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Explicit examples of constant curvature surfaces in the supersymmetric $\mathbb{C}P^2$ sigma model

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Abstract: The surfaces constructed from the holomorphic solutions of the supersymmetric (susy) $\mathbb{C}P^{N-1}$ sigma model are studied. By obtaining compact general expansion formulae having neat forms due to the properties of the superspace in which this model is described, the explicit expressions for the components of the radius vector as well as the elements of the metric and the Gaussian curvature are given in a rather natural manner. Several examples of constant curvature surfaces for the susy $\mathbb{C}P^2$ sigma model are presented.

Key words: Supersymmetric, curvature, sigma models

1. Introduction

Sigma models may be considered as generalizations of the nonlinear Lagrangians providing a phenomenological model of beta decay containing pions and a scalar meson which was then called sigma and hence the name sigma models. This phenomenological model was first introduced by Gürsey [24] and subsequently by Gell-Mann and Lévy [13]. They were further studied by introducing nonlinear terms in the pion field in [5, 6]. In three dimensions these nonlinear sigma models have been used to test various properties of the four-dimensional gauge theories which are centrally important in the description of elementary particles. An interesting class of these models is the $\mathbb{C}P^{N-1}$ sigma model that was first discovered by Eichenherr [11]. The nonlinear constraint defines $\mathbb{C}P^{N-1}$ as the target manifold and many of its interesting properties are due to its geometrical structure [38]. For example it has an associated linear scattering problem, an infinite set of conservation laws and classical solutions in the form of solitons and instantons, respectively, in (2+1)- and (1+1)-dimensions. Having such important properties this model has found wide range of applications in physics, to such areas as quantum field theory [1], fluid mechanics [4], two-dimensional gravity [16], statistical physics [35] and string theory [36].

Another important property of the $\mathbb{C}P^{N-1}$ sigma model (harmonic map in the mathematical literature) was found when the connection between the $\mathbb{C}P^1$ sigma model and the generalized Weierstrass representation has been established in $\mathbb{R}^3$ [3]. The expression describing minimal surfaces immersed in three-dimensional Euclidean space was first introduced by Enneper [12] and Weierstrass [39] and known as "Weierstrass representation". More than two decades ago this idea was used to generate surfaces in various multidimensional spaces by Konopelchenko et al. [32, 33]. Further studies were performed to obtain several variants of this rep-

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resentation [23, 25, 26, 31, 34]. However, it was soon realized that generalizing this idea for obtaining surfaces in higher-dimensional spaces was not an easy task and in order to systematically handle the problem the link between harmonic maps has been used [17–20]. In that systematic approach the idea was to write the Euler–Lagrange equations of the $\mathbb{C}P^{N-1}$ sigma model as a conservation law. This gave rise to a closed one-form and integral of it was identified as a two-dimensional surface in a real (N^2-1)-dimensional Euclidean space. From the subsequent development of the subject it was observed that the projector formalism of this model played an important role.

On the other hand in order to include the fermions in the theory these models were supersymmetrized [27, 40] and relating bosonic and fermionic fields had many far reaching consequences. Indeed, only after the generalization of this procedure to the supersymmetric case [27] the importance of the projector formalism in the construction of surfaces was fully understood. In this regard the possibility of relating the coordinates of the surface directly to the components of the projector was realized. Subsequently, by considering sums of the projectors (i.e. projectors of the Grassmanian sigma models) the idea of using projectors for the construction of surfaces was further developed in [7, 14, 15, 21, 22, 28, 37, 41]. In addition, constant curvature surfaces of these models were investigated using the generalized Veronese curve [8, 9] and particular generalizations to susy Grassmanian sigma models were recently investigated by the help of the gauge invariance of the theory [29, 30].

In this work we focus on the surfaces in $\mathbb{R}^{N^2-1}$ constructed out of the holomorphic solutions of the susy $\mathbb{C}P^{N-1}$ sigma model. We especially pay attention to the cases for which $N < 3$. In this regard after very briefly giving the basic notions of the classical $\mathbb{C}P^{N-1}$ sigma model and its generalization to the susy case described on a two-dimensional superspace, we obtain compact general expansion formulae. The latter are used for expressing the components of the radius vector explicitly. The nonvanishing elements of the metric and the Gaussian curvature are also given by the help of these expansion formulae in a manifest manner.

The outline of the paper is given as follows. In the next section, a very brief notion of the classical $\mathbb{C}P^{N-1}$ sigma model is discussed and the procedure of obtaining the canonical expressions for the components of the radius vector of the surface constructed out of the holomorphic solutions of this model is summarized. Then, in Section 3 the susy $\mathbb{C}P^{N-1}$ sigma model constructed on a superspace is described. Section 4 is devoted to the investigation of surfaces obtained from the holomorphic solutions of the two specific examples, namely, the susy $\mathbb{C}P^1$ and $\mathbb{C}P^2$ models. Finally, some conclusions and future directions are discussed in the last section.

2. The classical $\mathbb{C}P^{N-1}$ sigma model and the projector formalism

In order to maintain the self-containedness of the paper we give a brief summary of the $\mathbb{C}P^{N-1}$ sigma model. We basically follow [40].

Suppose that in Euclidean space we have an energy functional

$$S = \frac{1}{4}\int_\Omega (D_\mu z)^\dagger (D_\mu z) d\xi d\bar{\xi}\,, \qquad \begin{array}{l}\xi = \xi^1 + i\xi^2,\\ \bar{\xi} = \xi^1 - i\xi^2,\end{array} \qquad z = (z_0, z_1, \ldots, z_{N-1})^T\,, \tag{2.1}$$

with an additional constraint $z^\dagger \cdot z = 1$, then the stationary points of this functional are defined to be the $\mathbb{C}P^{N-1}$ sigma model equations. Here, we are interested in the maps $\mathbb{C} \ni (\xi, \bar{\xi}) \to (z_0, z_1, \ldots, z_{N-1})^T \in \mathbb{C}^N$, and

$$D_\mu z = \partial_\mu z - (z^\dagger \cdot \partial_\mu z) z, \qquad \partial_\mu = \partial_{\xi^\mu} \qquad \mu = 1, 2, \tag{2.2}$$

are covariant derivatives. They act on $z : \Omega \to \mathbb{C}P^{N-1}$ by the understanding that Ω is an open, connected subset of a complex plane $\mathbb{C}$, ξ and $\bar{\xi}$ are local coordinates in Ω and as usual the symbol $\dagger$ denotes Hermitian conjugation.

Having the advantage of the homogeneous coordinates $z = f(f^\dagger \cdot f)^{-1/2}$, $f \in \mathbb{C}^N$, we can introduce the gauge invariant projector formalism for the $\mathbb{C}P^{N-1}$ sigma model. Again following [40] we define the rank 1 orthogonal projector

$$P = \frac{f \otimes f^\dagger}{f^\dagger \cdot f}\,, \qquad P^\dagger = P\,, \qquad P^2 = P\,. \tag{2.3}$$

Of course, $P = (P_{ij}) \in \mathbb{C}^{N\times N}$ with $P_{ii} \in \mathbb{R}$ and $\bar{P}_{ij} = P_{ji}$, $i,j = 1,...,N$ is a Hermitian matrix. Then, the energy functional (2.1) can be expressed as

$$S = \int_\Omega \mathrm{tr}(\partial \mathrm{P}\, \bar{\partial}\mathrm{P})\, \mathrm{d}\xi \mathrm{d}\bar{\xi}\,, \qquad \partial = \frac{1}{2}\left(\partial_{\xi^1} - \mathrm{i}\partial_{\xi^2}\right), \;\; \bar{\partial} = \frac{1}{2}\left(\partial_{\xi^1} + \mathrm{i}\partial_{\xi^2}\right), \tag{2.4}$$

and the Euler–Lagrange equations become $[\partial\bar{\partial}P, P] = 0$, which could also be written as a conservation law

$$\partial[\bar{\partial}P, P] + \bar{\partial}[\partial P, P] = 0\,. \tag{2.5}$$

Among other things this formalism is important for the construction of surfaces in $\mathbb{R}^{N^2-1}$ obtained from the $\mathbb{C}P^{N-1}$ sigma model. Having expressed the Euler–Lagrange equations as a conservation law (2.5) we can construct an exact matrix-valued 1-form

$$dX = i(-[\partial P, P]d\xi + [\bar{\partial}P, P]d\bar{\xi})\,, \tag{2.6}$$

whose potential (i.e. the integral) determines a surface

$$X(\xi, \bar{\xi}) = i\int_\gamma (-[\partial P, P]d\xi + [\bar{\partial}P, P]d\bar{\xi})\,, \tag{2.7}$$

immersed in a real (N^2-1)-dimensional space. Due to the exactness of the 1-form (which indeed comes from (2.5)) the integral depends only on the end points of the curve γ and defines a mapping $X : \Omega \ni (\xi, \bar{\xi}) \to X(\xi, \bar{\xi}) \in su(N)$ where we consider $\mathbb{R}^{N^2-1} \cong su(N)$ by using the Lie algebra isomorphism (i.e. the (N^2-1)-dimensional Euclidean space is identified with the $su(N)$ algebra). This map X is called the generalized Weierstrass formula for immersion and each element of the real-valued $su(N)$ matrix function X is treated as coordinates of a two-dimensional surface immersed in $\mathbb{R}^{N^2-1}$.

However, as already mentioned in [28], the construction of surfaces in $\mathbb{R}^{N^2-1}$ based on the line integrals (2.7) can be bypassed and one can directly relate the surfaces by the so called "fundamental projector" of the holomorphic map through $X = P$. Due to the fact that in the generalization of this procedure to the susy case there happens to appear some obstacles because of the constraints of the model, in this paper neither the mixed solutions of the $\mathbb{C}P^{N-1}$ sigma model constructed out of holomorphic solutions, which are very well-known, [10, 38] nor the more fruitful approach [28, 41] of taking the sums of the projectors constructed from these mixed solutions are considered. Only the surfaces that are constructed out of the holomorphic solutions of the model are investigated.

Finally, let us finish this section by summarizing (following [28]) the procedure of obtaining the radius vector X in $\mathbb{R}^{N^2-1}$. By considering the real and imaginary parts of the off-diagonal entries of P, such as

$$\begin{aligned} X_n &:= (X_{ij})_+ = P_{ij} + \bar{P}_{ij}, \quad \text{for} \quad n = 1, \ldots, \frac{N^2-N}{2}, \ i \neq j, \\ X_n &:= (X_{ij})_- = i(P_{ij} - \bar{P}_{ij}), \ \text{for} \ n = \frac{N^2-N}{2}, \ldots, N^2-N, \ i \neq j, \end{aligned} \tag{2.8}$$

we can identify the $N(N-1)$ real components of X. They satisfy the following relation

$$\sum_{\substack{i,j=1 \\ i<j}}^{N} \left((X_{ij})_+^2 + (X_{ij})_-^2\right) = 4 \sum_{\substack{i,j=1 \\ i<j}}^{N} |P_{ij}|^2. \tag{2.9}$$

Then, the linear combination of the diagonal entries of P can be associated with the remaining components of X (notice that we are left with $(N-1)$ components). Indeed this freedom in choosing the last components give rise to different representations of the surface corresponding to the same solution of the $\mathbb{C}P^{N-1}$ sigma model. However, a canonical choice can be made by taking $X_{N^2-N+1} = P_{11} - P_{NN}$ and any orthogonal transformation made on the components of X obtained from the diagonal entries of P will leave it invariant. Such a surface can be characterized by a quadratic equation on the components of the radius vector X as:

$$\sum_{i=1}^{N^2-1} X_i^2 = \frac{2r}{N}(N-r)\,, \tag{2.10}$$

where r is the rank of the orthogonal projector P.

Of course, this equation for the surface should be understood together with the independent constraints. For the surfaces obtained from the fundamental projector (e.g., rank 1 projectors) all the 2×2 minors of P are vanishing and it is not difficult to see that among all the nonlinear constraints (due to the requirement $P^2 = P$ and $P^\dagger = P$)

$$\sum_{j=1}^{N} |P_{ij}|^2 + P_{ii}(P_{ii}-1) = 0\,, \qquad j \neq i\,, \quad i = 1, \ldots, N\,, \tag{2.11}$$

$$P_{ij}(P_{ii} + P_{jj} - 1) + \sum_{m=1}^{N} P_{im}P_{mj} = 0\,, \quad i<j, \ i \neq j \neq m, \ \ i,j = 1, \ldots, N, \tag{2.12}$$

only the following ones are independent

$$\begin{aligned} &|P_{1i}|^2 = P_{11}P_{ii}\,, && i = 2, \ldots, N, \\ &\begin{vmatrix} P_{1i} & P_{1j} \\ P_{ii} & P_{ij} \end{vmatrix} = 0\,, && j = 3, \ldots, N, \qquad i<j\,. \end{aligned} \tag{2.13}$$

Since we are embedding the surfaces obtained from solutions of the $\mathbb{C}P^{N-1}$ sigma model into the $\mathbb{R}^{N^2-1}$ with the above constraints (2.13) we are left with $2(N-1)$ real quantities (since $(N^2-1) - [N-1+(N-1)(N-$

2)] $= 2(N-1)$). This result coincides with the fact that the target space for the $\mathbb{C}P^{N-1}$ sigma model is $SU(N)/\big(SU(N-1)\times U(1)\big)$.

3. The susy $\mathbb{C}P^{N-1}$ sigma model and surfaces constructed from its holomorphic solutions

After presenting the procedure for obtaining the canonical expressions for the surfaces constructed from the holomorphic solutions of the $\mathbb{C}P^{N-1}$ sigma model, it is natural to ask what would be the analogues of those expressions for the supersymmetric case.

The $\mathbb{C}P^{N-1}$ sigma model has been supersymmetrized in [6]. Here, following [40] we first give a brief description of this model on a two-dimensional superspace and then give some general expansion formulae which have nice and simple expressions due to the properties of this superspace.

It is convenient to construct the susy $\mathbb{C}P^{N-1}$ sigma model on the two-dimensional superspace $(\xi_+,\xi_-,\theta_+,\theta_-)$ where the usual even coordinates $\xi,\bar{\xi}$ given in (2.1) are, respectively, denoted by ξ_+,ξ_- for convenience and $\theta_\pm$ are the odd coordinates defined by

$$\theta_+=\theta_1+i\theta_2\,,\qquad \theta_-=\theta_1-i\theta_2\,. \tag{3.1}$$

Here, we take θ_1 and θ_2 as real since they denote two components of a Majorana spinor θ. Then, we consider a bosonic superfield

$$\Phi(\xi_+,\xi_-,\theta_+,\theta_-)=z(\xi_+,\xi_-)+i\theta_+\chi_+(\xi_+,\xi_-)+i\theta_-\chi_-(\xi_+,\xi_-)-\frac{1}{2}\theta_+\theta_-F(\xi_+,\xi_-), \tag{3.2}$$

where z, F are N-component bosonic fields and $\chi_\pm$ are N-component fermionic fields. Taking into account that the odd fields χ_+ and χ_- anticommute with themselves as well as with the odd variables $\theta_\pm$, we write the Hermitian conjugate of Φ as

$$\Phi^\dagger(\xi_+,\xi_-,\theta_+,\theta_-)=z^\dagger(\xi_+,\xi_-)+i\theta_-\chi_+^\dagger(\xi_+,\xi_-)+i\theta_+\chi_-^\dagger(\xi_+,\xi_-)-\frac{1}{2}\theta_+\theta_-F^\dagger(\xi_+,\xi_-). \tag{3.3}$$

The constraints on even and odd component fields, which indeed follow from $\Phi^\dagger\cdot\Phi=1$, are given by

$$\begin{aligned} &z^\dagger\cdot z=1\,,\qquad \chi_\mp^\dagger\cdot z+z^\dagger\cdot\chi_\pm=0\,,\\ &F^\dagger\cdot z+z^\dagger\cdot F=2(\chi_-^\dagger\chi_--\chi_+^\dagger\chi_+)\,. \end{aligned} \tag{3.4}$$

Next, we introduce the supercovariant derivatives, the analogues of (2.2)

$$\check{D}_\pm=\check{\partial}_\pm-(\Phi^\dagger\cdot\check{\partial}_\pm\Phi)\,, \tag{3.5}$$

where $\check{\partial}_\pm$ are the generalizations of the usual derivatives ∂ and $\bar{\partial}$ (i.e. those given in (2.4) and for convenience will be denoted by $\partial_\pm$ for the rest of the article) to their super counterparts

$$\check{\partial}_\pm=-i\partial_{\theta_\pm}+\theta_\pm\partial_\pm\,. \tag{3.6}$$

The supercovariant derivatives could either act on bosonic or fermionic superfields and in terms of them the Lagrangian density and the equations of motion of the susy $\mathbb{C}P^{N-1}$ sigma model read, respectively,

$$\begin{aligned} &\mathcal{L}=2(|\check{D}_+\Phi|^2-|\check{D}_-\Phi|^2)\,,\\ &\check{D}_+\check{D}_-\Phi+|\check{D}_-\Phi|^2\,\Phi=0\,. \end{aligned} \tag{3.7}$$

Similarly to the case of the classical $\mathbb{C}P^{N-1}$ sigma model, we can give the gauge invariant projector formalism by defining the projector

$$\check{P} = \frac{\check{f} \otimes \check{f}^\dagger}{\check{f}^\dagger \cdot \check{f}}\,, \tag{3.8}$$

where we used $\Phi = \check{f}/(\check{f}^\dagger \cdot \check{f})^{-1/2}$, the analogue of $z = f(f^\dagger \cdot f)^{-1/2}$ for the nonsusy case. Then, the equations of motion could be written as a superconservation law

$$\check{\partial}_+[\check{\partial}_-\check{P}, \check{P}] + \check{\partial}_-[\check{\partial}_+\check{P}, \check{P}] = 0\,. \tag{3.9}$$

Having written the projector formalism for the supersymmetric case, we could easily apply our procedure for obtaining the canonical expressions for the supersurfaces constructed from the solutions of the susy $\mathbb{C}P^{N-1}$ sigma model. However, some obstacles start to appear when we try to generate nonholomorphic solutions from the holomorphic ones through the use of the analogue of the operator P_+ (see, e.g., [40]) due to the constraints of the model. Thus, we restrict ourselves to the holomorphic solutions

$$\check{f} = \check{f}(\xi_+, \theta_+)\,. \tag{3.10}$$

Before starting to give examples about generalization of our procedure to the supersymmetric case it would be helpful to give some general power formulae for the superfields. For a general bosonic superfield

$$a = a_0 + i\theta_+ a_1 + i\theta_- a_2 - \theta_+\theta_- a_3\,, \tag{3.11}$$

where a_0, a_3 are bosonic fields and a_1, a_2 are fermionic fields, the n^{th} power can be expressed as

$$a^n = \frac{1}{2} a_0^{n-2}\Big(n(n-1)a^2 - n(n-2)2a_0\,a + (n-1)(n-2)a_0^2\Big)\,, \tag{3.12}$$

for $n \geq 0$ and $a_0 \neq 0$ with a^2 given as

$$a^2 = 2a_0 a - a_0^2 + 2\theta_+\theta_- a_1 a_2\,. \tag{3.13}$$

For $a_0 = 0$ we have

$$a^2 = 2\theta_+\theta_- a_1 a_2\,, \qquad \text{and} \qquad a^n = 0\,, \quad n \geq 3\,. \tag{3.14}$$

The negative powers of a can be expressed as

$$a^{-n} = \frac{n}{2a_0^{n+2}}\left((n+1)a^2 - (n+2)2a_0 a + \left((n+3) + \frac{2}{n}\right)a_0^2\right)\,, \quad a_0 \neq 0\,, \tag{3.15}$$

where a^2 is given in (3.13). It is obvious that we cannot have an expression for a^{-n} if $a_0 = 0$.

4. Specific examples: the susy $\mathbb{C}P^1$ and $\mathbb{C}P^2$ cases

In this section we apply our procedure to the two specific cases, namely the susy $\mathbb{C}P^1$ and $\mathbb{C}P^2$ models. Although the susy $\mathbb{C}P^1$ case was discussed in [27], we briefly summarize it here for completeness.

4.1. The susy $\mathbb{C}P^1$ case

Using (3.8) and remembering that the overall gauge freedom allows us to choose $\check{f} = \begin{pmatrix} 1 \\ W \end{pmatrix}$, where W is a bosonic superfunction, we write the projector $\check{P}$ for this case as

$$\check{P} = \frac{1}{1+|W|^2} \begin{pmatrix} 1 & W^\dagger \\ W & |W|^2 \end{pmatrix} . \tag{4.1}$$

Then, making the canonical choice (actually for this case there is not any other choice) $\check{X}_3 = \check{P}_{11} - \check{P}_{22}$ we find the canonical expressions for the components of the radius vector

$$\check{X}_1 = \frac{W^\dagger + W}{1+|W|^2} , \quad \check{X}_2 = i\frac{W^\dagger - W}{1+|W|^2} , \quad \check{X}_3 = \frac{1-|W|^2}{1+|W|^2} . \tag{4.2}$$

Since we are only interested in the holomorphic solutions

$$W = \mathcal{F} + i\theta_+ \mathcal{G} , \qquad W^\dagger = \bar{\mathcal{F}} + i\theta_- \bar{\mathcal{G}} , \tag{4.3}$$

where $\mathcal{F}$ and $\mathcal{G}$ are, respectively, bosonic and fermionic functions of ξ_+, the explicit expressions for the components of the radius vector become

$$\begin{aligned}
\check{X}_1 &= \frac{\bar{\mathcal{F}}+\mathcal{F}}{1+|\mathcal{F}|^2} + i\theta_+ \frac{\mathcal{G}(1-\bar{\mathcal{F}}^2)}{(1+|\mathcal{F}|^2)^2} + i\theta_- \frac{\bar{\mathcal{G}}(1-\mathcal{F}^2)}{(1+|\mathcal{F}|^2)^2} + \theta_+\theta_- \frac{2|\mathcal{G}|^2(\bar{\mathcal{F}}+\mathcal{F})}{(1+|\mathcal{F}|^2)^3} , \\
\check{X}_2 &= i\frac{\bar{\mathcal{F}}-\mathcal{F}}{1+|\mathcal{F}|^2} + \theta_+ \frac{\mathcal{G}(1+\bar{\mathcal{F}}^2)}{(1+|\mathcal{F}|^2)^2} - \theta_- \frac{\bar{\mathcal{G}}(1+\mathcal{F}^2)}{(1+|\mathcal{F}|^2)^2} + i\theta_+\theta_- \frac{2|\mathcal{G}|^2(\bar{\mathcal{F}}-\mathcal{F})}{(1+|\mathcal{F}|^2)^3} , \\
\check{X}_3 &= \frac{1-|\mathcal{F}|^2}{1+|\mathcal{F}|^2} - i\theta_+ \frac{2\mathcal{G}\bar{\mathcal{F}}}{(1+|\mathcal{F}|^2)^2} - i\theta_- \frac{2\bar{\mathcal{G}}\mathcal{F}}{(1+|\mathcal{F}|^2)^2} + \theta_+\theta_- \frac{2|\mathcal{G}|^2(1-|\mathcal{F}|^2)}{(1+|\mathcal{F}|^2)^3},
\end{aligned} \tag{4.4}$$

where $|\mathcal{G}|^2$ denotes $\bar{\mathcal{G}}\mathcal{G}$. In order to get these explicit expressions we used (3.15) with $n = 1$ and $a = 1+|W|^2$. It is important to note that although the components of the radius vector are superfields, they satisfy the canonical expression for the surface

$$\check{X}_1^2 + \check{X}_2^2 + \check{X}_3^2 = 1 , \tag{4.5}$$

which is the analogue of the surface obtained from the solutions of the nonsusy $\mathbb{C}P^1$ model. The only nonzero element of the metric is $\check{g}_{+-}$ and could easily be calculated from the projector

$$\check{g}_{+-} = \frac{1}{2}\mathrm{tr}(\partial_+\check{P}\partial_-\check{P}) = \frac{\partial_+ W \partial_- W^\dagger}{2(1+|W|^2)^2} , \tag{4.6}$$

whose explicit form can immediately be written by using (3.15)

$$\begin{aligned}
\check{g}_{+-} = \frac{1}{2}\Bigg\{ &\frac{\partial_+\mathcal{F}\partial_-\bar{\mathcal{F}}}{(1+|\mathcal{F}|^2)^2} + i\theta_+\partial_+ \left(\frac{\mathcal{G}\partial_-\bar{\mathcal{F}}}{(1+|\mathcal{F}|^2)^2}\right) + i\theta_-\partial_- \left(\frac{\bar{\mathcal{G}}\partial_+\mathcal{F}}{(1+|\mathcal{F}|^2)^2}\right) \\
&-\theta_+\theta_-\partial_+\partial_- \left(\frac{|\mathcal{G}|^2}{(1+|\mathcal{F}|^2)^2}\right)\Bigg\} .
\end{aligned} \tag{4.7}$$

Since the other elements of the metric are equal to the zero (i.e. $\check{g}_{\pm\pm} = 0$), the Gaussian curvature is computed from the formula [2, 20]

$$\mathcal{K} = -\frac{1}{\check{g}_{+-}} \partial_+ \partial_- \ln \check{g}_{+-} \,, \tag{4.8}$$

and is found to be 4. Hence, we conclude that although the components of the metric and the radius vector are superfields, neither the surface nor its Gaussian curvature are changed due to those fermionic corrections.

4.2. The susy $\mathbb{C}P^2$ case

For this case using the overall gauge freedom we express the superfield as

$$\check{f} = \begin{pmatrix} 1 \\ W_1 \\ W_2 \end{pmatrix} \,, \tag{4.9}$$

where W_i $(i = 1, 2)$ are bosonic superfunctions and thus write the projector as

$$\check{P} = \frac{1}{1 + |W_1|^2 + |W_2|^2} \begin{pmatrix} 1 & W_1^\dagger & W_2^\dagger \\ W_1 & W_1 W_1^\dagger & W_1 W_2^\dagger \\ W_2 & W_2 W_1^\dagger & W_2 W_2^\dagger \end{pmatrix} \,. \tag{4.10}$$

Then using our procedure and making the canonical choice for those components of the radius vector which are obtained from the diagonal entries of $\check{P}$ (i.e. $\check{X}_7 = \check{P}_{11} - \check{P}_{33}$), we immediately get the components of the radius vector

$$\begin{aligned}
\check{X}_1 &= \frac{W_1^\dagger + W_1}{1 + |W_1|^2 + |W_2|^2} \,, & \check{X}_2 &= i \frac{W_1^\dagger - W_1}{1 + |W_1|^2 + |W_2|^2} \,, \\
\check{X}_3 &= \frac{W_2^\dagger + W_2}{1 + |W_1|^2 + |W_2|^2} \,, & \check{X}_4 &= i \frac{W_2^\dagger - W_2}{1 + |W_1|^2 + |W_2|^2} \,, \\
\check{X}_5 &= \frac{W_1 W_2^\dagger + W_2 W_1^\dagger}{1 + |W_1|^2 + |W_2|^2} \,, & \check{X}_6 &= i \frac{W_1 W_2^\dagger - W_2 W_1^\dagger}{1 + |W_1|^2 + |W_2|^2} \,, \\
\check{X}_7 &= \frac{1 - |W_2|^2}{1 + |W_1|^2 + |W_2|^2} \,, & \check{X}_8 &= \frac{2|W_1|^2 - |W_2|^2 - 1}{\sqrt{3}(1 + |W_1|^2 + |W_2|^2)} \,.
\end{aligned} \tag{4.11}$$

These components of the radius vector correspond exactly to the ones, found earlier for the nonsusy case.

Remark: It is worth mentioning that for the susy $\mathbb{C}P^2$ case the general expressions for those components of the radius vector which are obtained from the diagonal entries of the projector $\check{P}$ as given in [27]

$$\begin{aligned}
\check{X}_7 &= \pm 2\sqrt{3}\, d\, \check{P}_{11} \mp 2\sqrt{3}\, b\, \check{P}_{22} \pm 2\sqrt{3}\, \frac{b - d}{3} \,, \\
\check{X}_8 &= \mp 2\sqrt{3}\, c\, \check{P}_{11} \pm 2\sqrt{3}\, a\, \check{P}_{22} \pm 2\sqrt{3}\, \frac{c - a}{3} \,,
\end{aligned} \tag{4.12}$$

where

$$a=\frac{1}{\sqrt{3}}\cos\alpha\,,\qquad b=\frac{1}{\sqrt{3}}\sin\alpha\,,$$
$$c=\mp\frac{1}{2}\sin\alpha-\frac{1}{2\sqrt{3}}\cos\alpha\,,\quad d=-\frac{1}{2\sqrt{3}}\sin\alpha\pm\frac{1}{2}\cos\alpha\,, \tag{4.13}$$

can be transformed to the canonical form by the following transformation matrix

$$S=\begin{pmatrix} \pm\sqrt{3}\,d & \mp\sqrt{3}\,c \\ \mp(2b+d) & \pm(2a+c) \end{pmatrix}. \tag{4.14}$$

Writing the holomorphic bosonic superfunctions W_i as

$$W_i=\mathcal{F}_i+i\theta_+\mathcal{G}_i\,,\qquad W_i^\dagger=\bar{\mathcal{F}}_i+i\theta_-\bar{\mathcal{G}}_i\,,\qquad i=1,2\,, \tag{4.15}$$

where $\mathcal{F}_i$ and $\mathcal{G}_i$ are, respectively, bosonic and ferminonic functions of ξ_+, and using (3.15) the explicit forms for the components of the radius vector can be written as

$$\begin{aligned}\check{X}_i \;=\;& \frac{\eta_{i0}}{a_0}+i\theta_+\left(\frac{\eta_{i1}}{a_0}-\frac{\eta_{i0}a_1}{a_0^2}\right)+i\theta_-\left(\frac{\eta_{i2}}{a_0}-\frac{\eta_{i0}a_2}{a_0^2}\right)\\ & -\theta_+\theta_-\left(\frac{\eta_{i3}}{a_0}-\eta_{i0}\left(\frac{a_3}{a_0^2}+\frac{2a_1a_2}{a_0^3}\right)-\frac{\eta_{i2}a_1}{a_0^2}+\frac{\eta_{i1}a_2}{a_0^2}\right)\,,\end{aligned} \tag{4.16}$$

where

$$a_0=1+|\mathcal{F}_1|^2+|\mathcal{F}_2|^2\,,\qquad a_1=\bar{\mathcal{F}}_1\mathcal{G}_1+\bar{\mathcal{F}}_2\mathcal{G}_2\,,\qquad a_2=\bar{\mathcal{G}}_1\mathcal{F}_1+\bar{\mathcal{G}}_2\mathcal{F}_2\,,$$
$$a_3=|\mathcal{G}_1|^2+|\mathcal{G}_2|^2\,, \tag{4.17}$$

and

$$\begin{aligned}
&\eta_{10}=\bar{\mathcal{F}}_1+\mathcal{F}_1\,,\quad \eta_{11}=\mathcal{G}_1\,,\quad \eta_{12}=\bar{\mathcal{G}}_1\,,\quad \eta_{13}=0\,,\\
&\eta_{20}=i(\bar{\mathcal{F}}_1-\mathcal{F}_1)\,,\quad \eta_{21}=-i\mathcal{G}_1\,,\quad \eta_{22}=i\bar{\mathcal{G}}_1\,,\quad \eta_{23}=0\,,\\
&\eta_{30}=\bar{\mathcal{F}}_2+\mathcal{F}_2\,,\quad \eta_{31}=\mathcal{G}_2\,,\quad \eta_{32}=\bar{\mathcal{G}}_2\,,\quad \eta_{33}=0\,,\\
&\eta_{40}=i(\bar{\mathcal{F}}_2-\mathcal{F}_2)\,,\quad \eta_{41}=-i\mathcal{G}_2\,,\quad \eta_{42}=i\bar{\mathcal{G}}_2\,,\quad \eta_{43}=0\,,\\
&\eta_{50}=\bar{\mathcal{F}}_1\mathcal{F}_2+\bar{\mathcal{F}}_2\mathcal{F}_1,\ \eta_{51}=\bar{\mathcal{F}}_1\mathcal{G}_2+\bar{\mathcal{F}}_2\mathcal{G}_1,\ \eta_{52}=\bar{\eta}_{51},\ \eta_{53}=\bar{\mathcal{G}}_1\mathcal{G}_2+\bar{\mathcal{G}}_2\mathcal{G}_1,\\
&\eta_{60}=i(\bar{\mathcal{F}}_1\mathcal{F}_2-\bar{\mathcal{F}}_2\mathcal{F}_1),\ \eta_{61}=i(\bar{\mathcal{F}}_1\mathcal{G}_2-\bar{\mathcal{F}}_2\mathcal{G}_1),\ \eta_{62}=\bar{\eta}_{61},\ \eta_{63}=i(\bar{\mathcal{G}}_1\mathcal{G}_2-\bar{\mathcal{G}}_2\mathcal{G}_1)\,,\\
&\eta_{70}=1-|\mathcal{F}_2|^2\,,\quad \eta_{71}=-\bar{\mathcal{F}}_2\mathcal{G}_2\,,\quad \eta_{72}=-\bar{\mathcal{G}}_2\mathcal{F}_2\,,\quad \eta_{73}=-|\mathcal{G}_2|^2\,,\\
&\eta_{80}=\frac{2|\mathcal{F}_1|^2-|\mathcal{F}_2|^2-1}{\sqrt{3}},\ \eta_{81}=\frac{2\bar{\mathcal{F}}_1\mathcal{G}_1-\bar{\mathcal{F}}_2\mathcal{G}_2}{\sqrt{3}},\ \eta_{82}=\frac{2\bar{\mathcal{G}}_1\mathcal{F}_1-\bar{\mathcal{G}}_2\mathcal{F}_2}{\sqrt{3}},\\
&\qquad\qquad \eta_{83}=\frac{2|\mathcal{G}_1|^2-|\mathcal{G}_2|^2}{\sqrt{3}}\,.
\end{aligned} \tag{4.18}$$

Again the canonical expression for the surface

$$\sum_{i=1}^{8}\check{X}_i^2=\frac{4}{3}\,, \tag{4.19}$$

is recaptured for this case, albeit the components are now superfields. Of course, the equation for the surface should be understood together with the independent constraints which could be computed from the susy analogues of (2.13).

The components of the metric can be calculated from the projector (4.10) via the formulae

$$\check{g}_{\pm\pm} = \frac{1}{2}\mathrm{tr}(\partial_\pm \check{P} \partial_\pm \check{P}) , \qquad \check{g}_{+-} = \frac{1}{2}\mathrm{tr}(\partial_+ \check{P} \partial_- \check{P}) , \tag{4.20}$$

and it is easily seen that the only nonvanishing component is

$$\check{g}_{+-} = \frac{|\partial_+ W_1|^2 + |\partial_+ W_2|^2 + |W_2 \partial_+ W_1 - W_1 \partial_+ W_2|^2}{2(1 + |W_1|^2 + |W_2|^2)^2} . \tag{4.21}$$

As already stated in [27] this would be the energy density of the associated model if the derivatives $\partial_\pm$ are replaced by their super counterparts $\check{\partial}_\pm$. Using (3.15) the explicit form of $\check{g}_{+-}$ is given as

$$\begin{aligned} \check{g}_{+-} \quad = \quad & \frac{1}{2}\left\{ \frac{\gamma_0}{a_0^2} + i\theta_+ \left(\frac{\gamma_1}{a_0^2} - \frac{2\gamma_0 a_1}{a_0^3} \right) + i\theta_- \left(\frac{\gamma_2}{a_0^2} - \frac{2\gamma_0 a_2}{a_0^3} \right) \right. \\ & \left. -\theta_+\theta_- \left(\frac{\gamma_3}{a_0^2} - \frac{2\gamma_0}{a_0^3}\left(a_3 + \frac{3a_1 a_2}{a_0} \right) + \frac{2a_1\gamma_2}{a_0^3} + \frac{2\gamma_1 a_2}{a_0^3} \right) \right\} , \end{aligned} \tag{4.22}$$

where $a_i \ (i = 0, \ldots, 3)$ are given in (4.17) and $\gamma_i \ (i = 0, \ldots, 3)$ are given as

$$\begin{aligned} \gamma_0 &= |\mathcal{F}_1|^2|\mathcal{F}_2'|^2 + |\mathcal{F}_1'|^2|\mathcal{F}_2|^2 + |\mathcal{F}_1'|^2 + |\mathcal{F}_2'|^2 - \mathcal{F}_1\bar{\mathcal{F}}_1{}'\bar{\mathcal{F}}_2\mathcal{F}_2' - \bar{\mathcal{F}}_1\mathcal{F}_1'\mathcal{F}_2\bar{\mathcal{F}}_2{}', \\ \gamma_1 &= (|\mathcal{F}_2'|^2\bar{\mathcal{F}}_1 - \bar{\mathcal{F}}_2\mathcal{F}_2'\bar{\mathcal{F}}_1{}')\mathcal{G}_1 + (|\mathcal{F}_2|^2\bar{\mathcal{F}}_1{}' - \bar{\mathcal{F}}_1\mathcal{F}_2\bar{\mathcal{F}}_2{}' + \bar{\mathcal{F}}_1{}')\mathcal{G}_1{}' \\ &\quad +(|\mathcal{F}_1'|^2\bar{\mathcal{F}}_2 - \bar{\mathcal{F}}_1\mathcal{F}_1'\bar{\mathcal{F}}_2{}')\mathcal{G}_2 + (|\mathcal{F}_1|^2\bar{\mathcal{F}}_2{}' - \bar{\mathcal{F}}_2\mathcal{F}_1\bar{\mathcal{F}}_1{}' + \bar{\mathcal{F}}_2{}')\mathcal{G}_2{}', \\ \gamma_2 &= \bar{\gamma}_1, \\ \gamma_3 &= |\mathcal{F}_1|^2|\mathcal{G}_2'|^2 + |\mathcal{F}_1'|^2|\mathcal{G}_2|^2 + |\mathcal{F}_2|^2|\mathcal{G}_1'|^2 + |\mathcal{F}_2'|^2|\mathcal{G}_1|^2 + |\mathcal{G}_1'|^2 + |\mathcal{G}_2'|^2 \\ &\quad +\mathcal{F}_1\bar{\mathcal{F}}_2{}'\bar{\mathcal{G}}_1\mathcal{G}_2' + \bar{\mathcal{F}}_1\mathcal{F}_2'\bar{\mathcal{G}}_2{}'\mathcal{G}_1 + \mathcal{F}_2\bar{\mathcal{F}}_1{}'\bar{\mathcal{G}}_2\mathcal{G}_1' + \bar{\mathcal{F}}_2\mathcal{F}_1'\bar{\mathcal{G}}_1{}'\mathcal{G}_2 - \bar{\mathcal{F}}_2\mathcal{F}_1\bar{\mathcal{G}}_1{}'\mathcal{G}_2' \\ &\quad -\bar{\mathcal{F}}_2\mathcal{F}_2'\bar{\mathcal{G}}_1{}'\mathcal{G}_1 - \mathcal{F}_1\bar{\mathcal{F}}_1{}'\bar{\mathcal{G}}_2\mathcal{G}_2' - \bar{\mathcal{F}}_1{}'\mathcal{F}_2'\bar{\mathcal{G}}_2\mathcal{G}_1 - \bar{\mathcal{F}}_1\mathcal{F}_2\bar{\mathcal{G}}_2{}'\mathcal{G}_1' - \bar{\mathcal{F}}_1\mathcal{F}_1'\bar{\mathcal{G}}_2{}'\mathcal{G}_2 \\ &\quad -\mathcal{F}_2\bar{\mathcal{F}}_2{}'\bar{\mathcal{G}}_1\mathcal{G}_1' - \bar{\mathcal{F}}_2{}'\mathcal{F}_1'\bar{\mathcal{G}}_1\mathcal{G}_2. \end{aligned} \tag{4.23}$$

It is easily seen that the fermionic corrections to the metric cannot be written as total derivatives, hence, they do not vanish after integration over ξ_+ and ξ_-. The Gaussian curvature is again calculated from (4.8), however, the results of the computation are rather long to be presented here. In general it is not constant and moreover, in contrast to the susy $\mathbb{C}P^1$ case, the fermionic corrections to this curvature do not cancel. This result is expected since the Gaussian curvature for the holomorphic solutions of the bosonic $\mathbb{C}P^2$ model is also not constant in general. However, for some specific examples we do have a constant Gaussian curvature. Let us now investigate these situations.

In purely bosonic case it has been shown that the solutions of the $\mathbb{C}P^2$ model obtained from the Veronese sequence lead to a constant Gaussian curvature [2, 20, 41]. Thus, for the holomorphic solutions (4.15) we choose the bosonic part from the Veronese sequence

$$\mathcal{F}_1 = \sqrt{2}\,\xi_+ , \qquad \mathcal{F}_2 = \xi_+^2 , \tag{4.24}$$

and leave the fermionic part as general as possible,

$$\mathcal{G}_1 = \Lambda_1 \mathcal{H}_1(\xi_+)\,, \qquad \mathcal{G}_2 = \Lambda_2 \mathcal{H}_2(\xi_+)\,, \tag{4.25}$$

where Λ_1, Λ_2 are arbitrary real Grassmann constants and $\mathcal{H}_1$, $\mathcal{H}_2$ are arbitrary bosonic functions of ξ_+. The analysis of requiring the vanishing of all the fermionic contributions in (4.8) gives that the two Grassmann constants Λ_1, Λ_2 should be proportional $\Lambda_2 = \sqrt{2}\,\Lambda_1$ and the two arbitrary bosonic functions should be related as $\mathcal{H}_2(\xi_+) = \xi_+ \mathcal{H}_1(\xi_+)$. Hence, for the special holomorphic solutions of the form

$$\begin{aligned} W_1 &= \sqrt{2}\,\xi_+ + i\theta_+ \Lambda_1 \mathcal{H}_1(\xi_+)\,, \qquad W_1^\dagger = \sqrt{2}\,\xi_- + i\theta_- \Lambda_1 \bar{\mathcal{H}}_1(\xi_-)\,, \\ W_2 &= \xi_+^2 + i\theta_+ \sqrt{2}\,\Lambda_1 \xi_+ \mathcal{H}_1(\xi_+)\,, \quad W_2^\dagger = \xi_-^{\,2} + i\theta_- \sqrt{2}\,\Lambda_1 \xi_- \bar{\mathcal{H}}_1(\xi_-), \end{aligned} \tag{4.26}$$

we have a constant Gaussian curvature

$$\mathcal{K} = 2\,. \tag{4.27}$$

For the above solutions (4.26) the only nonvanishing component of the induced metric and the components of the radius vector have the following explicit expressions

$$\begin{aligned} \breve{g}_{+-} = \frac{1}{(1+|\xi_+|^2)^2} \quad &+ \quad i\theta_+ \Lambda_1 \frac{(1+|\xi_+|^2)\mathcal{H}_1' - 2\xi_- \mathcal{H}_1}{\sqrt{2}(1+|\xi_+|^2)^3} \\ &+ \quad i\theta_- \Lambda_1 \frac{(1+|\xi_+|^2)\bar{\mathcal{H}}_1' - 2\xi_+ \bar{\mathcal{H}}_1}{\sqrt{2}(1+|\xi_+|^2)^3}, \end{aligned} \tag{4.28}$$

$$\begin{aligned}
\check{X}_1 &= \frac{\sqrt{2}(\xi_+ + \xi_-)}{(1+|\xi_+|^2)^2} - i\theta_+\Lambda_1 \frac{\big(\xi_-(\xi_+ + 2\xi_-) - 1\big)\mathcal{H}_1}{(1+|\xi_+|^2)^3} \\
&\quad - i\theta_-\Lambda_1 \frac{\big(\xi_+(\xi_- + 2\xi_+) - 1\big)\bar{\mathcal{H}}_1}{(1+|\xi_+|^2)^3}\,, \\
\check{X}_2 &= -i\frac{\sqrt{2}(\xi_+ - \xi_-)}{(1+|\xi_+|^2)^2} - \theta_+\Lambda_1 \frac{\big(\xi_-(\xi_+ - 2\xi_-) - 1\big)\mathcal{H}_1}{(1+|\xi_+|^2)^3} \\
&\quad + \theta_-\Lambda_1 \frac{\big(\xi_+(\xi_- - 2\xi_+) - 1\big)\bar{\mathcal{H}}_1}{(1+|\xi_+|^2)^3}\,, \\
\check{X}_3 &= \frac{\xi_+^2 + \xi_-^2}{(1+|\xi_+|^2)^2} + i\theta_+\Lambda_1 \frac{\sqrt{2}(\xi_+ - \xi_-^3)\mathcal{H}_1}{(1+|\xi_+|^2)^3} + i\theta_-\Lambda_1 \frac{\sqrt{2}(\xi_- - \xi_+^3)\bar{\mathcal{H}}_1}{(1+|\xi_+|^2)^3}\,, \\
\check{X}_4 &= -i\frac{\xi_+^2 - \xi_-^2}{(1+|\xi_+|^2)^2} + \theta_+\Lambda_1 \frac{\sqrt{2}(\xi_+ + \xi_-^3)\mathcal{H}_1}{(1+|\xi_+|^2)^3} - \theta_-\Lambda_1 \frac{\sqrt{2}(\xi_- + \xi_+^3)\bar{\mathcal{H}}_1}{(1+|\xi_+|^2)^3}\,, \\
\check{X}_5 &= \frac{\sqrt{2}|\xi_+|^2(\xi_+ + \xi_-)}{(1+|\xi_+|^2)^2} + i\theta_+\Lambda_1 \frac{\xi_-\big(\xi_- - \xi_+(\xi_-^2 - 2)\big)\mathcal{H}_1}{(1+|\xi_+|^2)^3} \\
&\quad + i\theta_-\Lambda_1 \frac{\xi_+\big(\xi_+ - \xi_-(\xi_+^2 - 2)\big)\bar{\mathcal{H}}_1}{(1+|\xi_+|^2)^3}\,, \\
\check{X}_6 &= i\frac{\sqrt{2}|\xi_+|^2(\xi_+ - \xi_-)}{(1+|\xi_+|^2)^2} - \theta_+\Lambda_1 \frac{\xi_-\big(\xi_+(2 + \xi_-^2) - \xi_-\big)\mathcal{H}_1}{(1+|\xi_+|^2)^3} \\
&\quad + \theta_-\Lambda_1 \frac{\xi_+\big(\xi_-(2 + \xi_+^2) - \xi_+\big)\bar{\mathcal{H}}_1}{(1+|\xi_+|^2)^3}\,, \\
\check{X}_7 &= \frac{1 - |\xi_+|^2}{1 + |\xi_+|^2} - i\theta_+\Lambda_1 \frac{\sqrt{2}\,\xi_-\mathcal{H}_1}{(1+|\xi_+|^2)^2} - i\theta_-\Lambda_1 \frac{\sqrt{2}\,\xi_+\bar{\mathcal{H}}_1}{(1+|\xi_+|^2)^2}\,, \\
\check{X}_8 &= \frac{|\xi_+|^2(4 - |\xi_+|^2) - 1}{\sqrt{3}(1+|\xi_+|^2)^2} + i\theta_+\Lambda_1 \frac{\sqrt{6}\xi_-(1 - |\xi_+|^2)\mathcal{H}_1}{(1+|\xi_+|^2)^3} \\
&\quad + i\theta_-\Lambda_1 \frac{\sqrt{6}\xi_+(1 - |\xi_+|^2)\bar{\mathcal{H}}_1}{(1+|\xi_+|^2)^3}\,. \qquad (4.29)
\end{aligned}$$

For another example of constant curvature surfaces let us consider the following class of the bosonic part of the holomorphic solutions of the susy $\mathbb{C}P^2$ model:

$$\mathcal{F}_1 = c_1\,\xi_+^m\,, \qquad \mathcal{F}_2 = c_2\xi_+^n\,, \tag{4.30}$$

where c_1 and c_2 are complex constants and m and n are real constants. In purely bosonic case it has been shown that for following values of c_1, c_2, m and n

(i) $c_1 = 0$, $c_2 = 0$, $m = 0$, $n = 0$ and $m = n$ or a combination thereof,

(ii) $n = 2m$ and $|c_1|^2 = \pm 2|c_2|$ simultaneously,

the Gaussian curvature $\mathcal{K}$ is constant [20]. Following this example we choose the special holomorphic solutions of the form

$$W_1 = \xi_+ + i\theta_+\Lambda_1\xi_+ \,, \qquad W_1^\dagger = \xi_- + i\theta_-\Lambda_1\xi_- \,,$$
$$W_2 = \frac{1}{2}\xi_+^{\,2} + i\theta_+\Lambda_1\xi_+^{\,2} \,, \quad W_2^\dagger = \frac{1}{2}\xi_-^{\,2} + i\theta_-\Lambda_1\xi_-^{\,2}, \tag{4.31}$$

and obtain a constant Gaussian curvature $\mathcal{K} = 2$. The explicit expressions for the components of the induced metric and the components of the radius vector can easily be obtained by the help of the general expansion formulae.

5. Conclusion

In this paper the surfaces obtained from the holomorphic solutions of the susy $\mathbb{C}P^{N-1}$ sigma model are investigated. For this purpose the relation between the fundamental projector of these susy harmonic maps and the surfaces is used. This way of approaching the problem has the advantage of being more direct over the construction of surfaces based on line integrals and further gives the possibility of obtaining canonical expressions for the components of the radius vector. The crucial contribution of the paper is explicitly expressing these components of the radius vector as well as the components of the metric and the Gaussian curvature in a natural form by making use of the compact general expansion formulae having nice and simple expressions due to the properties of the superspace on which the susy $\mathbb{C}P^{N-1}$ sigma model is described.

In the susy $\mathbb{C}P^1$ case we conclude that although the components of the metric and the radius vector are superfields, neither the surface nor its Gaussian curvature are altered due to the fermionic corrections and hence the surface is again a two-sphere as expected. In contrast to this case, in the susy $\mathbb{C}P^2$ case, the fermionic corrections to the curvature do not vanish. This is also expected and welcome since the Gaussian curvature for the holomorphic solutions of the nonsusy $\mathbb{C}P^2$ model is not constant in general. However, for some specific examples we do have a constant Gaussian curvature and for those we have provided the explicit expressions for the components of the radius vector and the metric.

An interesting next step would be a search for more explicit examples of susy $\mathbb{C}P^{N-1}$ sigma model by considering the sum of the projectors constructed from the mixed solutions (i.e., beyond the holomorphic (nonholomorphic) solutions). In that way various other surfaces and their properties would be studied and their geometrical properties could be explicitly given. We hope to report on the developments along these lines elsewhere.

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