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The dual spaces of variable anisotropic Hardy–Lorentz spaces and continuity of a class of linear operators

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Abstract: In this paper, the authors obtain the continuity of a class of linear operators on variable anisotropic Hardy–Lorentz spaces. In addition, the authors also obtain that the dual space of variable anisotropic Hardy–Lorentz spaces is the anisotropic BMO-type spaces with variable exponents. This result is still new even when the exponent function $p(\cdot)$ is p.

Key words: Anisotropy, Hardy–Lorentz space, atom, Calderón–Zygmund operator, BMO space

1. Introduction

As is known to all, Hardy space on the Euclidean space $\mathbb{R}^n$ is a good substitutes of Lebesgue space $L^p(\mathbb{R}^n)$ when $p \in (0,1]$, and plays an important role in haronmic analysis and **PDE**s; see, for examples, [5, 11, 14, 20–22, 25]. Moreover, when studying the boundedness of some operators in the critical case, the weak Hardy space $w\mathcal{H}^p(\mathbb{R}^n)$ naturally appears and it is a good substitute of $\mathcal{H}^p(\mathbb{R}^n)$. $w\mathcal{H}^p(\mathbb{R}^n)$ with $p \in (0, 1)$ was first introduced by Fefferman and Soria [10] to find out the biggest space from which the Riesz transform is bounded to the weak Lebesgue space $wL^1(\mathbb{R}^n)$. In 2007, Abu-Shammala and Torchinsky [1] introduced the Hardy–Lorentz spaces $\mathcal{H}^{p,r}(\mathbb{R}^n)$ for the full range $p \in (0, 1]$ and $r \in (0, \infty]$, and obtained some real-variable characterizations of this space. In 2016, Liu et al. [16] introduced the anisotropic Hardy–Lorentz space $\mathcal{H}^{p,r}_A(\mathbb{R}^n)$ associated with a general expansive dilation A, including the classical isotropic Hardy–Lorentz space of Abu-Shammala and Torchinsky.

As a generalization, variable exponent function spaces have their applications in fluid dynamics [2], image processing [4], **PDE**s and variational calculus [9, 25]. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a variable exponent function. Recently, Liu et al. [17] introduced the variable anisotropic Hardy–Lorentz space $\mathcal{H}^{p(\cdot),r}_A(\mathbb{R}^n)$, via the radial grand maximal function, and then established its some real-variable characterizations, respectively, in terms of atom, the radial and the nontangential maximal functions. For more information about variable function spaces, see [6–8, 13, 18, 19, 24, 27].

To complete the theory of the variable anisotropic Hardy–Lorentz space $\mathcal{H}^{p(\cdot),r}_A(\mathbb{R}^n)$, in this article, we obtain the boundedness of a class of Calderón–Zygmund operators from $\mathcal{H}^{p(\cdot),r}_A(\mathbb{R}^n)$ to variable Lorentz space

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$L^{p(\cdot),r}(\mathbb{R}^n)$ and from $\mathcal{H}_A^{p(\cdot),r}(\mathbb{R}^n)$ to itself. In addition, we also obtain the dual space of $\mathcal{H}_A^{p(\cdot),r}(\mathbb{R}^n)$ is the anisotropic BMO-type space with variable exponents.

Precisely, this article is organized as follows.

In Section 2, we recall some notations and definitions concerning expansive dilations, the variable Lorentz space $L^{p(\cdot),r}(\mathbb{R}^n)$ and the variable anisotropic Hardy–Lorentz space $\mathcal{H}_A^{p(\cdot),r}(\mathbb{R}^n)$, via the radial grand maximal function.

Section 3 is devoted to establishing the boundedness of anisotropic convolutional δ-type Calderón–Zygmund operators from $\mathcal{H}_A^{p(\cdot),r}(\mathbb{R}^n)$ to $L^{p(\cdot),r}(\mathbb{R}^n)$ and from $\mathcal{H}_A^{p(\cdot),r}(\mathbb{R}^n)$ to itself.

In Section 4, we prove that the dual space of $\mathcal{H}_A^{p(\cdot),r}(\mathbb{R}^n)$ is the anisotropic BMO-type space with variable exponents (see Theorem 4.6). For this purpose, we first introduce a new kind of anisotropic BMO-type spaces with variable exponents $\mathcal{BMO}_A^{p(\cdot),q,s}(\mathbb{R}^n)$ in Definition 4.1, which includes the space $\mathrm{BMO}(\mathbb{R}^n)$ of John and Nirenberg [12]. It is worth pointing out that this result is also new, when $\mathcal{H}_A^{p(\cdot),r}(\mathbb{R}^n)$ is reduced to $\mathcal{H}_A^{p,r}(\mathbb{R}^n)$.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. In this article, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. For any $q \in [1, \infty]$, we denote by q' its conjugate index. For any $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the *maximal integer* not larger than a. The *symbol* $D \lesssim F$ means that $D \le CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. If a set $E \subset \mathbb{R}^n$, we denote by χ_E its *characteristic function*. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by $\mathcal{X}$.

2. Preliminaries

Firstly, we recall the definitions of anisotropic dilations on $\mathbb{R}^n$; see [3, p. 5]. A real $n \times n$ matrix A is called an *anisotropic dilation*, shortly a *dilation*, if $\min_{\lambda \in \sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the set of all *eigenvalues* of A. Let λ_- and λ_+ be two *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \le \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

By [3, Lemma 2.2], we know that, for a given dilation A, there exist a number $r \in (1, \infty)$ and a set $\Delta := \{x \in \mathbb{R}^n : |Px| < 1\}$, where P is some nondegenerate $n \times n$ matrix, such that

$$\Delta \subset r\Delta \subset A\Delta,$$

and we can assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n-dimensional Lebesgue measure of the set Δ. Let $B_k := A^k \Delta$ for $k \in \mathbb{Z}$. Then B_k is open,

$$B_k \subset rB_k \subset B_{k+1} \quad \text{and} \quad |B_k| = b^k,$$

here and hereafter, $b := |\det A|$. An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ is called a *dilated ball*. Denote

$$\mathfrak{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}. \tag{2.1}$$

Throughout the whole paper, let σ be the *smallest integer* such that $2B_0 \subset A^\sigma B_0$ and, for any subset E of $\mathbb{R}^n$, let $E^\complement := \mathbb{R}^n \setminus E$. Then, for all $k, j \in \mathbb{Z}$ with $k \le j$, it holds true that

$$B_k + B_j \subset B_{j+\sigma}, \tag{2.2}$$

$$B_k + (B_{k+\sigma})^{\complement} \subset (B_k)^{\complement}, \tag{2.3}$$

where $E + F$ denotes the *algebraic sum* $\{x + y : x \in E, y \in F\}$ of sets $E, F \subset \mathbb{R}^n$.

Recall a *quasi-norm*, associated with dilation A, is a Borel measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$, satisfying

(i) $\rho(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, here and hereafter, $\vec{0}_n$ denotes the origin of $\mathbb{R}^n$;

(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$, where, as above, $b := |\det A|$;

(iii) $\rho(x + y) \le H[\rho(x) + \rho(y)]$ for all $x, y \in \mathbb{R}^n$, where $H \in [1, \infty)$ is a constant independent of x and y.

By [3, Lemma 2.4], we know that all homogeneous quasi-norms associated with a given dilation A are equivalent. Therefore, for a fixed dilation A, in what follows, for convenience, we always use the *step homogeneous quasi-norm* ρ defined by setting, for all $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{k\in\mathbb{Z}} b^k \chi_{B_{k+1}\setminus B_k}(x) \text{ if } x \ne \vec{0}_n, \quad \text{or else} \quad \rho(\vec{0}_n) := 0.$$

By (2.2), we know that, for all $x, y \in \mathbb{R}^n$,

$$\rho(x + y) \le b^\sigma[\rho(x) + \rho(y)];$$

Moreover, $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [5], where dx denotes the n*-dimensional Lebesgue measure*.

A measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ is called a *variable exponent*. For any variable exponent $p(\cdot)$, let

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x). \tag{2.4}$$

Denote by $\mathcal{P}$ the set of all variable exponents $p(\cdot)$ satisfying $0 < p_- \le p_+ < \infty$.

Let f be a measurable function on $\mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}$. Define

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda \in (0, \infty) : \varrho_{p(\cdot)}(f/\lambda) \le 1\right\},$$

where

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx.$$

Moreover, the *variable Lebesgue space* $L^{p(\cdot)}$ is defined to be the set of all measurable functions f satisfying that $\varrho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}}$.

Remark 2.1 [17] Let $p(\cdot) \in \mathcal{P}$.

(i) For any $r \in (0, \infty)$ and $f \in L^{p(\cdot)}$, $\||f|^r\|_{L^{p(\cdot)}} = \|f\|^r_{L^{rp(\cdot)}}$. Moreover, for any $\mu \in \mathbb{C}$ and $f, g \in L^{p(\cdot)}$, $\|\mu f\|_{L^{p(\cdot)},} = |\mu|\, \|f\|_{L^{p(\cdot)}}$ and $\|f + g\|^{\underline{p}}_{L^{p(\cdot)}} \le \|f\|^{\underline{p}}_{L^{p(\cdot)}} + \|g\|^{\underline{p}}_{L^{p(\cdot)}}$, where

$$\underline{p} := \min\{p_-, 1\} \tag{2.5}$$

with p_- as in (2.4).

(ii) For any function $f \in L^{p(\cdot)}$ with $\|f\|_{L^{p(\cdot)}} > 0$, $\varrho_{p(\cdot)}(f/\|f\|_{L^{p(\cdot)}}) = 1$ and, for $\|f\|_{L^{p(\cdot)}} \leq 1$, then $\varrho_{p(\cdot)}(f) \leq \|f\|_{L^{p(\cdot)}}$.

Definition 2.2 *Let $p(\cdot) \in \mathcal{P}$. The variable Lorentz space $L^{p(\cdot),r}$ is defined to be the set of all measurable functions f such that*

$$\|f\|_{L^{p(\cdot),r}} := \begin{cases} \left[\int_0^\infty \lambda^r \left\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\right\|_{L^{p(\cdot)}}^r \frac{d\lambda}{\lambda}\right]^{1/r}, & r \in (0,\infty), \\ \sup_{\lambda\in(0,\infty)} \left[\lambda \left\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\right\|_{L^{p(\cdot)}}\right], & r = \infty \end{cases}$$

is finite.

We say that $p(\cdot) \in \mathcal{P}$ satisfy the *globally log-Hölder continuous condition*, denoted by $p(\cdot) \in C^{\log}$, if there exist two positive constants $C_{\log}(p)$ and C_∞, and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/\rho(x-y))}$$

and

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + \rho(x))}.$$

A C^∞ function φ is said to belong to the Schwartz class $\mathcal{S}$ if, for every integer $\ell \in \mathbb{Z}_+$ and multiindex α, $\|\varphi\|_{\alpha,\ell} := \sup_{x\in\mathbb{R}^n} [\rho(x)]^\ell |\partial^\alpha \varphi(x)| < \infty$. The dual space of $\mathcal{S}$, namely, the space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-$*$ topology, is denoted by $\mathcal{S}'$. For any $N \in \mathbb{Z}_+$, let

$$\mathcal{S}_N := \{\varphi \in \mathcal{S} : \ \|\varphi\|_{\alpha,\ell} \leq 1, \ |\alpha| \leq N, \ \ \ell \leq N\}.$$

In what follows, for $\varphi \in \mathcal{S}$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\varphi_k(x) := b^{-k}\varphi\left(A^{-k}x\right)$.

Definition 2.3 *Let $\varphi \in \mathcal{S}$ and $f \in \mathcal{S}'$. For any given $N \in \mathbb{N}$, the radial grand maximal function $M_N(f)$ of $f \in \mathcal{S}'$ is defined by setting, for any $x \in \mathbb{R}^n$,*

$$M_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N} \sup_{k\in\mathbb{Z}} |f * \varphi_k(x)|.$$

Definition 2.4 [17] Let $p(\cdot) \in C^{\log}$, $r \in (0,\infty)$, A be a dilation and $N \in [\lfloor (1/\underline{p} - 1)\ln b/\ln \lambda_- \rfloor + 2, \infty)$, where $\underline{p}$ is as in (2.5). The *variable anisotropic Hardy–Lorentz space* $\mathcal{H}_A^{p(\cdot),r}$ is defined as

$$\mathcal{H}_A^{p(\cdot),r} := \left\{f \in \mathcal{S}' : M_N(f) \in L^{p(\cdot),r}\right\}$$

and, for any $f \in \mathcal{H}_A^{p(\cdot),r}$, let $\|f\|_{\mathcal{H}_A^{p(\cdot),r}} := \|M_N(f)\|_{L^{p(\cdot),r}}$.

Remark 2.5 *Let $p(\cdot) \in C^{\log}$, $r \in (0,\infty)$.*

(i) *When* $p(\cdot) := p$, *where* $p \in (0, \infty)$, *the space* $\mathcal{H}_A^{p(\cdot), r}$ *is reduced to the anisotropic Hardy–Lorentz space* $\mathcal{H}_A^{p, r}$ *studied in [16].*

(ii) *When* $A := 2\mathrm{I}_{n\times n}$ *and* $p(\cdot) := p$, *the space* $\mathcal{H}_A^{p(\cdot), r}$ *is reduced to the Hardy–Lorentz space* $\mathcal{H}^{p, r}$ *studied in [1].*

Definition 2.6 [17] Let $p(\cdot) \in \mathcal{P}$, $q \in (1, \infty]$ and

$$s \in [\lfloor (1/p_- - 1)\ln b/\ln \lambda_- \rfloor, \infty) \cap \mathbb{Z}_+$$

with p_- as in (2.4). An *anisotropic* $(p(\cdot), q, s)$*-atom* is a measurable function a on $\mathbb{R}^n$ satisfying

(i) (support) $\operatorname{supp} a := \overline{\{x \in \mathbb{R}^n : a(x) \neq 0\}} \subset B$, where $B \in \mathfrak{B}$ and $\mathfrak{B}$ is as in (2.1);

(ii) (size) $\|a\|_{L^q} \leq \frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}}$;

(iii) (vanishing moment) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$.

Definition 2.7 [17] Let $p(\cdot) \in C^{\log}$, $r \in (0, \infty)$, $q \in (1, \infty]$, $s \in [\lfloor (1/p_- - 1)\ln b/\ln \lambda_- \rfloor, \infty) \cap \mathbb{Z}_+$ with p_- as in (2.4) and A be a dilation. The *anisotropic variable atomic Hardy–Lorentz space* $\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}$ is defined to be the set of all distributions $f \in \mathcal{S}'$ satisfying that there exists a sequence of $(p(\cdot), q, s)$-atoms, $\{a_i^k\}_{i\in\mathbb{N}, k\in\mathbb{Z}}$, supported, respectively, on $\{x_i^k + B_{\ell_i^k}\}_{i\in\mathbb{N},k\in\mathbb{Z}} \subset \mathfrak{B}$ and a positive constant $\widetilde{C}$ such that $\sum_{i\in\mathbb{N}} \chi_{x_i^k + A^{j_0} B_{\ell_i^k}}(x) \leq \widetilde{C}$ for any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, with some $j_0 \in \mathbb{Z} \setminus \mathbb{N}$, and

$$f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \quad \text{in} \quad \mathcal{S}',$$

where $\lambda_i^k \sim 2^k \|\chi_{x_i^k + B_{\ell_i^k}}\|_{L^{p(\cdot)}}$ for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$ with the equivalent positive constants independent of k and i.

Moreover, for any $f \in \mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}$, define

$$\|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}} := \inf \left[\sum_{k\in\mathbb{Z}} \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i^k \chi_{x_i^k + B_{\ell_i^k}}}{\left\| \chi_{x_i^k + B_{\ell_i^k}} \right\|_{L^{p(\cdot)}}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}}^r \right]^{1/r}.$$

3. The continuity of Calderón–Zygmund operators

In this section, we get the continuity of anisotropic convolutional δ-type Calderón–Zygmund operators from $\mathcal{H}_A^{p(\cdot), r}$ to $L^{p(\cdot), r}$ or from $\mathcal{H}_A^{p(\cdot), r}$ to itself.

Let $\delta \in (0, \frac{\ln \lambda_+}{\ln b})$. We call a linear operator T is an anisotropic convolutional δ-type Calderón–Zygmund operator, if T is bounded on L^2 with kernel $\mathcal{K} \in \mathcal{S}'$ coinciding with a locally integrable function on $\mathbb{R}^n \setminus \{\vec{0}_n\}$, and satisfying that there exists a positive constant C such that, for any $x,\, y\in\mathbb{R}^n$ with $\rho(x)>b^{2\sigma}\rho(y)$,

$$|\mathcal{K}(x-y)-\mathcal{K}(x)|\le C\frac{[\rho(y)]^{\delta}}{[\rho(x)]^{1+\delta}}.$$

For any $f\in L^2$, define $T(f)(x):=\text{p.v.}\,\mathcal{K}*f(x)$.

Theorem 3.1 *Let $p(\cdot)\in C^{\log}$, $r\in(0,\,\infty)$ and $\delta\in(0,\,\frac{\ln\lambda_+}{\ln b})$. Assume that T is an anisotropic convolutional δ-type Calderón–Zygmund operator. If $p_-\in(\frac{1}{1+\delta},\,1)$ with p_- as in (2.4), then there exists a positive constant C such that, for any $\mathcal{H}_A^{p(\cdot),\,r}$,*

(i) $\|T(f)\|_{L^{p(\cdot),\,r}}\le C\|f\|_{\mathcal{H}_A^{p(\cdot),\,r}}$;

(ii) $\|T(f)\|_{\mathcal{H}_A^{p(\cdot),\,r}}\le C\|f\|_{\mathcal{H}_A^{p(\cdot),\,r}}$.

Remark 3.2 *When $p(\cdot):=p$, Theorem 3.1 coincides with [16, Theorem 6.16].*

To prove Theorem 3.1, we need some technical lemmas.

Lemma 3.3 [17, Theorem 4.8] Let $p(\cdot)\in C^{\log}$, $r\in(0,\,\infty)$, $q\in(\max\{p_+,\,1\},\,\infty]$ with p_+ as in (2.4) and $s\in[\lfloor(1/p_--1)\ln b/\ln\lambda_-\rfloor,\,\infty)\cap\mathbb{Z}_+$ with p_- as in (2.4). Then

$$\mathcal{H}_A^{p(\cdot),\,r}=\mathcal{H}_{A,\,\text{atom}}^{p(\cdot),\,q,\,s,\,r}$$

with equivalent quasi-norms.

By the proof of [17, Theorem 4.8], we obtain the following conclusion, which plays an important role in this section.

Lemma 3.4 *Let $p(\cdot)\in C^{\log}$, $r\in(0,\,\infty)$, $q\in(1,\,\infty)$ and $s\in[\lfloor(1/p_--1)\ln b/\ln\lambda_-\rfloor,\,\infty)\cap\mathbb{Z}_+$ with p_- as in (2.4). Then, for any $f\in\mathcal{H}_A^{p(\cdot),\,r}\cap L^q$, there exist $\{\lambda_i^k\}_{i\in\mathbb{N},k\in\mathbb{Z}}\subset\mathbb{C}$, dilated balls $\{x_i^k+B_{\ell_i^k}\}_{i\in\mathbb{N},k\in\mathbb{Z}}\subset\mathfrak{B}$ and $(p(\cdot),\,\infty,\,s)$-atoms $\{a_i^k\}_{i\in\mathbb{N},k\in\mathbb{Z}}$ such that*

$$f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k\ \text{ in } L^q \text{ and } \mathcal{H}_A^{p(\cdot),\,r},$$

where the series also converges almost everywhere.

Proof Let $f\in\mathcal{H}_A^{p(\cdot),\,r}\cap L^q$. For any $k\in\mathbb{Z}$, by the proof of [17, Theorem 4.8], we know that there exist

$$\left\{x_i^k\right\}_{i\in\mathbb{N}}\subset\Omega_k:=\left\{x\in\mathbb{R}^n:\ M_Nf(x)>2^k\right\},\quad \left\{\ell_i^k\right\}_{i\in\mathbb{N},k\in\mathbb{Z}}\subset\mathbb{Z},$$

a sequence of $(p(\cdot),\,\infty,\,s)$-atoms, $\{a_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}}$, supported on $\{x_i^k+B_{\ell_i^k+4\sigma}\}_{k\in\mathbb{Z},i\in\mathbb{N}}$, respectively, and $\{\lambda_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}}\subset\mathbb{C}$, such that

$$f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k=:\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}b_i^k\quad\text{in}\ \ \mathcal{S}',\tag{3.1}$$

and for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, $\operatorname{supp} b_i^k \subset x_i^k + B_{\ell_i^k+4\sigma} \subset \Omega_k$,

$$\|b_i^k\|_{L^\infty} \lesssim 2^k \quad \text{and} \quad \sharp\left\{j \in \mathbb{N}:\ (x_i^k + B_{\ell_i^k+4\sigma}) \cap (x_j^k + B_{\ell_j^k+4\sigma}) \neq \emptyset\right\} \leq R, \tag{3.2}$$

where R is as in [17, Lemma 4.7]. Moreover, by $f \in \mathcal{H}_A^{p(\cdot),r} \cap L^q$, we have, for almost every $x \in \Omega_k$, there exists a $k(x) \in \mathbb{Z}$ such that $2^{k(x)} < M_N f(x) \leq 2^{k(x)+1}$. From this, $\operatorname{supp} b_i^k \subset \Omega_k$ and (3.2), we deduce that, for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left|b_i^k(x)\right| &\sim \sum_{k\in\mathbb{Z},k\in(-\infty,k(x)]}\sum_{i\in\mathbb{N}}\left|b_i^k(x)\right| \lesssim \sum_{k\in\mathbb{Z},k\in(-\infty,k(x)]}\sum_{i\in\mathbb{N}} 2^k \chi_{x_i^k+B_{\ell_i^k+4\sigma}}(x) \\ &\sim \sum_{k\in(-\infty,k(x)]\cap\mathbb{Z}} 2^k \sim M_N f(x).\end{aligned} \tag{3.3}$$

Therefore, there exists a subsequence of the series $\{\sum_{|k|<K}\sum_{i\in\mathbb{Z}} b_i^k\}_{K\in\mathbb{N}}$, denoted still by itself without loss of generality, which converges to some measurable function F almost everywhere in $\mathbb{R}^n$.

It follows from (3.3) that, for any $K \in \mathbb{N}$ and a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned}\left|F(x) - \sum_{|k|<K} b_i^k(x)\right| &\lesssim |F(x)| + \sum_{k\in\mathbb{Z},k\in(-\infty,k(x)]}\sum_{i\in\mathbb{N}}\left|b_i^k(x)\right| \\ &\lesssim |F(x)| + M_N f(x) \lesssim M_N f(x).\end{aligned}$$

From this, the fact that $M_N(f) \in L^q$, and the dominated convergence theorem, we conclude that $F = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} b_i^k$ in L^q. By this and (3.3), we know $f = F \in L^q$ and hence

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} b_i^k \quad \text{in} \quad L^q \text{ and } \mathcal{H}_A^{p(\cdot),r},$$

and also almost everywhere. □

In what follows, we also need the definition of *anisotropic Hardy–Littlewood maximal function* $\mathcal{M}(f)$. For any $f \in L^1_{\mathrm{loc}}$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{x\in B\in\mathfrak{B}} \frac{1}{|B|}\int_B |f(z)|\,dz, \tag{3.4}$$

where $\mathfrak{B}$ is as in (2.1).

Lemma 3.5 [17, Lemma 4.3] Let $q \in (1, \infty)$. Assume that $p(\cdot) \in C^{\log}$ satisfies $1 < p_- \leq p_+ < \infty$, where p_- and p_+ are as in (2.4). Then there exists a positive constant C such that, for any sequence $\{f_k\}_{k\in\mathbb{N}}$ of measurable functions,

$$\left\|\left\{\sum_{k\in\mathbb{N}}[\mathcal{M}(f_k)]^q\right\}^{1/q}\right\|_{L^{p(\cdot)}} \leq C\left\|\left(\sum_{k\in\mathbb{N}}|f_k|^q\right)^{1/q}\right\|_{L^{p(\cdot)}},$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator as in (3.4).

Lemma 3.6 [15, Lemma 4.5] Let $p(\cdot) \in C^{\log}$, $r \in (0, \infty)$ and $q \in (\max\{p_+, 1\}, \infty)$. Then $\mathcal{H}_A^{p(\cdot), r} \cap L^q$ is dense in $\mathcal{H}_A^{p(\cdot), r}$.

The following Lemma show that variable anisotropic Hardy–Lorentz space $\mathcal{H}_A^{p(\cdot), r}$ is complete. Its proof is similar to [26, Lemma 3.9], we only need to make some minor changes. To limit the length of this article, we omit the concrete details.

Lemma 3.7 *Let* $p(\cdot) \in C^{\log}$, $r \in (0, \infty)$*. Then* $\mathcal{H}_A^{p(\cdot), r}$ *is complete.*

Proof [Proof of Theorem 3.1]

By the density, we only prove that (i) holds true for any $f \in \mathcal{H}_A^{p(\cdot), r} \cap L^q$ with $q \in (1, \infty) \cap (p_+, \infty)$. For any $f \in \mathcal{H}_A^{p(\cdot), r} \cap L^q$, from Lemma 3.4, we know that there exist numbers $\{\lambda_i^k\}_{i\in\mathbb{N}, k\in\mathbb{Z}} \subset \mathbb{C}$ and a sequence of $(p(\cdot), q, s)$-atom, $\{a_i^k\}_{i\in\mathbb{N}, k\in\mathbb{Z}}$, supported, respectively, on $\{x_i^k + B_{\ell_i^k}\}_{i\in\mathbb{N}, k\in\mathbb{Z}} \subset \mathfrak{B}$ such that

$$f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \quad \text{in } L^q$$

where $\lambda_i^k \sim 2^k \|\chi_{x_i^k + B_{\ell_i^k}}\|_{L^{p(\cdot)}}$ for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, $\sum_{i\in\mathbb{N}} \chi_{x_i^k + A^{j_0} B_{\ell_i^k}}(x) \lesssim 1$ with some $j_0 \in \mathbb{Z} \setminus \mathbb{N}$ for any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, and,

$$\|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\, q,\, s,\, r}} \sim \left[\sum_{k\in\mathbb{Z}} 2^{kr} \left\|\sum_{i\in\mathbb{N}} \chi_{x_i^k + B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r}. \tag{3.5}$$

By the fact that T is bounded on L^q, we have

$$T(f) = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^k T(a_i^k) \quad \text{in } L^q.$$

Set

$$f = \sum_{k=-\infty}^{k_0-1} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k + \sum_{k=k_0}^{\infty} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k =: F_1 + F_2 \text{ in } \ L^q.$$

Then

$$\begin{aligned} \left\|\chi_{\{x\in\mathbb{R}^n:\, T(f)(x) > 2^{k_0}\}}\right\|_{L^{p(\cdot)}} &\lesssim \left\|\chi_{\{x\in\mathbb{R}^n:\, T(F_1)(x) > 2^{k_0-1}\}}\right\|_{L^{p(\cdot)}} + \left\|\chi_{\{x\in E_{k_0}:\, T(F_2)(x) > 2^{k_0-1}\}}\right\|_{L^{p(\cdot)}} \\ &\quad + \left\|\chi_{\{x\in (E_{k_0})^{\complement}:\, T(f_2)(x) > 2^{k_0-1}\}}\right\|_{L^{p(\cdot)}} \\ &=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3\,, \end{aligned} \tag{3.6}$$

where

$$E_{k_0} := \bigcup_{k=k_0}^{\infty} \bigcup_{i\in\mathbb{N}} \left(x_i^k + A^{\sigma} B_{\ell_i^k}\right).$$

Therefore,

$$
\begin{aligned}
\mathrm{I}_1 &\lesssim \left\| \chi_{\{x\in\mathbb{R}^n:\, \sum_{k=-\infty}^{k_0-1}\sum_{i\in\mathbb{N}}\lambda_i^k T(a_i^k)(x)\chi_{x_i^k+A^\sigma B_{\ell_i^k}}(x)>2^{k_0-2}\}} \right\|_{L^{p(\cdot)}} \\
&\quad + \left\| \chi_{\{x\in\mathbb{R}^n:\, \sum_{k=-\infty}^{k_0-1}\sum_{i\in\mathbb{N}}\lambda_i^k T(a_i^k)(x)\chi_{(x_i^k+A^\sigma B_{\ell_i^k})^\complement}(x)>2^{k_0-2}\}} \right\|_{L^{p(\cdot)}} \\
&=: \mathrm{I}_{1,1} + \mathrm{I}_{1,2}\,.
\end{aligned}
\tag{3.7}
$$

For the term $\mathrm{I}_{1,1}$, from the fact that T is bounded on L^q, Remark 2.1, (3.5) and a similar proof of [17, (4.7)], we deduce that

$$
\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{I}_{1,1}\right)^r\right]^{1/r} \lesssim \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left\|\sum_{i\in\mathbb{N}}\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r} \sim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}. \tag{3.8}
$$

For the term $\mathrm{I}_{1,2}$, from the Hölder inequality and the size condition of $a_i^k(x)$, we conclude that, for any $x\in(x_i^k+A^\sigma B_{\ell_i^k})^\complement$,

$$
\begin{aligned}
\left|Ta_i^k(x)\right| &\le \int_{x_i^k+B_{\ell_i^k+\sigma}} \left|\mathcal{K}(x-y)-\mathcal{K}(x-x_i^k)\right|\left|a_i^k(y)\right|dy \\
&\lesssim \int_{x_i^k+B_{\ell_i^k+\sigma}} \frac{\rho(y-x_i^k)^\delta}{\rho(x-x_i^k)^{1+\delta}}\left|a_i^k(y)\right|\,dy \lesssim \frac{\left|x_i^k+B_{\ell_i^k}\right|^\delta}{\rho(x-x_i^k)^{1+\delta}}\left\|a_i^k\right\|_{L^q}\left|x_i^k+B_{\ell_i^k}\right|^{1/q'} \\
&\lesssim \frac{\left|x_i^k+B_{\ell_i^k}\right|^{1+\delta}}{\rho(x-x_i^k)^{1+\delta}}\frac{1}{\left\|\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}} \lesssim \left[\mathcal{M}(\chi_{x_i^k+B_{\ell_i^k}})(x)\right]^{1+\delta}\frac{1}{\left\|\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}}.
\end{aligned}
$$

By this and a similar estimate of [17, p. 374], we obtain

$$
\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{I}_{1,2}\right)^r\right]^{1/r} \lesssim \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left\|\sum_{i\in\mathbb{N}}\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r} \sim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}.
$$

Therefore, it follows from (3.7) and (3.8) that

$$
\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{I}_1\right)^r\right]^{1/r} \lesssim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}. \tag{3.9}
$$

For I_2 and I_3, by a proof similar to those of [17, (4.12) and (4.13)], we obtain

$$
\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{I}_2\right)^r\right]^{1/r} \lesssim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}} \text{ and } \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{I}_3\right)^r\right]^{1/r} \lesssim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}. \tag{3.10}
$$

Combining the estimate of (3.6), (3.9) and (3.10), we obtain

$$\|T(f)\|_{L^{p(\cdot),\,r}} \sim \left[\sum_{k\in\mathbb{Z}} 2^{kr} \left\|\chi_{\{x\in\mathbb{R}^n:\,|Tf(x)|>2^k\}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r}$$

$$\lesssim \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{I}_1\right)^r\right]^{1/r} + \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{I}_2\right)^r\right]^{1/r} + \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{I}_3\right)^r\right]^{1/r}$$

$$\lesssim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}} \sim \|f\|_{\mathcal{H}_A^{p(\cdot),\,r}},$$

which implies that $T(f)\in L^{p(\cdot),\,r}$. This finishes the proof of Theorem 3.1(i).

Now we show Theorem 3.1(ii). By Lemma 3.6, we only need to prove that (ii) holds true for any $f\in\mathcal{H}_A^{p(\cdot),r}\cap L^q$ with $q\in(1,\infty)\cap(p_+,\infty)$. Let $f\in\mathcal{H}_A^{p(\cdot),r}\cap L^q$. From Lemma 3.4, we know that there exist numbers $\{\lambda_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}\subset\mathbb{C}$ and a sequence of $(p(\cdot),\,q,\,s)$-atom, $\{a_i^k\}_{i\in\mathbb{N},k\in\mathbb{Z}}$, supported, respectively, on $\{x_i^k+B_{\ell_i^k}\}_{i\in\mathbb{N},k\in\mathbb{Z}}\subset\mathfrak{B}$ such that

$$f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k \quad \text{in } L^q$$

where $\lambda_i^k\sim 2^k\|\chi_{x_i^k+B_{\ell_i^k}}\|_{L^{p(\cdot)}}$ for any $k\in\mathbb{Z}$ and $i\in\mathbb{N}$, $\sum_{i\in\mathbb{N}}\chi_{x_i^k+A^{j_0}B_{\ell_i^k}}(x)\lesssim 1$ with some $j_0\in\mathbb{Z}\setminus\mathbb{N}$ for any $x\in\mathbb{R}^n$ and $k\in\mathbb{Z}$, and,

$$\|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}} \sim \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left\|\sum_{i\in\mathbb{N}}\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r}. \tag{3.11}$$

By the fact that T is bounded on L^q, we have

$$T(f)=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k T(a_i^k) \quad \text{in } L^q.$$

Set

$$f=\sum_{k=-\infty}^{k_0-1}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k+\sum_{k=k_0}^{\infty}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k =: F_1+F_2 \text{ in } \ L^q.$$

Then

$$\begin{aligned}
&\left\|\chi_{\left\{x\in\mathbb{R}^n:\,M_N(T(f))(x)>2^{k_0}\right\}}\right\|_{L^{p(\cdot)}} \\
&\lesssim \left\|\chi_{\left\{x\in\mathbb{R}^n:\,M_N(T(F_1))(x)>2^{k_0-1}\right\}}\right\|_{L^{p(\cdot)}} + \left\|\chi_{\left\{x\in G_{k_0}:\,M_N(T(F_2))(x)>2^{k_0-1}\right\}}\right\|_{L^{p(\cdot)}} \\
&\quad + \left\|\chi_{\left\{x\in(G_{k_0})^{\complement}:\,M_N(T(f_2))(x)>2^{k_0-1}\right\}}\right\|_{L^{p(\cdot)}} \\
&=: \mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3,
\end{aligned} \tag{3.12}$$

where M_N is as in Definition 2.3 and

$$G_{k_0} := \bigcup_{k=k_0}^{\infty}\bigcup_{i\in\mathbb{N}}\left(x_i^k+A^{\sigma}B_{\ell_i^k}\right).$$

Therefore,

$$\begin{aligned} \mathrm{J}_1 &\lesssim \left\| \chi_{\{x\in\mathbb{R}^n:\, \sum_{k=-\infty}^{k_0-1}\sum_{i\in\mathbb{N}} \lambda_i^k M_N(T(a_i^k))(x)\chi_{x_i^k+A^\sigma B_{\ell_i^k}}(x)>2^{k_0-2}\}} \right\|_{L^{p(\cdot)}} \\ &\quad + \left\| \chi_{\{x\in\mathbb{R}^n:\, \sum_{k=-\infty}^{k_0-1}\sum_{i\in\mathbb{N}} \lambda_i^k M_N(T(a_i^k))(x)\chi_{(x_i^k+A^\sigma B_{\ell_i^k})^{\complement}}(x)>2^{k_0-2}\}} \right\|_{L^{p(\cdot)}} \\ &=: \mathrm{J}_{1,1} + \mathrm{J}_{1,2}\,. \end{aligned} \tag{3.13}$$

For the term $\mathrm{J}_{1,1}$, from the fact that M_N and T are bounded on L^q, Remark 2.1, (3.11) and a similar proof of [17, (4.7)], we deduce that

$$\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{J}_{1,1}\right)^r\right]^{1/r} \lesssim \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left\|\sum_{i\in\mathbb{N}}\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r} \sim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}. \tag{3.14}$$

For the term $\mathrm{J}_{1,2}$, from the Hölder inequality, the size condition of $a_i^k(x)$, and a similar proof of [20, p.117, Lemma], we conclude that, for any $x \in (x_i^k + A^\sigma B_{\ell_i^k})^{\complement}$,

$$\begin{aligned} M_N(Ta_i^k)(x) &= \sup_{\varphi\in\mathcal{S}_N}\sup_{j\in\mathbb{Z}}\left|(\varphi_j * Ta_i^k)(x)\right| = \sup_{\varphi\in\mathcal{S}_N}\sup_{j\in\mathbb{Z}}\left|(\varphi_j*\mathcal{K}*a_i^k)(x)\right| \\ &\le \sup_{\varphi\in\mathcal{S}_N}\sup_{j\in\mathbb{Z}}\int_{x_i^k+B_{\ell_i^k+\sigma}} \left|(\varphi_j*\mathcal{K})(x-y)-(\varphi_j*\mathcal{K})(x-x_i^k)\right|\left|a_i^k(y)\right|dy \\ &\lesssim \int_{x_i^k+B_{\ell_i^k+\sigma}} \frac{\rho(y-x_i^k)^\delta}{\rho(x-x_i^k)^{1+\delta}}\left|a_i^k(y)\right|dy \lesssim \frac{\left|x_i^k+B_{\ell_i^k}\right|^\delta}{\rho(x-x_i^k)^{1+\delta}}\left\|a_i^k\right\|_{L^q}\left|x_i^k+B_{\ell_i^k}\right|^{1/q'} \\ &\lesssim \frac{\left|x_i^k+B_{\ell_i^k}\right|^{1+\delta}}{\rho(x-x_i^k)^{1+\delta}}\frac{1}{\left\|\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}} \lesssim \left[\mathcal{M}(\chi_{x_i^k+B_{\ell_i^k}})(x)\right]^{1+\delta}\frac{1}{\left\|\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}}. \end{aligned}$$

From this and a similar estimate of [17, p. 374], we deduce that

$$\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{J}_{1,2}\right)^r\right]^{1/r} \lesssim \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left\|\sum_{i\in\mathbb{N}}\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r} \sim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}.$$

Therefore, it follows from (3.13) and (3.14) that

$$\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{J}_1\right)^r\right]^{1/r} \lesssim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}. \tag{3.15}$$

For J_2 and J_3, by a proof similar to those of [17, (4.12) and (4.13)], we obtain

$$\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{J}_2\right)^r\right]^{1/r} \lesssim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}} \text{ and } \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{J}_3\right)^r\right]^{1/r} \lesssim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}. \tag{3.16}$$

It follows from the estimates of (3.12), (3.15) and (3.16) that

$$\begin{aligned}\|T(f)\|_{\mathcal{H}_A^{p(\cdot),\,r}} &= \|M_N(T(f))\|_{L^{p(\cdot),\,r}}\\ &\sim \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left\|\chi_{\{x\in\mathbb{R}^n:\,|M_N(Tf)(x)|>2^k\}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r}\\ &\lesssim \left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{J}_1\right)^r\right]^{1/r}+\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{J}_2\right)^r\right]^{1/r}+\left[\sum_{k\in\mathbb{Z}} 2^{kr}\left(\mathrm{J}_3\right)^r\right]^{1/r}\\ &\lesssim \|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}} \sim \|f\|_{\mathcal{H}_A^{p(\cdot),\,r}},\end{aligned}$$

which implies that $T(f)\in\mathcal{H}_A^{p(\cdot),\,r}$. Therefore, we complete the proof of Theorem 3.1. □

4. The dual space of variable anisotropic Hardy–Lorentz space $\mathcal{H}_A^{p(\cdot),\,r}$

In this section, we establish the dual space of $\mathcal{H}_A^{p(\cdot),\,r}$. More precisely, we prove that the dual space of $\mathcal{H}_A^{p(\cdot),\,r}$ is the variable anisotropic BMO-type space $\mathcal{BMO}_A^{p(\cdot),\,q,\,s}$.

Now, we define two new variable anisotropic BMO-type space as follows. In this article, for any $m\in\mathbb{Z}_+$, we use P_m to denote the set of polynomials on $\mathbb{R}^n$ with order not more than m. For any $B\in\mathfrak{B}$ and any locally integrable function g on $\mathbb{R}^n$, we use $P_B^m(g)$ to denote the minimizing polynomial of g with degree not greater than m, which means that $P_B^m(g)$ is the unique polynomial $f\in P_m$ such that, for any $h\in P_m$,

$$\int_B h(x)(g(x)-f(x))\,dx=0.$$

Definition 4.1 *Let A be a given dilation, $p(\cdot)\in\mathcal{P}$, s be a nonnegative integer and $q\in[1,\infty)$. Then the variable anisotropic BMO-type space $\mathcal{BMO}_A^{p(\cdot),\,q,\,s}$ is defined to be the set of all $f\in L^q_{\mathrm{loc}}$ such that*

$$\|f\|_{\mathcal{BMO}_A^{p(\cdot),\,q,\,s}} := \sup_{B\in\mathfrak{B}}\inf_{P\in P_s}\frac{|B|^{1-1/q}}{\|\chi_B\|_{L^{p(\cdot)}}}\left[\int_B |f(x)-P(x)|^q\,dx\right]^{1/q}<\infty$$

where $\mathfrak{B}$ is as in (2.1).

Definition 4.2 *Let A be a given dilation, $p(\cdot)\in\mathcal{P}$, s be a nonnegative integer and $q\in[1,\infty)$. Then the variable anisotropic BMO-type space $\widetilde{\mathcal{BMO}}_A^{p(\cdot),\,q,\,s}$ is defined to be the set of all $f\in L^q_{\mathrm{loc}}$ such that*

$$\|f\|_{\widetilde{\mathcal{BMO}}_A^{p(\cdot),\,q,\,s}} := \sup_{B\in\mathfrak{B}}\frac{|B|^{1-1/q}}{\|\chi_B\|_{L^{p(\cdot)}}}\left[\int_B |f(x)-P_B^s(f)(x)|^q\,dx\right]^{1/q}<\infty$$

where $\mathfrak{B}$ is as in (2.1).

Lemma 4.3 [3, (8.9)] Let $q\in[1,\infty]$, A be a given dilation, $f\in L^q_{\mathrm{loc}}$ and s be a nonnegative integer and $B\in\mathfrak{B}$. Then there exists a positive constant C, independent of f and B, such that

$$\sup_{x\in B}|P_B^s(f)(x)|\le C\frac{\int_B |f(x)|\,dx}{|B|}.$$

Lemma 4.4 *Let A be a given dilation, $p(\cdot)\in\mathcal{P}$, s be a nonnegative integer and $q\in[1,\infty)$. Then*

$$\mathcal{BMO}_A^{p(\cdot),\,q,\,s}=\widetilde{\mathcal{BMO}}_A^{p(\cdot),\,q,\,s}$$

with equivalent quasi-norms.

Proof By the above definition, it is easy to see that

$$\mathcal{BMO}_A^{p(\cdot),\,q,\,s}\supseteq\widetilde{\mathcal{BMO}}_A^{p(\cdot),\,q,\,s}.$$

Conversely, from Lemma 4.3 and the Hölder inequality, we conclude that, for any $B\in\mathfrak{B}$, $Q\in P_s$,

$$\begin{aligned}\left[\frac{1}{|B|}\int_B|P_B^s(Q-f)(x)|^q\,dx\right]^{1/q}&\lesssim\frac{1}{|B|}\int_B|Q(x)-f(x)|\,dx\\&\lesssim\left[\frac{1}{|B|}\int_B|Q(x)-f(x)|^q\,dx\right]^{1/q}.\end{aligned}$$

Therefore, by the Minkowski inequality, we obtain

$$\begin{aligned}&\frac{|B|}{\|\chi_B\|_{L^{p(\cdot)}}}\left[\frac{1}{|B|}\int_B|P_B^s(f)(x)-f(x)|^q\,dx\right]^{1/q}\\=&\frac{|B|}{\|\chi_B\|_{L^{p(\cdot)}}}\left[\frac{1}{|B|}\int_B|P_B^s(Q-f)(x)+f(x)-Q(x)|^q\,dx\right]^{1/q}\\\lesssim&\frac{|B|}{\|\chi_B\|_{L^{p(\cdot)}}}\left[\frac{1}{|B|}\int_B|Q(x)-f(x)|^q\,dx\right]^{1/q},\end{aligned}$$

which implies that

$$\mathcal{BMO}_A^{p(\cdot),\,q,\,s}\subseteq\widetilde{\mathcal{BMO}}_A^{p(\cdot),\,q,\,s}.$$

This completes the proof of Lemma 4.4. □

Lemma 4.5 *Let A be a given dilation, $p(\cdot)\in C^{\log}$, $r\in(0,1]$, s be a nonnegative integer and $q\in[1,\infty)$. Then, for any continuous linear functional $\mathcal{L}$ on $\mathcal{H}_A^{p(\cdot),\,r}=\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}$,*

$$\begin{aligned}\|\mathcal{L}\|_{(\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*}:=&\sup\left\{|\mathcal{L}(f)|:\|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}\le 1\right\}\\=&\sup\{|\mathcal{L}(a)|:a\text{ is }(p(\cdot),\,q,\,s)\text{−atom}\},\end{aligned}$$

where $(\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*$ denotes the dual space of $\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}$.

Proof Let a be a $(p(\cdot),\,q,\,s)$-atom. Then we have that $\|a\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}\le 1$. Therefore,

$$\sup\{|\mathcal{L}(a)|:a\text{ is }(p(\cdot),\,q,\,s)\text{−atom}\}\le\sup\left\{|\mathcal{L}(f)|:\|f\|_{\mathcal{H}_A^{p(\cdot),\,q,\,s,\,r}}\le 1\right\}.$$

Moreover, let $f \in \mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}$ and $\|f\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}} \le 1$. Then, for any $\varepsilon > 0$, we know that there exist $\{\lambda_i^k\}_{i\in\mathbb{N},k\in\mathbb{Z}} \subset \mathbb{C}$ and a sequence of $(p(\cdot),\,q,\,s)$-atoms, $\{a_i^k\}_{i\in\mathbb{N},k\in\mathbb{Z}}$, supported, respectively, on $\{x_i^k + B_{\ell_i^k}\}_{i\in\mathbb{N},k\in\mathbb{Z}} \subset \mathfrak{B}$ such that

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \quad \text{in } \mathcal{S}' \text{ and a. e.}$$

and

$$\left[\sum_{k\in\mathbb{Z}} 2^{kr} \left\|\sum_{i\in\mathbb{N}} \chi_{x_i^k + B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r} \le 1+\varepsilon.$$

Therefore, from the boundedness of $\mathcal{L}$, $\lambda_i^k \sim 2^k \|\chi_{x_i^k+B_{\ell_i^k}}\|_{L^{p(\cdot)}}$ and $r \in (0,\,1]$, we further conclude that

$$\begin{aligned}
|\mathcal{L}(g)| &\le \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} |\lambda_i^k|\,|\mathcal{L}(a_i^k)| \le \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} |\lambda_i^k| \sup\{|\mathcal{L}(a)| : a \text{ is } (p(\cdot),\,q,\,s)\text{−atom}\}\\
&\lesssim \left[\sum_{k\in\mathbb{Z}} 2^{kr} \left\|\sum_{i\in\mathbb{N}} \chi_{x_i^k + B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r} \sup\{|\mathcal{L}(a)| : a \text{ is } (p(\cdot),\,q,\,s)\text{−atom}\}\\
&\lesssim (1+\varepsilon)\sup\{|\mathcal{L}(a)| : a \text{ is } (p(\cdot),\,q,\,s)\text{−atom}\}.
\end{aligned}$$

Combined with the arbitrariness of ε and hence finishes the proof of Lemma 4.5. □

For any $q \in [1,\,\infty]$ and $s \in \mathbb{Z}_+$. Denote by L^q_{comp} the set of all functions $f \in L^q$ with compact support and

$$L^{q,\,s}_{\mathrm{comp}} := \left\{ f \in L^q_{\mathrm{comp}} : \int_{\mathbb{R}^n} f(x) x^\alpha \, dx = 0,\, |\alpha| \le s \right\}.$$

The main result of this section is as follows.

Theorem 4.6 *Let A be a given dilation, $r \in (0,\,1]$, $p(\cdot) \in C^{\log}$, $p_+ \in (0,\,1]$, $q \in (\max\{p_+,\,1\},\,\infty)$ and $s \in [\lfloor (1/p_- - 1)\ln b/\ln\lambda_-\rfloor,\,\infty) \cap \mathbb{Z}_+$ with p_- as in (2.4). Then*

$$(\mathcal{H}_A^{p(\cdot),\,r})^* = (\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^* = \mathcal{BMO}_A^{p(\cdot),\,q',\,s} = \widetilde{\mathcal{BMO}}_A^{p(\cdot),\,q',\,s}$$

in the following sense: for any $\psi \in \mathcal{BMO}_A^{p(\cdot),\,q',\,s}$ or $\widetilde{\mathcal{BMO}}_A^{p(\cdot),\,q',\,s}$, the linear functional

$$\mathcal{L}_\psi(g) := \int_{\mathbb{R}^n} \psi(x) g(x)\, dx, \tag{4.1}$$

initial defined for all $g \in L^{q,\,s}_{\mathrm{comp}}$, has a bounded extension to $\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r} = \mathcal{H}_A^{p(\cdot),\,r}$.

Conversely, if $\mathcal{L}$ is a bounded linear functional on $\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r} = \mathcal{H}_A^{p(\cdot),\,r}$, then $\mathcal{L}$ has the form as in (4.1) with a unique $\psi \in \mathcal{BMO}_A^{p(\cdot),\,q',\,s}$ or $\widetilde{\mathcal{BMO}}_A^{p(\cdot),\,q',\,s}$. Moreover,

$$\|\psi\|_{\widetilde{\mathcal{BMO}}_A^{p(\cdot),\,q',\,s}} \sim \|\psi\|_{\mathcal{BMO}_A^{p(\cdot),\,q',\,s}} \sim \|\mathcal{L}_\psi\|_{(\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*},$$

where the implicit positive constants are independent of ψ.

Remark 4.7 *We should point that, when $p(\cdot) := p \in (0,\,1]$, this result is also new.*

Proof [Proof of Theorem 4.6] By Lemmas 3.3 and 4.4, we only need to show

$$\mathcal{BMO}_A^{p(\cdot),\,q',\,s} = (\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*.$$

Firstly, we prove that

$$\mathcal{BMO}_A^{p(\cdot),\,q',\,s} \subset (\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*.$$

Let $\psi \in \mathcal{BMO}_A^{p(\cdot),\,q',\,s}$ and a be a $(p(\cdot),\,q,\,s)$-atom with $\operatorname{supp} a \subset B \in \mathfrak{B}$. Then, by the vanishing moment condition of a, Hölder's inequality and the size condition of a, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \psi(x)a(x)\,dx\right| &= \inf_{P\in P_s}\left|\int_B (\psi(x)-P(x))a(x)\,dx\right| \qquad (4.2)\\ &\le \|a\|_{L^q}\inf_{P\in P_s}\left[\int_B |\psi(x)-P(x)|^{q'}\,dx\right]^{1/q'}\\ &\le \frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}}\inf_{P\in P_s}\left[\int_B |\psi(x)-P(x)|^{q'}\,dx\right]^{1/q'}\\ &\le \|\psi\|_{\mathcal{BMO}_A^{p(\cdot),\,q',\,s}}.\end{aligned}$$

Therefore, for $\{\lambda_i^k\}_{i\in\mathbb{N},k\in\mathbb{Z}} \subset \mathbb{C}$ and a sequence $\{a_i^k\}_{i\in\mathbb{N},k\in\mathbb{Z}}$ of $(p(\cdot),\,q,\,s)$-atoms supported, respectively, on $\{x_i + B_{\ell_i^k}\}_{i\in\mathbb{N},k\in\mathbb{Z}} \subset \mathfrak{B}$ and

$$g = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \in \mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r},$$

from (4.2), we deduce that

$$\begin{aligned}|\mathcal{L}_\psi(g)| = \left|\int_{\mathbb{R}^n}\psi(x)g(x)\,dx\right| &\le \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left|\lambda_i^k\right|\left|\int_B |\psi(x)-P(x)|\left|a_i^k(x)\right|\,dx\right|\\ &\le \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left|\lambda_i^k\right|\|\psi\|_{\mathcal{BMO}_A^{p(\cdot),\,q',\,s}}\\ &\lesssim \left[\sum_{k\in\mathbb{Z}}2^{kr}\left\|\sum_{i\in\mathbb{N}}\chi_{x_i^k+B_{\ell_i^k}}\right\|_{L^{p(\cdot)}}^r\right]^{1/r}\|\psi\|_{\mathcal{BMO}_A^{p(\cdot),\,q',\,s}}\\ &\lesssim \|g\|_{\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r}}\|\psi\|_{\mathcal{BMO}_A^{p(\cdot),\,q',\,s}}.\end{aligned}$$

This implies that $\mathcal{BMO}_A^{p(\cdot),\,q',\,s} \subset (\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*$.

Next we show that $(\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^* \subset \mathcal{BMO}_A^{p(\cdot),\,q',\,s}$. For any $B \in \mathfrak{B}$, let

$$S_B : L^1(B) \to P_s$$

be the natural projection satisfying, for any $g \in L^1$ and $Q \in P_s$,

$$\int_B S_B(g)(x)Q(x)\,dx = \int_B g(x)Q(x)\,dx.$$

By a similar proof of [3, (8.9)], we obtain that, for any $B\in\mathfrak{B}$ and $g\in L^1(B)$,

$$\sup_{x\in B}|S_B(g)(x)|\lesssim\frac{\int_B|g(z)|\,dz}{|B|}.$$

Define

$$L_0^q(B):=\{g\in L^q(B):S_B(g)(x)=0\text{ and }g\text{ is not zero almost everywhere}\},$$

where $L^q(B):=\{f\in L^q:\operatorname{supp}f\subset B\}$ with $q\in(1,\infty]$ and $B\in\mathfrak{B}$,

For any $g\in L_0^q(B)$, set

$$a(x):=\frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}}\|g\|_{L^q(B)}^{-1}g(x)\chi_B(x)$$

Then a is a $(p(\cdot),\,q,\,s)$-atom. By this and Lemma 4.5, we obtain, for any $\mathcal{L}\in(\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*$ and $g\in L_0^q(B)$,

$$|\mathcal{L}(g)|\le\frac{\|\chi_B\|_{L^{p(\cdot)}}}{|B|^{1/q}}\|g\|_{L^q(B)}\|\mathcal{L}\|_{(\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*}. \tag{4.3}$$

Thus, by the Hahn-Banach theorem, it can be extended to a bounded linear functional on $L^q(B)$ with the same norm.

If $q\in(1,\infty]$, by the duality of $L^q(B)$ is $L^{q'}(B)$, we see that there exists a $\Phi\in L^{q'}(B)$ such that, for any $f\in L_0^q(B)$, $\mathcal{L}(f)=\int_B f(x)\Phi(x)\,dx$. In what follows, for any $B\in\mathfrak{B}$, let $P_s(B)$ denote all the P_s elements vanishing outside B. Now we prove that, if there exists another function $\Phi'\in L^{q'}(B)$ such that, for any $f\in L_0^q(B)$ and $\mathcal{L}(f)=\int_B f(x)\Phi'(x)\,dx$, then $\Phi'-\Phi\in P_s(B)$. For this, we only need to show that, if $\Phi,\,\Phi'\in L^1(B)$ such that, for any $f\in L_0^\infty(B)$, $\int_B f(x)\Phi'(x)\,dx=\int_B f(x)\Phi(x)\,dx$, then $\Phi-\Phi'\in P_s(B)$. In fact, for any $f\in L_0^\infty(B)$, we have

$$\begin{aligned}0&=\int_B[f(x)-S_B(f)(x)][\Phi'(x)-\Phi(x)]\,dx\\&=\int_B f(x)[\Phi'(x)-\Phi(x)]\,dx-\int_B f(x)S_B(\Phi'(x)-\Phi(x))\,dx\\&=\int_B f(x)[\Phi'(x)-\Phi(x)-S_B(\Phi'-\Phi)(x)]\,dx.\end{aligned}$$

Therefore, for a.e. $x\in B\in\mathfrak{B}$, we have

$$\Phi'(x)-\Phi(x)=S_B(\Phi'-\Phi)(x).$$

Hence $\Phi'-\Phi\in P_s(B)$. From this, we see that, for any $q\in(1,\infty]$ and $f\in L_0^q(B)$, there exists a unique $\Phi\in L^{q'}(B)/P_s(B)$ such that $\mathcal{L}(f)=\int_B f(x)\Phi(x)\,dx$.

For any $j\in\mathbb{N}$ and $g\in L_0^q(B_j)$ with $q\in(1,\infty)$, let $f_j\in L^{q'}(B_j)/P_s(B_j)$ be a unique function such that $\mathcal{L}(g)=\int_{B_j}f_j(x)g(x)\,dx$. Then, for any $i,\,j\in\mathbb{N}$ with $i<j$, $f_j|_{B_i}=f_i$. From this and the fact that, for any $g\in(\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*$, there exists a number $j_0\in\mathbb{N}$ such that $g\in L_0^q(B_{j_0})$, we conclude that, for any

$g \in (\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*$, we have

$$\mathcal{L}(g) = \int_B \psi(x)g(x)\,dx, \tag{4.4}$$

where $\psi(x) := f_j(x)$ with $x \in B_j$.

Next we show that $\psi \in \mathcal{BMO}_A^{p(\cdot),\,q',\,s}$. By [3, (8.12)], (4.3) and (4.4), we have that, for any $q \in (1,\,\infty)$, $B \in \mathfrak{B}$,

$$\inf_{P\in P_s} \|\psi - P\|_{L^{q'}(B)} = \|\psi\|_{(L_0^q(B))^*} \le \frac{\|\chi_B\|_{L^{p(\cdot)}}}{|B|^{1/q}} \|\mathcal{L}\|_{(\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*}.$$

Therefore, we have that, for any $q \in (1,\,\infty)$,

$$\begin{aligned}\|\psi\|_{\mathcal{BMO}_A^{p(\cdot),\,q',\,s}} &= \sup_{B\in\mathfrak{B}} \frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}} \inf_{P\in P_s} \|\psi - P\|_{L^{q'}(B)} = \sup_{B\in\mathfrak{B}} \frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}} \|\psi\|_{(L_0^q(B))^*} \\ &\le \|\mathcal{L}\|_{(\mathcal{H}_{A,\,\mathrm{atom}}^{p(\cdot),\,q,\,s,\,r})^*},\end{aligned}$$

which implies $\psi \in \mathcal{BMO}_A^{p(\cdot),\,q',\,s}$. This finishes the proof of Theorem 4.6. □

From Theorem 4.6, we easily obtain the following two conclusions. Moreover, the proof of Corollary 4.8 is similar to [28, Lemma 2.21], we omit the details.

Corollary 4.8 *Let A be a given dilation, $p(\cdot) \in C^{\log}$ and $s \in [\lfloor (1/p_- - 1)\ln b/\ln\lambda_-\rfloor,\,\infty) \cap \mathbb{Z}_+$ with p_- as in (2.4). Assume that $f \in \mathcal{BMO}_A^{p(\cdot),\,1,\,s}$ and $p_+ \in (0,\,1]$. Then there exist two positive constants c_1 and c_2, such that, for any $B \in \mathfrak{B}$ and $\lambda \in (0,\,\infty)$,*

$$|x \in B : |f(x) - P_B^s(f)(x)| > \lambda| \le c_1 \exp\left\{\frac{c_2\lambda|B|}{\|f\|_{\mathcal{BMO}_A^{p(\cdot),\,1,\,s}} \|\chi_B\|_{L^{p(\cdot)}}}\right\}.$$

Corollary 4.9 *Let A be a given dilation, $p(\cdot) \in C^{\log}$, $p_+ \in (0,\,1]$, $q \in (1,\,\infty)$ and $s \in [\lfloor (1/p_- - 1)\ln b/\ln\lambda_-\rfloor,\,\infty) \cap \mathbb{Z}_+$ with p_+, p_- as in (2.4). Then*

$$\mathcal{BMO}_A^{p(\cdot),\,1,\,s} = \mathcal{BMO}_A^{p(\cdot),\,q,\,s}$$

with equivalent quasi-norms.

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