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On the solutions of fractional integro-differential equations involving Ulam–Hyers–Rassias stability results via ψ-fractional derivative with boundary value conditions

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Abstract: In this paper, we study boundary value problems for the impulsive integro-differential equations via ψ-fractional derivative. The contraction mapping concept and Schaefer's fixed point theorem are used to produce the main results. The results reported here are more general than those found in the literature, and some special cases are presented. Furthermore, we discuss the Ulam–Hyers–Rassias stability of the solution to the proposed system.

Key words: Fractional integro-differential equation, Schaefer's fixed point theorem, Ulam–Hyers–Rassias stability

1. Introduction

In this paper, we investigate the existence and Ulam–Hyers stability results for ψ-fractional impulsive integro-differential equation with boundary condition

$$^{c}\mathbf{D}^{\beta;\psi}\mathbf{x}(\xi) = \mathcal{G}\left(\xi, \mathbf{x}(\xi), \int_0^{\xi} \mathfrak{p}(\xi, s, \mathbf{x}(s))ds\right), \quad \text{for each} \quad \xi \in \mathbb{F} := [0,t],\ \beta \in (0,1), \tag{1.1}$$

$$\Delta\mathbf{x}|_{\xi=\xi_j} = \mathbb{I}_j(\mathbf{x}(\xi_k^-)), \quad \xi = \xi_j, \quad j = 1,2,....,n, \tag{1.2}$$

$$l\mathbf{x}(0) + m\mathbf{x}(t) = c, \tag{1.3}$$

where the ψ-Caputo fractional derivative $^{c}\mathbf{D}^{\beta;\psi}$ of order β. Let the continuous functions be $\mathcal{G} : \mathbb{F}\times\mathbb{E} \to \mathbb{E}$, $\mathfrak{p} : \Delta\times\mathbb{E} \to \mathbb{E}$ and the real constants be l, m, n with $l + m \neq 0$. Now $\Delta = \{(\xi, s) : 0 \leq s \leq \xi \leq t\}$. For the purpose of brevity, we make use of

$$\mathbb{P}\mathbf{x}(\xi) = \int_0^{\xi} \mathfrak{p}(\xi, s, \mathbf{x}(s))ds.$$

Fractional differential equations (FDEs) have newly confirmed to be important aid in modelling of numerable phenomena in different fields of science and engineering. Viscoelasticity, electrochemistry, control,

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porous media, electromagnetic, and other difficulties have a variety of applications (see [4, 21] and references in that). In latterly, there has been a dramatic increase in the number of ODE and PDE incorporating both Riemann–Liouville and Caputo fractional derivatives; see the monographs of Hilfer [24], Podlubny [33] and Samko et al. [34]. The theoretical study of these kinds of differential equations is significant for the applicability on the reality. For that reason, as a part of theoretical study, the preknowledge of the existence of a solution to FDEs is the first action for finding the analytic solution. Many natural phenomena can be formulated by BVPs of FDEs. We mention here some works on FDEs with boundary conditions (see [5],[7]–[20] and references therein). By creating various fractional integral inequalities with applying the nonlinear alternative Leray–Schauder type, Aghajani et al. investigated the solvability of a huge group of nonlinear fractional integro-differential equations. Balachandran and Kiruthika analysed the presence of results of nonlinear fractional integro-differential equations of Sobolev type with nonlocal condition in Banach spaces [6]. Very recently, Almeida [2] introduced the ψ-fractional derivative with respect to some other function. For more information on ψ-type derivatives, see [23], [35].

The remainder of this paper is formatted as follows. In the second section, we go through some helpful preliminaries. In the third section, we establish certain necessary criteria of the existence of the solutions and fourth section includes Ulam–Hyers stability.

2. Preliminaries

This part includes some basic definitions and results used throughout this paper. All continuous functions from $\mathbb{F}$ into $\mathbb{E}$ are represented in the Banach space by $C(\mathbb{F},\mathbb{E})$ and

$$\|\mathbf{x}\|_\infty := \sup\{|\mathbf{x}(\xi)| : \mathbf{x} \in C(\mathbb{F},\mathbb{E}),\ \xi \in \mathbb{F}\}.$$

Definition 2.1 *[22] A family $\mathcal{A}$ in $C(\mathbb{F},\mathbb{E})$ is equicontinuous at t in $\mathbb{F}$ if for each $\varepsilon > 0$ there exists $\delta(\varepsilon,t) > 0$ such that, for each $s \in \mathbb{F}$ with $\|t-s\| < \delta(\varepsilon,t)$, we have $\|f(t)-f(s)\| < \varepsilon$, uniformly with respect to $f \in \mathcal{A}$.*

For a complete study on ψ-fractional derivative, we are referring to [2],[36].

Definition 2.2 *Let $\beta > 0$, $\mathbb{F} = [0,t]$ be either a finite or infinite interval, and $\mathcal{G}$ be an integrable function defined on $\mathbb{F}$ with an increasing function $\psi \in C^1(\mathbb{F},\mathbb{E})$ such that $\psi^{'}(\xi) \neq 0$ for all $\xi \in \mathbb{F}$. The followings are fractional integrals and fractional derivatives of a function $\mathcal{G}$ with respect to another function ψ*

$$\begin{aligned} I^{\beta;\psi}\mathcal{G}(\xi) &:= \frac{1}{\mu(\beta)}\int_0^\xi \psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(s)ds, \\ \mathbf{D}^{\beta;\psi}\mathcal{G}(\xi) &:= \left(\frac{1}{\psi^{'}(t)}\frac{d}{d\xi}\right)^r I^{r-\beta;\psi}\mathcal{G}(\xi) \\ &= \frac{1}{\mu(r-\beta)}\left(\frac{1}{\psi^{'}(\xi)}\frac{d}{d\xi}\right)^r \int_0^\xi \psi^{'}(s)(\psi(\xi)-\psi(s))^{r-\beta-1}\mathcal{G}(s)ds, \end{aligned}$$

respectively, where $r = [\beta]+1$.

Gronwall's lemma for ψ-fractional derivative is generalised as follows. It plays a vital role in Ulam–Hyers stability's proof.

Lemma 2.3 *[36] Let* $\mathbf{H},\mathbf{L}:[0,t]\to[0,\infty)$ *be continuous functions where* $t\leq\infty$. *If* $\mathbf{L}$ *is nondecreasing,* $k\geq 0$ *and* $0<\beta<1$ *are constants such that*

$$\mathbf{H}(\xi)\leq\mathbf{L}(\xi)+k\int_0^{\xi}\psi^{'}(\xi)(\psi(\xi)-\psi(s))^{\beta-1}\mathbf{H}(s)ds,\quad \xi\in[0,t],$$

then

$$\mathbf{H}(\xi)\leq\mathbf{L}(\xi)+\int_0^{\xi}\left(\sum_{n=1}^{\infty}\frac{(k\mu(\beta))^r}{\mu(n\beta)}(\psi(\xi)-\psi(s))^{n\beta-1}\mathbf{L}(s)\right)ds,\quad \xi\in[0,t].$$

Remark 2.4 *[36] According to Lemma 2.3's hypothesis, for a nondecreasing function* $\mathbf{L}(\xi)$ *on* $[0,t]$, *we have*

$$\mathbf{H}(\xi)\leq\mathbf{L}(\xi)E_{\beta;\psi}(k\mu(\beta)(\psi(\xi))^{\beta}).$$

Theorem 2.5 *(Banach fixed point theorem) Let* C *be a nonempty closed subset of a Banach space* X. *Then there is a single fixed point for any contraction mapping* $\mathcal{P}$ *of* C *into itself.*

Theorem 2.6 *(Schaefer's fixed point theorem) Let completely continuous operator be* $\mathcal{P}:C(\mathbb{F},\mathbb{E})\to C(\mathbb{F},\mathbb{E})$. *If the set*

$$\kappa=\{\mathbf{x}\in C(\mathbb{F},\mathbb{E}):\mathbf{x}=\mu\mathcal{P}(u)\quad \textit{for any}\quad \mu\in(0,t)\}$$

is bounded, then at least, $\mathcal{P}$ *has a fixed point.*

For recent interesting results on existence and uniqueness problems of fixed points, see [10–19, 31, 32].

3. Existence results

Let us start by defining what we mean when we say the solution of the problem (1.1)–(1.3).

Definition 3.1 $u\in C^1(\mathbb{F},\mathbb{E})$ *is a function and is said to be a solution of* (1.1)–(1.3) *if u fulfilled*

$${}^{c}\mathbf{D}^{\beta;\psi}\mathbf{x}(\xi)=\mathcal{G}(t,\mathbf{x}(\xi))$$

on $\mathbb{F}$, $\Delta\mathbf{x}|_{\xi=\xi_k}=\mathbb{I}_k(\mathbf{x}(\xi_j^-))$, $\xi=\xi_j$, $j=1,2,....,n$ *with the condition* $l\mathbf{x}(0)+m\mathbf{x}(t)=n$.

The following lemma is required to obtain the existence of solutions for the equation (1.1)–(1.3).

Lemma 3.2 *[35] Let* $\beta\in(0,1)$, $\mathcal{G},\psi:\mathbb{F}\to\mathbb{E}$ *and* $\mathfrak{p}:\Delta\times\mathbb{E}\to\mathbb{E}$ *be a continuous function.* $\mathbf{x}$ *is the solution to the fractional integral equation* ψ

$$\mathbf{x}(\xi)=\mathbf{x}_0+\frac{1}{\mu(\beta)}\int_0^{\xi}\psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(s,\mathbf{x}(s),\mathbb{P}\mathbf{x}(s))ds$$

if and only if $\mathbf{x}$ *is the solution to the IVP for the* ψ*-fractional differential equation.*

$${}^{c}\mathbf{D}^{\beta;\psi}\mathbf{x}(\xi)=\mathcal{G}(\xi,\mathbf{x}(\xi),\mathbb{P}\mathbf{x}(\xi)),\quad \textit{for each}\quad \xi\in\mathbb{F}:=[0,t],\quad \beta\in(0,1),\quad \mathbf{x}(0)=\mathbf{x}_0.$$

We have the following lemma as a result of Lemma 3.2 which will be applied in the later.

Lemma 3.3 *Let* $\beta \in (0,1)$, $\mathcal{G}, \psi : I \to \mathbb{E}$ *and* $\mathfrak{p} : \Delta \times \mathbb{E} \to \mathbb{E}$ *be a continuous function. The fractional impulsive integral equation* ψ *has a solution called* $\mathbf{x}$

$$\begin{aligned}\mathbf{x}(\xi) &= \frac{1}{\mu(\beta)}\int_0^{\xi}\psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(s,\mathbf{x}(s),\mathbb{P}\mathbf{x}(s))ds\\ &\quad -\frac{1}{l+m}\left[\frac{m}{\mu(\beta)}\int_0^{t}\psi^{'}(s)(\psi(t)-\psi(s))^{\beta-1}\mathcal{G}(s,\mathbf{x}(s),\mathbb{P}\mathbf{x}(s))ds-n\right]\\ &\quad +\sum_{0<\xi_j<\xi}\mathbb{I}_j(\mathbf{x}(\xi_j^-))\end{aligned}$$

if and only if $\mathbf{x}$ *is the solution to the BVP for the* ψ*-fractional differential equation*

$${}^{c}\mathbf{D}^{\beta;\psi}\mathbf{x}(\xi) = \mathcal{G}(\xi,\mathbf{x}(\xi),\mathbb{P}\mathbf{x}(\xi)), \quad \textit{for each} \quad \xi \in \mathbb{F} := [0,t], \quad \beta \in (0,1),$$

$$\Delta\mathbf{x}|_{\xi=\xi_j} = \mathbb{I}_j(\mathbf{x}(\xi_j^-)), \quad \xi = \xi_j, \quad j = 1,2,....,n,$$

$$l\mathbf{x}(0) + m\mathbf{x}(t) = n.$$

We impose the following assumptions:

(H1) $\mathcal{G} : \mathbb{F} \times \mathbb{E} \to \mathbb{E}$ is a continuous function.

(H2) $\mathbf{K} > 0$ is constant such that

$$|\mathcal{G}(\xi,\mathcal{U}) - \mathcal{G}(\xi,\overline{\mathcal{U}})| \le \mathbf{K}|\mathcal{U} - \overline{\mathcal{U}}|, \quad \text{for all} \quad \xi \in \mathbb{F}, \quad \forall\, \mathcal{U},\overline{\mathcal{U}} \in \mathbb{E}.$$

(H3) $\mathbf{K} > 0$ is constant such that

$$|\mathcal{G}(\xi,\mathcal{U},\overline{\mathcal{U}})| \le \mathbf{K} \quad \text{for each} \quad \xi \in \mathbb{F}, \quad \forall\, \mathcal{U},\overline{\mathcal{U}} \in \mathbb{E}.$$

(H4) $\mathfrak{p} : \Delta \times \mathbb{E} \to \mathbb{E}$ is a continuous function and $\mathbb{P}_1 > 0$ is a constant such that

$$|\mathfrak{p}(\xi,s,\mathcal{U}) - f(\xi,s,\overline{\mathcal{U}})| \le \mathbb{P}_1|\mathcal{U} - \overline{\mathcal{U}}|, \quad \forall\, \mathcal{U},\overline{\mathcal{U}} \in \mathbb{E}.$$

(H5) There exists $\rho > 0$ that says

$$\|I_k(x) - I_k(y)\| \le \rho\|x - y\|, \quad \text{for all} \quad x,y \in X \quad \textit{with} \quad k = 1,2,...,m.$$

Also, there exists $\rho_1 > 0$ that says

$$\|I_k(x)\| \le \rho_1, \quad \forall \quad x \in X \quad \textit{and} \quad k = 1,2,...,m.$$

Banach fixed point theorem gives our first result.

Theorem 3.4 *Suppose (H1), (H2), (H4) and (H5) hold. If*

$$\frac{\mathbf{K}(1+\mathbb{P}_1)(\psi(t))^{\beta}}{\mu(\beta+1)}\left(1 + \frac{|m|}{|l+m|}\right) + \rho\|\mathbf{x} - \mathfrak{v}\|_{\infty} < 1, \tag{3.1}$$

on $\mathbb{F}$, *there is only one solution to the BVP* (1.1)–(1.3).

Proof Transform (1.1)–(1.3) into a fixed point problem. Denote $\Phi = C(\mathbb{F}, \mathbb{E})$. Let $\mathcal{P} : \Phi \to \Phi$ be a operator and defined by

$$\begin{aligned} \mathcal{P}\mathbf{x}(\xi) &= \frac{1}{\mu(\beta)} \int_0^{\xi} \psi^{'}(s)(\psi(\xi) - \psi(s))^{\beta-1} \mathcal{G}(s, \mathbf{x}(s), \mathbb{P}\mathbf{x}(s))ds \\ &\quad - \frac{1}{l+m} \left[\frac{m}{\mu(\beta)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\beta-1} \mathcal{G}(s, \mathbf{x}(s), \mathbb{P}\mathbf{x}(s))ds - n \right] \\ &\quad + \sum_{0<\xi_j<\xi} \mathbb{I}_j(\mathbf{x}(\xi_j^-)). \end{aligned} \tag{3.2}$$

Noticeably, the operator $\mathcal{P}$'s fixed points are the solutions to the problem (1.1)–(1.3). To verify that $\mathcal{P}$ defined by (3.2) has a fixed point, we will apply Banach contraction principle. Firstly, we show that $\mathcal{P}$ is a contraction.

If $\mathbf{x}, \mathfrak{v} \in \Phi$, then for every $\xi \in \mathbb{F}$, we have

$$\begin{aligned} |\mathcal{P}(\mathbf{x})(\xi) - \mathcal{P}(\mathfrak{v})(\xi)| &\leq \frac{1}{\mu(\beta)} \int_0^{\xi} \psi^{'}(s)(\psi(\xi) \\ &\quad -\psi(s))^{\beta-1} |\mathcal{G}(s, \mathbf{x}(\xi), \mathbb{P}\mathbf{x}(s)) - \mathcal{G}(s, \mathfrak{v}(\xi), \mathbb{P}\mathfrak{v}(s))|ds \\ &\quad - \frac{|m|}{|l+m|\mu(\beta)} \int_0^t \psi^{'}(s)(\psi(t) \\ &\quad -\psi(s))^{\beta-1} |\mathcal{G}(s, \mathbf{x}(\xi), \mathbb{P}\mathbf{x}(s)) - \mathcal{G}(s, \mathfrak{v}(s), \mathbb{P}\mathfrak{v}(s))|ds \\ &\quad + \sum_{0<\xi_j<\xi} \|\mathbb{I}_j\mathbf{x}(\xi_j^-)) - \mathbb{I}_j\mathfrak{v}(\xi_j^-))\| \\ &\leq \frac{\mathbf{K}(1+\mathbb{P}_1)\|\mathbf{x} - \mathfrak{v}\|_\infty}{\mu(\beta)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\beta-1} ds \\ &\quad - \frac{|m|\mathbf{K}(1+\mathbb{P}_1)\|\mathbf{x} - \mathfrak{v}\|_\infty}{|l+m|\mu(\beta)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\beta-1} ds \\ &\quad + \rho\|\mathbf{x} - \mathfrak{v}\| \\ &\leq \left[\frac{\mathbf{K}(1+\mathbb{P}_1)(\psi(t))^{\beta}}{\mu(\beta+1)} \left(1 + \frac{|m|}{|l+m|}\right) + \rho \right] \|\mathbf{x} - \mathfrak{v}\|_\infty. \end{aligned}$$

Therefore we obtain

$$\|\mathcal{P}(\mathbf{x}) - \mathcal{P}(\mathfrak{v})\|_\infty \leq \left[\frac{\mathbf{K}(1+\mathbb{P}_1)(\psi(t))^{\beta}}{\mu(\beta+1)} \left(1 + \frac{|m|}{|l+m|}\right) + \rho \right] \|\mathbf{x} - \mathfrak{v}\|_\infty. \tag{3.3}$$

Hence $\mathcal{P}$ is a contraction by (3.1). We derive that $\mathcal{P}$ has a fixed point, which is a solution of the problem (1.1)–(1.3) based on a consequence of Banach fixed point theorem. □

The following result comes from Schaefer's fixed point theorem.

Theorem 3.5 *Suppose (H1), (H3), (H4) and (H5) hold. Then there exists at least one solution on* $\mathbb{F}$ *to the BVP* (1.1)–(1.3).

Proof Schaefer's fixed point theorem will be used to verify that $\mathcal{P}$ defined by (3.2) has a fixed point. Proof will be presented in stages.

Claim 1: Let $\mathcal{P}$ be a continuous operator and $\{\mathbf{x}_n\}$ be a sequence such that $\mathbf{x}_n \to \mathbf{x}$ in Φ. Then for each $\xi \in \mathbb{F}$, we find

$$\begin{aligned}
|\mathcal{P}(\mathbf{x_n})(\xi) - \mathcal{P}(\mathbf{x})(\xi)| \leq\ & \frac{1}{\mu(\beta)} \int_0^\xi \psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1} |\mathcal{G}(s, \mathbf{x}_n(s), \mathbb{P}\mathbf{x}_n(s)) - \mathcal{G}(s, \mathbf{x}(s), \mathbb{P}\mathbf{x}(s))| ds \\
& + \frac{|m|}{|l+m|\mu(\beta)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\beta-1} |\mathcal{G}(s, \mathbf{x}_n(s), \mathbb{P}\mathbf{x}_n(s)) - \mathcal{G}(s, \mathbf{x}(s), \mathbb{P}\mathbf{x}(s))| ds \\
& + \sum_{0<\xi_j<\xi} \|\mathbb{I}_j(\mathbf{x}_n(\xi_j^-)) - \mathbb{I}_j(\mathbf{x}(\xi_j^-))\| \\
\leq\ & \frac{1}{\mu(\beta)} \int_0^\xi \psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1} \sup_{s\in\mathbb{F}} |\mathcal{G}(s, \mathbf{x}_n(s), \mathbb{P}\mathbf{x}_n(s)) - \mathcal{G}(s, \mathbf{x}(s), \mathbb{P}\mathbf{x}(s))| ds \\
& + \frac{|m|}{|l+m|\mu(\beta)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\beta-1} \sup_{s\in\mathbb{F}} |\mathcal{G}(s, \mathbf{x}_n(s), \mathbb{P}\mathbf{x}_n(s)) - \mathcal{G}(s, \mathbf{x}(s), \mathbb{P}\mathbf{x}(s))| ds \\
& + \sum_{0<\xi_j<\xi} \|\mathbb{I}_j(\mathbf{x}_n(\xi_j^-)) - \mathbb{I}_j(\mathbf{x}(\xi_j^-))\| \\
\leq\ & \frac{1}{\mu(\beta)} \left[\int_0^\xi \psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1} ds + \frac{|m|}{|l+m|} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\beta-1} ds \right] \\
& \|\mathcal{G}(\cdot, \mathbf{x}_n(\cdot), \mathbb{P}\mathbf{x}_n(\cdot)) - \mathcal{G}(\cdot, \mathbf{x}(\cdot), \mathbb{P}\mathbf{x}(\cdot))\|_\infty + \|\mathbf{x}_n - \mathbf{x}\|\rho \\
\leq\ & \frac{(\psi(t))^\beta}{\mu(\beta+1)} \left(1 + \frac{|m|}{|l+m|}\right) \|\mathcal{G}(\cdot, \mathbf{x}_n(\cdot), \mathbb{P}\mathbf{x}_n(\cdot)) - \mathcal{G}(\cdot, \mathbf{x}(\cdot), \mathbb{P}\mathbf{x}(\cdot))\|_\infty + \|\mathbf{x}_n - \mathbf{x}\|\rho.
\end{aligned}$$

Since $\mathcal{G}$ is a continuous function, then we get

$$\|\mathcal{P}(\mathbf{x}_n) - \mathcal{P}(\mathbf{x})\|_\infty \leq \frac{(\psi(t))^\beta}{\mu(\beta+1)} \left(1 + \frac{|m|}{|l+m|}\right) \|\mathcal{G}(\cdot, \mathbf{x}_n(\cdot), \mathbb{P}\mathbf{x}_n(\cdot)) - \mathcal{G}(\cdot, \mathbf{x}(\cdot), \mathbb{P}\mathbf{x}(\cdot))\|_\infty + \|\mathbf{x}_n - \mathbf{x}\|\rho.$$

Claim 2: In Φ, the operator $\mathcal{P}$ maps bounded sets into bounded sets. In fact, it is sufficient to prove that there exists a positive constant ζ for each $\mathbf{x} \in \mathbf{D}_q = \{\mathbf{x} \in \Phi : \|\mathbf{x}\|_\infty \leq q\}$ where $q > 0$ such that

$\|\mathcal{P}(\mathbf{x})\|_\infty \leq \zeta$. By (H3), we have

$$\begin{aligned}
|\mathcal{P}(\mathbf{x})(\xi)| &\leq \frac{1}{\mu(\beta)}\int_0^\xi \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}|\mathcal{G}(s,\mathbf{x}(s),\mathbb{P}\mathbf{x}(s))|ds \\
&\quad +\frac{|m|}{|l+m|}\frac{1}{\mu(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}|\mathcal{G}(s,\mathbf{x}(s),\mathbb{P}\mathbf{x}(s))|ds+\frac{|n|}{|l+m|} \\
&\quad +\sum_{0<\xi_j<\xi}\|\mathbb{I}_j(\mathbf{x}(\xi_j^-))\| \\
&\leq \frac{\mathbf{K}}{\mu(\beta)}\int_0^\xi \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}ds \\
&\quad +\frac{|m|}{|l+m|}\frac{\mathbf{K}}{\mu(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds+\frac{|n|}{|l+m|}+\rho_1 \\
&\leq \frac{\mathbf{K}}{\mu(\beta+1)}(\psi(t))^\beta+\frac{\mathbf{K}|m|}{|\beta+1|\mu(l+m)}(\psi(t))^\beta+\frac{|n|}{|l+m|}+\rho_1
\end{aligned}$$

for every $\xi \in \mathbb{F}$. Therefore we find

$$\|\mathcal{P}(\mathbf{x})\|_\infty \leq \frac{\mathbf{K}}{\mu(\beta+1)}(\psi(t))^\beta+\frac{\mathbf{K}|m|}{|l+m|\mu(\beta+1)}(\psi(t))^\beta+\frac{|n|}{|l+m|}+\rho_1 := \zeta$$

Claim 3: The operator $\mathcal{P}$ maps bounded sets into equicontinuous sets of Φ. Let $\xi_1,\xi_2 \in \mathbb{F}$ with $\xi_1<\xi_2$ and $\mathbf{D}_q$ be the bounded set with $\mathbf{x}\in\mathbf{D}_q$ in Φ as in Claim 2. Then

$$\begin{aligned}
|\mathcal{P}(\mathbf{x})(\xi_2)-\mathcal{P}(\mathbf{x})(\xi_1)| &\leq \left|\frac{1}{\mu(\beta)}\int_0^{\xi_1}\psi'(s)\left[(\psi(\xi_2)-\psi(s))^{\beta-1}-(\psi(\xi_1)-\psi(s))^{\beta-1}\right]\mathcal{G}(s,\mathbf{x}(s),\mathbb{P}\mathbf{x}(s))ds\right. \\
&\quad \left.+\frac{1}{\mu(\beta)}\int_{\xi_1}^{\xi_2}\psi'(s)(\psi(\xi_2)-\psi(s))^{\beta-1}\mathcal{G}(s,\mathbf{x}(s),\mathbb{P}\mathbf{x}(s))ds\right| \\
&\quad +\sum_{0<\xi_j<\xi}\|\mathbb{I}_j(\mathbf{x}(\xi_2^-))-\mathbb{I}_j(\mathbf{x}(\xi_1^-))\| \\
&\leq \frac{\mathbf{K}}{\mu(\beta)}\int_0^{\xi_1}\psi'(s)\left[(\psi(\xi_2)-\psi(s))^{\beta-1}-(\psi(\xi_1)-\psi(s))^{\beta-1}\right]ds \\
&\quad +\frac{\mathbf{K}}{\mu(\beta)}\int_{\xi_1}^{\xi_2}\psi'(s)(\psi(\xi_2)-\psi(s))^{\beta-1}ds+\|\mathbf{x}(\xi_2)-\mathbf{x}(\xi_1)\|\rho \\
&\leq \frac{\mathbf{K}}{\mu(\beta+1)}(\psi(\xi_2)-\psi(\xi_1))^\beta+\frac{\mathbf{K}}{\mu(\beta+1)}((\psi(\xi_1))^\beta-(\psi(\xi_2))^\beta)\|\mathbf{x}(\xi_2)-\mathbf{x}(\xi_1)\|\rho.
\end{aligned}$$

Since $\xi_1\to\xi_2$, the above inequality's right-hand side tends to zero. We will able to finish that $\mathcal{G}:\Phi\to\Phi$ is continuous because of a result of Stages 1 to 3 together with the Arzela–Ascoli theorem.

Claim 4: A priori bounds. Now it remains to prove that

$$\kappa=\{\mathbf{x}\in\Phi:\mathbf{x}=\mu\mathcal{P}(\mathbf{x})\quad \text{for any}\quad \mu\in(0,1)\}$$

is bounded. If $\mathbf{x} \in \kappa$, then $\mu\mathcal{P}(\mathbf{x})$ for any $\mu \in (0,1)$. Hence, for each $\xi \in \mathbb{F}$, we get the following:

$$\begin{aligned}\mathbf{x}(\xi) &= \mu\Bigg[\frac{1}{\mu(\beta)}\int_0^{\xi}\psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(s,\mathbf{x}(s),\mathbb{P}\mathbf{x}(s))ds \\ &\quad -\frac{1}{l+m}\left[\frac{m}{\mu(\beta)}\int_0^{t}\psi^{'}(s)(\psi(t)-\psi(s))^{\beta-1}\mathcal{G}(s,\mathbf{x}(s),\mathbb{P}\mathbf{x}(s))ds-n\right]\Bigg] \\ &\quad +\sum_{0<\xi_j<\xi}\mathbb{I}_j(\mathbf{x}(\xi_j^-)).\end{aligned}$$

We complete this stage by considering the estimation in Claim 2. As a result of Schaefer's fixed point theorem, we finish the proof that $\mathcal{P}$ has fixed point which is the solution of the problem (1.1)–(1.3). □

4. Ulam–Hyers–Rassias stability

In this part, we study the Ulam stability of BVP for ψ-fractional differential equations (1.1)–(1.3). There are many works on the Ulam stability of solutions for fractional differential equations. We mention here some works [3],[25]–[30]; also see the references cited therein. A similar idea can be found in [7]. However, for the next problem, we will look at Ulam stability.

$$^{c}\mathbf{D}^{\beta;\psi}\mathbf{x}(\xi) = \mathcal{G}(\xi,\mathbf{x}(\xi),\mathbb{P}\mathbf{x}(\xi)), \quad \xi \in \mathbb{F} := [0,t]. \tag{4.1}$$

Let $\mathcal{G} : \mathbb{F}\times\mathbb{E} \to \mathbb{E}$, $\mathfrak{p} : \Delta\times\mathbb{E} \to \mathbb{E}$ be continuous functions. For the simplicity, we make use of

$$\mathbb{P}\mathbf{x}(\xi) = \int_0^{\xi}\mathfrak{p}(\xi,s,\mathbf{x}(s))ds.$$

We pay attention to the topic of a novel operator with respect to another function, as it covers many fractional systems that are special cases for various values. More precisely, the existence and the Ulam–Hyers–Rassias stability of solutions to the system (4.1) are obtained in weighted spaces by using standard Schaefer's fixed point theorem and the following inequalities:

$$|^{c}\mathbf{D}^{\beta;\psi}\mathbf{H}(\xi) - \mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))| \le \epsilon, \quad \xi \in \mathbb{F}, \tag{4.2}$$

$$|^{c}\mathbf{D}^{\beta;\psi}\mathbf{H}(\xi) - \mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))| \le \epsilon\varphi(\xi), \quad \xi \in \mathbb{F}, \tag{4.3}$$

$$|^{c}\mathbf{D}^{\beta;\psi}\mathbf{H}(\xi) - \mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))| \le \varphi(\xi), \quad \xi \in \mathbb{F}. \tag{4.4}$$

Definition 4.1 *Equation (4.1) is Ulam–Hyers stable if there exists $C_f > 0$ which is a real number such that for each $\epsilon > 0$ and for every solution $\mathbf{H} \in \Phi$ of inequality (4.2) there exists a solution $\mathbf{x} \in \Phi$ of Equation (4.1) along with*

$$|\mathbf{H}(\xi) - \mathbf{x}(\xi)| \le C_{f^{\epsilon}}, \quad \xi \in J.$$

Definition 4.2 *Equation (4.1) is generalized Ulam–Hyers stable if there exists* $\psi_f \in C([0,\infty),[0,\infty)), \psi_f(0)=0$ *such that for one solution* $\mathbf{H}\in\Phi$ *of inequality (4.2) there exists a solution* $\mathbf{x}\in\Phi$ *of Equation (4.1) with*

$$|\mathbf{H}(\xi)-\mathbf{x}(\xi)| \le \psi_{f^\epsilon}, \quad \xi \in J.$$

Definition 4.3 *Equation (4.1) is Ulam–Hyers–Rassias stable with respect to* $\varphi\in\Phi$ *if there exists a real number* $C_f>0$ *such that for each* $\epsilon>0$ *and for any solution* $\mathbf{H}\in\Phi$ *of inequality (4.3) there exists a solution* $\mathbf{x}\in\Phi$ *of Equation (4.1) with*

$$|\mathbf{H}(\xi)-\mathbf{x}(\xi)| \le C_{f^\epsilon}\varphi(\xi), \quad \xi \in J.$$

Definition 4.4 *Equation (4.1) is generalized Ulam–Hyers–Rassias stable with respect to* $\varphi\in\Phi$ *if there exists a real number* $C_{f,\varphi}>0$ *such that for each solution* $\mathbf{H}\in\Phi$ *of inequality (4.4) there exists a solution* $\mathbf{x}\in\Phi$ *of Equation (4.1) with*

$$|\mathbf{H}(\xi)-\mathbf{x}(\xi)| \le C_{f,\varphi}\varphi(\xi), \quad \xi \in J.$$

Remark 4.5 *A function* $\mathbf{H}\in\Phi$ *is a solution of (4.2) if and only if there exists a function* $g\in\Phi$ *(which depend on* $\mathbf{H}$*) such that*

$$(1)\ \ |g(\xi)|\le\epsilon, \quad \xi\in\mathbb{F}; \quad (2)\ \ {}^c\mathbf{D}^{\beta;\psi}\mathbf{H}(\xi)=\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))+g(\xi), \quad \xi\in\mathbb{F}.$$

Remark 4.6 *Let* $\beta\in(0,1)$*. If* $\mathbf{H}\in\Phi$ *is a solution of the inequality (4.2), then the inequality has a solution of* $\mathbf{H}$

$$\left|\mathbf{H}(\xi)-\mathfrak{A}_{\mathbf{H}}-\frac{1}{\lambda(\beta)}\int_0^\xi \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))ds\right| \le \epsilon\frac{(\psi(t))^\beta}{\lambda(\beta+1)}\left(1+\frac{|m|}{|l+m|}\right).$$

In fact, by Remark 4.5, we have

$${}^c\mathbf{D}^{\beta;\psi}\mathbf{H}(\xi)=\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))+g(\xi), \quad \textit{where } \xi\in\mathbb{F},$$

$$\begin{aligned}\mathbf{H}(\xi) \;=\; & \mathfrak{A}_{\mathbf{H}}+\frac{1}{\lambda(\beta)}\int_0^\xi \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))ds \\ & +\frac{1}{\lambda(\beta)}\int_0^\xi \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}g(s)ds \\ & -\left(\frac{m}{l+m}\right)\frac{1}{\lambda(\beta)}\int_0^\xi \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}g(s)ds, \quad \xi\in\mathbb{F},\end{aligned}$$

with

$$\mathfrak{A}_{\mathbf{H}}=\frac{1}{l+m}\left[n-\frac{m}{\lambda(\beta)}\int_0^t \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))ds\right].$$

From this, it follows that

$$\left|\mathbf{H}(\xi) - \mathfrak{A}_{\mathbf{H}} - \frac{1}{\lambda(\beta)}\int_0^{\xi}\psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))ds\right|$$

$$\begin{aligned} &= |\frac{1}{\lambda(\beta)}\int_0^{\xi}\psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}g(s)ds \\ &\quad -\left(\frac{m}{l+m}\right)\frac{1}{\lambda(\beta)}\int_0^{\xi}\psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}g(s)ds| \\ &\leq \frac{1}{\lambda(\beta)}\int_0^{\xi}\psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}|g(s)|ds \\ &\quad -\left(\frac{m}{l+m}\right)\int_0^{\xi}\psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}|g(s)|ds \\ &\leq \epsilon\frac{(\psi(t))^{\beta}}{\lambda(\beta+1)}\left(1+\frac{|m|}{|l+m|}\right). \end{aligned}$$

Remark 4.7 *It is obvious that Definition 4.1 ⇒ Definition 4.2 and Definition 4.3 ⇒ Definition 4.4.*

Remark 4.8 *A solution of the ψ-fractional diferential equations with boundary condition inequality (4.2) is said to be an ϵ-solution of problem (4.1).*

Theorem 4.9 *Suppose (H1), (H2), (H4), (H5) and (3.1) hold. Then, the problem (1.1)–(1.3) is Ulam–Hyers stable.*

Proof Let $\epsilon > 0$, $\mathbf{H} \in \Phi$ be a function satisfying inequality (4.2) and $\mathbf{x} \in \Phi$ be the unique solution of the following problem

$$^{c}\mathbf{D}^{\beta;\psi}\mathbf{x}(\xi) = \mathcal{G}(\xi,\mathbf{x}(\xi),\mathbb{P}\mathbf{x}(\xi)), \quad \xi \in \mathbb{F}, \quad \beta \in (0,1),$$

$$\Delta\mathbf{x}|_{\xi=\xi_j} = \mathbb{I}_j(\mathbf{x}(\xi_k^-)), \quad \xi = \xi_j, \quad j = 1,2,....,n,$$

$$\mathbf{x}(0) = \mathbf{H}(0), \quad \mathbf{x}(t) = \mathbf{H}(t).$$

Using Lemma 3.3, we obtain

$$\mathbf{x}(\xi) = \mathfrak{A}_{\mathbf{x}} + \frac{1}{\lambda(\beta)}\int_0^{\xi}\psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(\xi,\mathbf{x}(\xi),\mathbb{P}\mathbf{x}(\xi))ds + \sum_{0<\xi_j<\xi}\mathbb{I}_j(\mathbf{x}(\xi_j^-)).$$

Alternatively, if $\mathbf{x}(0) = \mathbf{H}(0)$, $\mathbf{x}(t) = \mathbf{H}(t)$, then $\mathfrak{A}_{\mathbf{x}} = \mathfrak{A}_{\mathbf{H}}$. In fact,

$$\begin{aligned} |\mathfrak{A}_{\mathbf{x}} - \mathfrak{A}_{\mathbf{H}}| &\leq \frac{|m|}{|l+m|\lambda(\beta)}\int_0^{t}\psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}|\mathcal{G}(\xi,\mathbf{x}(\xi),\mathbb{P}\mathbf{x}(\xi)) \\ &\quad -\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))|ds + \sum_{0<\xi_j<\xi}\|\mathbb{I}_j\mathbf{x}(\xi_j^-)) - \mathbb{I}_j\mathbf{H}(\xi_j^-))\| \\ &\leq \frac{\mathbf{K}(1+\mathfrak{H}_1)|m|}{|l+m|}I^{\beta;\psi}|\mathbf{x}(t)-\mathbf{H}(t)| + \rho\|\mathbf{x}-\mathbf{H}\| = 0. \end{aligned}$$

Therefore we obtain $\mathfrak{A}_{\mathbf{x}} = \mathfrak{A}_{\mathbf{H}}$ and so

$$\mathbf{x}(\xi) = \mathfrak{A}_{\mathbf{x}} + \frac{1}{\lambda(\beta)} \int_0^{\xi} \psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(\xi,\mathbf{x}(\xi),\mathbb{P}\mathbf{x}(\xi))ds + \sum_{0<\xi_j<\xi} \mathbb{I}_j(\mathbf{x}(\xi_j^-)).$$

By integration of inequality (4.2) and using Remark 4.6, we conclude that

$$\left|\mathbf{H}(\xi) - \mathfrak{A}_{\mathbf{H}} - \frac{1}{\lambda(\beta)} \int_0^{\xi} \psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))ds\right| + \sum_{0<\xi_j<\xi} \mathbb{I}_j(\mathbf{H}(\xi_j^-))$$

$$\leq \epsilon\frac{(\psi(t))^{\beta}}{\lambda(\beta+1)}\left(1+\frac{|m|}{|l+m|}\right)+\rho.$$

For any $\xi \in \mathbb{F}$, we find

$$\begin{aligned}
|\mathbf{H}(\xi)-\mathbf{x}(\xi)| &\leq \left|\mathbf{H}(\xi) - \mathfrak{A}_{\mathbf{H}} - \frac{1}{\lambda(\beta)} \int_0^{\xi} \psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi))ds\right| \\
&\quad + \frac{1}{\lambda(\beta)} \int_0^{\xi} \psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}|\mathcal{G}(\xi,\mathbf{H}(\xi),\mathbb{P}\mathbf{H}(\xi)) \\
&\quad -\mathcal{G}(\xi,\mathbf{x}(\xi),\mathbb{P}\mathbf{x}(\xi))|ds + \sum_{0<\xi_j<\xi} \|\mathbb{I}_j\mathbf{H}(\xi_j^-)) - \mathbb{I}_j\mathbf{x}(\xi_j^-))\| \\
&\leq \epsilon\frac{(\psi(t))^{\beta}}{\lambda(\beta+1)}\left(1+\frac{|m|}{|l+m|}\right) \\
&\quad + \frac{\mathbf{K}(1+\mathfrak{H}_1)}{\lambda(\beta)} \int_0^{\xi} \psi^{'}(s)(\psi(\xi)-\psi(s))^{\beta-1}|\mathbf{H}(s)-\mathbf{x}(s)|ds + \rho\|\mathbf{H}-\mathbf{x}\|.
\end{aligned}$$

Using the Gronwall inequality, Lemma 2.3 and Remark 2.4, we obtain

$$|\mathbf{H}(\xi)-\mathbf{x}(\xi)| \leq \left(1+\frac{|m|}{|l+m|}\right)\frac{\epsilon(\psi(t))^{\beta}}{\lambda(\beta+1)}E_{\beta;\psi}(\mathbf{K}(1)+\mathfrak{H}_1(\psi(t))^{\beta}) + \rho\|\mathbf{H}-\mathbf{x}\|.$$

Thus, Problem (1.1)–(1.3) is Ulam–Hyers stable. □

Theorem 4.10 *Suppose (H1)–(H2), inequality (3.1) and (H4) hold. If there exists an increasing function $\varphi \in \Phi$ and $\mu_{\varphi} > 0$, then the system (1.1)-(1.3) is Ulam–Hyers–Rassias stable.*

Proof By using Theorem 4.9, we get

$$I^{\beta;\psi}\varphi(\xi) \leq \mu_{\varphi}\varphi(\xi)$$

for each $\xi \in \mathbb{F}$. Thus, in view of Definition 4.4, then the system (1.1)–(1.3) is Ulam–Hyers stable. □

Remark 4.11 *Under the assumptions of Theorem 4.9, we consider problem (1.1)–(1.3) and inequality (4.4). One can repeat the same process to verify that problem (1.1)–(1.3) is Ulam–Hyers–Rassias stable.*

5. Conclusion

FDEs have recently piqued the interest of a number of scholars due to their numerous applications, particularly those involving generalised fractional operators. We are interested in fractional systems with generalised ψ-fractional derivatives. In the literature, derivatives cover a wide range of systems and have a kernel with various properties. As a further contribution to this discussion, existence and the Ulam–Hyers–Rassias stability results are presented for a new class of linked systems. The generalised ψ-fractional integro-differential equations is investigated. The results are analysed using Schaefer's fixed point theorem.

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