m}{m!} = 2^{-2d}(2d)!\gamma^d \sum_{m=0}^{\infty} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{m}{k} \binom{k}{j} d^{m-k} y_2(j,d;\gamma)\, \mathcal{G}_{k-j}^{(C,2d)}(x,y;\gamma) \frac{w^m}{m!}.$$

Comparing the coefficients of $\frac{w^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.12 *Let* $m, d \in \mathbb{N}_0$ *with* $m \geq 2d$. *Then we have*

$$S_{m-2d}(x,y) = \frac{(2d)!\gamma^d}{2^{2d}(m)_{2d}} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{m}{k} \binom{k}{j} d^{m-k} y_2(j,d;\gamma)\, \mathcal{G}_{k-j}^{(S,2d)}(x,y;\gamma).$$

Proof By aid of (1.13), (1.15) and (1.29), we get

$$w^{2d} R_{\mathrm{S}}(w,x,y) = 2^{-2d}(2d)!\gamma^d e^{wd} R_{y_2}(w,d;\gamma) R_{\mathrm{GS}}(w,x,y;2d,\gamma).$$

With the help of the above functional equation, the proof of the Theorem 2.12 is completed by following exactly the same lines as the proof of the assertion of the Theorem 2.11, and so we omit it. □

3. Some identities involving Vieta polynomials and parametric type polynomials

In this section, with the help of the results obtained in the previous section, we give many novel formulas and identities involving the Vieta-Fibonacci polynomials, the Vieta-Lucas polynomials, the numbers $y_2(m, v; \gamma)$, the Apostol type numbers, and the two-parametric types of the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials, the Apostol-Genocchi polynomials.

Setting $y = \sqrt{1-x^2}$ in (1.16) and (1.17), then combining the final equations with (1.7) and (1.9), we have the following identities, respectively

$$v_r(2x) = 2C_r\left(x, \sqrt{1-x^2}\right) \tag{3.1}$$

and

$$V_r(2x) = \frac{S_r\left(x, \sqrt{1-x^2}\right)}{\sqrt{1-x^2}}. \tag{3.2}$$

Note that throughout in this section we assume that $|x| < 1$.

Substituting $y = \sqrt{1-x^2}$ into the following identity

$$C_r(mx, my) = m^r C_r(x, y)$$

(*cf.* [7]), and combining the final equation with (3.1), we obtain the following theorem:

Theorem 3.1 *Let $m, r \in \mathbb{N}_0$. Then we have*

$$C_r\left(mx, m\sqrt{1-x^2}\right) = \frac{v_r(2x)\, m^r}{2}.$$

Combining (2.1) with (3.1), we get a formula including the numbers $y_2(s, d; \gamma)$, the polynomials $v_r(2x)$ and the two-parametric types of the Apostol-Euler polynomials by the following theorem:

Theorem 3.2 *Let $r, d \in \mathbb{N}_0$. Then we have*

$$v_r(2x) = \gamma^d 2^{1-2d}(2d)! \sum_{j=0}^{r} \sum_{s=0}^{j} \binom{r}{j} \binom{j}{s} d^{r-j} y_2(s, d; \gamma)\, \mathcal{E}_{j-s}^{(C,2d)}\left(x, \sqrt{1-x^2}; \gamma\right).$$

Combining (2.2) with (3.2), we obtain the following theorem:

Theorem 3.3 *Let $r, d \in \mathbb{N}_0$. Then we have*

$$V_r(2x) = \frac{\gamma^d (2d)!}{2^{2d}\sqrt{1-x^2}} \sum_{k=0}^{r} \sum_{j=0}^{k} \binom{r}{k} \binom{k}{j} d^{r-k} y_2(j, d; \gamma)\, \mathcal{E}_{k-j}^{(S,2d)}\left(x, \sqrt{1-x^2}; \gamma\right).$$

Combining (2.6) with (3.1), we obtain a relation including the numbers $\mathcal{B}_s^{(-v)}(\gamma)$, the polynomials $v_r(2x)$ and the two-parametric types of the Apostol-Bernoulli polynomials by the following theorem:

Theorem 3.4 *Let $r,d \in \mathbb{N}_0$. Then we have*

$$v_r(2x) = 2\sum_{s=0}^{r}\binom{r}{s}\mathcal{B}_s^{(-d)}(\gamma)\,\mathcal{B}_{r-s}^{(C,d)}\left(x,\sqrt{1-x^2};\gamma\right).$$

Combining (2.7) with (3.2), we arrive at the following theorem:

Theorem 3.5 *Let $r,d \in \mathbb{N}_0$. Then we have*

$$V_r(2x) = \frac{1}{\sqrt{1-x^2}}\sum_{j=0}^{r}\binom{r}{j}\mathcal{B}_j^{(-d)}(\gamma)\,\mathcal{B}_{r-j}^{(S,d)}\left(x,\sqrt{1-x^2};\gamma\right).$$

Combining Theorem 2.11 with Equation (3.1), we derive a relation among the numbers $y_2(n,d;\gamma)$, the polynomials $v_n(2x)$ and the two-parametric types of the Apostol-Genocchi polynomials by the following theorem:

Theorem 3.6 *Let $r,d \in \mathbb{N}_0$ with $r \geq 2d$. Then we have*

$$v_{r-2d}(2x) = \frac{(2d)!\gamma^d}{2^{2d-1}(r)_{2d}}\sum_{k=0}^{r}\binom{r}{k}\sum_{j=0}^{k}\binom{k}{j}d^{r-k}y_2(j,d;\gamma)\,\mathcal{G}_{k-j}^{(C,2d)}\left(x,\sqrt{1-x^2};\gamma\right).$$

Combining Theorem 2.12 with Equation (3.2), we arrive at the following theorem:

Theorem 3.7 *Let $r,d \in \mathbb{N}_0$ with $r \geq 2d$. Then we have*

$$V_{r-2d}(2x) = \frac{(2d)!\gamma^d}{2^{2d}(r)_{2d}\sqrt{1-x^2}}\sum_{k=0}^{r}\binom{r}{k}\sum_{j=0}^{k}\binom{k}{j}d^{r-k}y_2(j,d;\gamma)\,\mathcal{G}_{k-j}^{(S,2d)}\left(x,\sqrt{1-x^2};\gamma\right).$$

4. Formulas for Fibonacci and Lucas numbers and parametric type polynomials

In this section, we derive some identities and formulas involving the Fibonacci numbers, the Lucas numbers, the numbers $y_2(m,v;\gamma)$, the Apostol Bernoulli numbers of the negative higher order, the two-parametric types of the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials, the Apostol-Genocchi polynomials.

By using the following well-known identities

$$T_r\left(\frac{i}{2}\right) = \frac{i^r}{2}L_r$$

and

$$U_r\left(\frac{i}{2}\right) = i^r F_{r+1},$$

where $F_0 = 0$, $F_1 = 1$, $F_{r+2} = F_{r+1} + F_r$ and $L_0 = 2$, $L_1 = 1$, $L_{r+2} = L_{r+1} + L_r$ (*cf.* [9, 12]), we have the following identities including the Fibonacci numbers F_r, the Lucas numbers L_r, the Vieta-Fibonacci polynomials $V_r(x)$, and the Vieta-Lucas polynomials $v_r(x)$:

$$v_r(i) = i^r L_r \tag{4.1}$$

and

$$V_r\left(i\right) = i^{r-1} F_r \tag{4.2}$$

(*cf.* [12, Eqs. (47.1) and (47.2)]; see also [5]).

Substituting $x = \frac{i}{2}$ into the Theorem 3.2, and using (4.1), we obtain the following result:

Corollary 4.1 *Let* $r, d \in \mathbb{N}_0$. *Then we have*

$$L_r = \frac{\gamma^d \left(2d\right)!}{i^r 2^{2d-1}} \sum_{j=0}^{r} \sum_{s=0}^{j} \binom{r}{j} \binom{j}{s} d^{r-j} y_2\left(s, d; \gamma\right) \mathcal{E}_{j-s}^{(C,2d)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}; \gamma\right).$$

Substituting $x = \frac{i}{2}$ into the Theorem 3.3, and using (4.2), we get the following result:

Corollary 4.2 *Let* $r, d \in \mathbb{N}_0$. *Then we have*

$$F_r = \frac{\gamma^d 2^{1-2d} \left(2d\right)!}{i^{r-1}\sqrt{5}} \sum_{k=0}^{r} \sum_{j=0}^{k} \binom{r}{k} \binom{k}{j} d^{r-k} y_2\left(j, d; \gamma\right) \mathcal{E}_{k-j}^{(S,2d)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}; \gamma\right).$$

Substituting $x = \frac{i}{2}$ into the Theorems 3.4 and 3.5, and using (4.1) and (4.2), we obtain the following corollaries:

Corollary 4.3 *Let* $r, d \in \mathbb{N}_0$. *Then we have*

$$L_r = \frac{2}{i^r} \sum_{s=0}^{r} \binom{r}{s} \mathcal{B}_s^{(-d)}\left(\gamma\right) \mathcal{B}_{r-s}^{(C,d)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}; \gamma\right).$$

Corollary 4.4 *Let* $r, d \in \mathbb{N}_0$. *Then we have*

$$F_r = \frac{2 i^{1-r}}{\sqrt{5}} \sum_{j=0}^{r} \binom{r}{j} \mathcal{B}_j^{(-d)}\left(\gamma\right) \mathcal{B}_{r-j}^{(S,d)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}; \gamma\right).$$

Substituting $x = \frac{i}{2}$ into the Theorems 3.6 and 3.7, and using (4.1) and (4.2), we have the following corollaries:

Corollary 4.5 *Let* $r, d \in \mathbb{N}_0$ *with* $r \geq 2d$. *Then we have*

$$L_{r-2d} = \frac{\left(-1\right)^d \left(2d\right)! \gamma^d}{i^r 2^{2d-1} \left(r\right)_{2d}} \sum_{k=0}^{r} \sum_{j=0}^{k} \binom{r}{k} \binom{k}{j} d^{r-k} y_2\left(j, d; \gamma\right) \mathcal{G}_{k-j}^{(C,2d)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}; \gamma\right).$$

Corollary 4.6 *Let* $r, d \in \mathbb{N}_0$ *with* $r \geq 2d$. *Then we have*

$$F_{r-2d} = \frac{\left(-1\right)^d 2^{1-2d} \left(2d\right)! \gamma^d}{i^{r-1}\sqrt{5} \left(r\right)_{2d}} \sum_{k=0}^{r} \sum_{j=0}^{k} \binom{r}{k} \binom{k}{j} d^{r-k} y_2\left(j, d; \gamma\right) \mathcal{G}_{k-j}^{(S,2d)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}; \gamma\right).$$

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References

[1] Alkan M. The generalized Fibonacci sequences on an integral domain. Montes Taurus Journal of Pure and Applied Mathematics 2021; 3 (2): 60-69.

[2] Comtet L. Advanced Combinatorics. Dordrecht-Holland, Boston: D. Reidel Publication Company, 1974.

[3] Djordjević GB, Milovanović GV. Special Classes of Polynomials. Leskovac: University of Niš, Faculty of Technology, 2014.

[4] Fox L, Parker IB. Chebyshev Polynomials in Numerical Analysis. London: Oxford University Press, 1968.

[5] Horadam AF. Vieta polynomials. Fibonacci Quarterly 2002; 40 (3): 223-232.

[6] Kilar N, Simsek Y. Relations between Apostol-type polynomials and combinatorial numbers. In: Proceedings Book of the Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2019); Paris, France; 2019. pp. 18-22.

[7] Kilar N, Simsek Y. Relations on Bernoulli and Euler polynomials related to trigonometric functions. Advanced Studies in Contemporary Mathematics 2019; 29 (2): 191-198.

[8] Kilar N, Simsek Y. A special approach to derive new formulas for some special numbers and polynomials. Turkish Journal of Mathematics 2020; 44 (6): 2217-2240.

[9] Kilar N, Simsek Y. Identities and relations for Hermite-based Milne-Thomson polynomials associated with Fibonacci and Chebyshev polynomials. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A: Matemáticas RACSAM 2021; 115. doi: 10.1007/s13398-020-00968-3

[10] Kilar N, Simsek Y. Computational formulas and identities for new classes of Hermite-based Milne-Thomson type polynomials: Analysis of generating functions with Euler's formula. Mathematical Methods in the Applied Sciences 2021; 44 (8): 6731-6762.

[11] Kilar N, Simsek Y. Formulas and relations of special numbers and polynomials arising from functional equations of generating functions. Montes Taurus Journal of Pure and Applied Mathematics 2021; 3 (1): 106-123.

[12] Koshy T. Fibonacci and Lucas Numbers with Applications. Volume 2, USA: John Wiley Sons, Inc., 2019.

[13] Kucukoglu I, Simsek Y. New formulas and numbers arising from analyzing combinatorial numbers and polynomials. Montes Taurus Journal of Pure and Applied Mathematics 2021; 3 (3): 238-259.

[14] Masjed-Jamei M, Milovanović GV, Dağlı MC. A generalization of the array type polynomials. Mathematica Moravica 2022; 26 (1): 37-46.

[15] Masjed-Jamei M, Koepf W. Symbolic computation of some power-trigonometric series. Journal of Symbolic Computation 2017; 80: 273-284.

[16] Milovanović GV, Simsek Y, Stojanović VS. A class of polynomials and connections with Bernoulli's numbers. The Journal of Analysis 2019; 27: 709-726.

[17] Ozdemir G, Simsek Y. Generating functions for two-variable polynomials related to a family of Fibonacci type polynomials and numbers. Filomat 2016; 30 (4): 969-975.

[18] Ozdemir G, Simsek Y, Milovanović GV. Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials. Mediterranean Journal of Mathematics 2017; 14: 1-17.

[19] Simsek Y. New families of special numbers for computing negative order Euler numbers and related numbers and polynomials. Applicable Analysis and Discrete Mathematics 2018; 12: 1-35.

[20] Simsek Y. Explicit formulas for p-adic integrals: Approach to p-adic distributions and some families of special numbers and polynomials. Montes Taurus Journal of Pure and Applied Mathematics 2019; 1 (1): 1-76.

[21] Simsek Y. Interpolation functions for new classes special numbers and polynomials via applications of p-adic integrals and derivative operator. Montes Taurus Journal of Pure and Applied Mathematics 2021; 3 (1): 38-61.

[22] Srivastava HM, Choi J. Zeta and q-Zeta Functions and Associated Series and Integrals. Amsterdam: Elsevier Science Publishers, 2012.

[23] Srivastava HM, Masjed-Jamei M, Beyki MR. Some new generalizations and applications of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Rocky Mountain Journal of Mathematics 2019; 49: 681-697.