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# Curvature identities for Einstein manifolds of dimensions 5 and 6

**Yunhee EUH, Jihun KIM, JeongHyeong PARK***
Department of Mathematics, Sungkyunkwan University, Suwon, South Korea

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**Abstract:** Patterson discussed the curvature identities on Riemannian manifolds based on the skew-symmetric properties of the generalized Kronecker delta, and a curvature identity for any 6-dimensional Riemannian manifold was independently derived from the Chern-Gauss-Bonnet Theorem. In this paper, we provide the explicit formulae of Patterson's curvature identity that holds on 5-dimensional and 6-dimensional Einstein manifolds. We confirm that the curvature identities on the Einstein manifold derived from the Chern-Gauss-Bonnet Theorem are the same as the curvature identities deduced from Patterson's result. We also provide examples that support the theorems.

**Key words:** Einstein, curvature identity, curvature tensor

## 1. Introduction

Let $M = (M, g)$ be an $m$-dimensional Riemannian manifold and $\nabla$ be the Levi-Civita connection of $g$. The curvature tensor $R$ on $M$ is defined by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for $X$, $Y$, $Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on $M$. The Ricci tensor of $M$ is defined by $\rho(X,Y) = \mathrm{Tr}(Z \to R(Z,X)Y)$ and the scalar curvature of $M$ is obtained by $\tau = \mathrm{Tr}\,\rho$. A symmetric (0,2)-tensor field $\overline{R}$ on $M$ is defined by $\overline{R}(X,Y) = \sum_{i,j,k=1}^{m} R(X,e_i,e_j,e_k)R(Y,e_i,e_j,e_k)$ for a local orthonormal frame $\{e_i\}$. An $m$-dimensional Riemannian manifold $(M,g)$ is said to be *Einstein* if $\rho = \frac{\tau}{m}g$. The $m$-dimensional Einstein manifold $(M,g)$ is said to be *super-Einstein* if the following condition is satisfied on $M$

$$\overline{R}(X,Y) = \frac{||R||^2}{m} g(X,Y) \tag{1.1}$$

with constant $||R||^2$ (see [3, 6, 11]). The condition (1.1) has some geometric meanings: For a compact manifold, an Einstein metric is critical for the functional $\int_M ||R||^2 dv_M$ restricted to $vol(M) = 1$ if and only if (1.1) satisfies. (see [2, Corollary 4.72]). Boeckx and Vanhecke [3] showed that an Einstein manifold $M$ is super-Einstein if and only if the unit tangent sphere bundle $T_1M$ equipped with the standard contact metric structure has constant scalar curvature ([3, Proposition 3.6]).

Let $M_q(r,f)$ denote the mean-value of a real-valued function $f$ over a geodesic sphere $S(q;r)$ with center

*Correspondence: parkj@skku.edu
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$q$ and radius $r$ in an $m$-dimensional Riemannian manifold $M$:

$$M_q(r,f) := \frac{1}{\int_{S(q;r)} dv_{S(q;r)}} \int_{p\in S(q;r)} f(p)\ dv_{S(q;r)}.$$

Gray and Willmore [11] showed the mean-value properties for an Einstein manifold and a super-Einstein manifold: They proved, from the expansions of $M_q(r,f)$, that the harmonic function $f$ near $q$ has the mean-value properties

$$M_q(r,f) = f(q) + O(r^6), \quad \text{as} \quad r \to 0$$

for an Einstein manifold ([11, Theorem 1.1]) and

$$M_q(r,f) = f(q) + O(r^8), \quad \text{as} \quad r \to 0$$

for a super-Einstein manifold with $\dim M \geq 4$ ([11, Theorem 6.1]), respectively.

A Riemannian manifold $M$ is said to be 2*-stein* if there exist two functions $\mu_1, \mu_2$ on $M$ such that $\mathrm{Tr}\, R_X = \mu_1(p)||X||^2$ and $\mathrm{Tr}(R_X^2) = \mu_2(p)||X||^4$, for all $p \in M$ and all $X \in T_pM$. Here, the Jacobi operator $R_X$ is defined by $R_XY = R(Y,X)X$ for tangent vectors $X$, $Y$ at a point $p \in M$. A unit vector field $V$ on $M$ is said to be a *harmonic vector field* if it is a critical point for the energy functional in the set of all unit vector fields of $M$ [16]. A contact metric manifold whose characteristic vector field $\xi$ is a harmonic vector field is called an *H-contact manifold*. Nikolayevsky and Park [13] showed that for a Riemannian manifold $M$, $T_1M$ equipped with the standard contact metric structure is $H$-contact if and only if $M$ is 2-stein. Gilkey, Swann, and Vanhecke [10] showed that a 4-dimensional manifold $M$ is 2-stein if and only if locally there is a choice of orientation of $M$ for which the metric is self-dual and Einstein ([10, Theorem 2.6]). From the definition of a 2-stein manifold, we can derive the super-Einstein conditions (see [1, Chapter 6, §E]). Thus, a 2-stein manifold is necessarily super-Einstein.

Euh, Park, and Sekigawa [6] derived a curvature identity on any 4-dimensional manifold from the Chern-Gauss-Bonnet theorem. There are many applications of this identity (see [5, 7, 9]). On the other hand, Deszcz, Hotloś, and Sentürk [4] gave some curvature properties of 4-dimensional semi-Riemannian manifolds as an application of Patterson's curvature identity.

In this paper, we introduce the curvature identities on some 5- and 6-dimensional Riemannian manifolds such as Einstein, super-Einstein manifolds. Throughout the paper, we assume that the components of the tensor fields are with respect to a local orthonormal frame $\{e_i\}$ and we also adopt the Einstein convention on sum over repeated indices unless otherwise specified. Our main results are the following.

**Theorem A** *Let $M = (M,g)$ be a 5-dimensional Riemannian manifold.*

(a) *If $M$ is Einstein, then the following curvature identity holds on $M$:*

$$2\tau\overline{R}_{ij} + 4\check{R}_{ij} + 4\hat{R}_{ij} - 8\mathring{R}_{ij} = \Big(\frac{\tau}{5}||R||^2 + \frac{\tau^3}{25}\Big)g_{ij}.$$

(b) *If $M$ is super-Einstein, then the following curvature identity holds on $M$:*

$$4\mathring{R}_{ij} - 2\hat{R}_{ij} = \Big(\frac{9}{50}\tau||R||^2 - \frac{\tau^3}{50}\Big)g_{ij}.$$

**Theorem B** *Let $M = (M, g)$ be a 6-dimensional Riemannian manifold.*

(a) *If $M$ is Einstein, then the following curvature identity holds on $M$:*

$$4\tau\overline{R}_{ij} + 12\check{R}_{ij} + 12\hat{R}_{ij} - 24\mathring{R}_{ij} = (\tau||R||^2 - 4\mathring{R} + 2\hat{R})g_{ij}.$$

(b) *If $M$ is super-Einstein, then the following curvature identity holds on $M$:*

$$2\mathring{R}_{ij} - \hat{R}_{ij} = \frac{1}{6}(2\mathring{R} - \hat{R})g_{ij}.$$

*Here, we set*

$$\check{R}_{ij} = R_{iuvj}R_{abcu}R_{abcv}, \quad \hat{R}_{ij} = R_{ibcd}R_{jbuv}R_{cduv}, \quad \mathring{R}_{ij} = R_{ibcd}R_{jucv}R_{budv},$$
$$\hat{R} = R_{abcd}R_{abuv}R_{cduv}, \quad \mathring{R} = R_{abcd}R_{aucv}R_{budv}.$$

We derive the curvature identities from Patterson's curvature identity. In Section 2, we recall the previous results for a 6-dimensional Riemannian manifold [8] and introduce Patterson's curvature identity. In Sections 3 and 4, we prove Theorem A and Theorem B, respectively. We also give some examples supporting the theorems. In Appendix, we attach the detailed computation for the proof of Lemma 4.1.

## 2. Curvature identities

On a 6-dimensional compact oriented Riemannian manifold, Chern-Gauss-Bonnet theorem states that Euler characteristic $\chi(M)$ of $M$ is given by the following integral formula.

**Proposition 2.1 ([15])** *Let $M = (M, g)$ be a 6-dimensional compact oriented Riemannian manifold. Then the Euler characteristic $\chi(M)$ of $M$ is given by*

$$\chi(M) = \frac{1}{384\pi^3}\int_M \{\tau^3 - 12\tau||\rho||^2 + 3\tau||R||^2 + 16\rho_{ab}\rho_{ac}\rho_{bc} - 24\rho_{ab}\rho_{cd}R_{acbd} - 24\rho_{uv}R_{abcu}R_{abcv} + 8R_{abcd}R_{aucv}R_{bvdu} - 2R_{abcd}R_{abuv}R_{cduv}\}dv_M.$$

In [8], the authors consider the one-parameter deformation $g(t)$ of $g$, using the fact that Euler characteristic is a topological invariant for the deformation, they got the universal curvature identity which holds on any 6-dimensional Riemannian manifold. In particular, they obtained the following.

**Theorem 2.2 ([8])** *Let $M = (M, g)$ be a 6-dimensional Einstein manifold. Then the following identity holds on $M$:*

$$(-\tau||R||^2 + 4\mathring{R} - 2\hat{R})g_{ij} + 12\check{R}_{ij} + 12\hat{R}_{ij} - 24\mathring{R}_{ij} + 4\tau\overline{R}_{ij} = 0.$$

For an integer $N \geq 1$, the *generalized Kronecker delta* is given by

$$\delta^{j_1 j_2 \cdots j_N}_{i_1 i_2 \cdots i_N} = \begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_N}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_N}_{i_2} \\ \vdots & \vdots & \vdots & \vdots \\ \delta^{j_1}_{i_N} & \delta^{j_2}_{i_N} & \cdots & \delta^{j_N}_{i_N} \end{vmatrix}.$$

By using the skew-symmetric properties of the generalized Kronecker delta, Patterson [14] obtained the curvature identity for an $m$-dimensional Riemannian manifold as follows:

$$\delta^{jj_1j_2\cdots j_m}_{ii_1i_2\cdots i_m} R^{i_1i_2}{}_{j_1j_2}\cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} = 0, \tag{2.1}$$

where $r$ is any integer such that $1 \le r \le \frac{m}{2}$ ($m$ even) or $1 \le r \le \frac{m-1}{2}$ ($m$ odd).

We note that the identity (2.1) holds with respect to the Weyl curvature tensor $W$ even if we replace $R$ by $W$ in (2.1) ([14, Section 8]).

## 3. Proof of Theorem A

Let $(M, g)$ be a 5-dimensional Riemannian manifold. Making use of (2.1), we have the curvature identity with respect to the Weyl curvature tensor $W$ for the case when $r = 2$ as follows:

$$\begin{aligned} &||W||^2(g_{ik}g_{jl} - g_{il}g_{jk}) - 4\Big\{W_{abcj}W_{abcl}g_{ik} - W_{abcj}W_{abck}g_{il} + W_{abci}W_{abck}g_{jl} \\ &- W_{abci}W_{abcl}g_{jk} - 2W_{iabl}W_{kabj} + 2W_{iabk}W_{labj} - W_{abij}W_{abkl}\Big\} = 0. \end{aligned} \tag{3.1}$$

Since the Weyl tensor $W$ on a 5-dimensional Riemannian manifold is given by

$$W_{pqrs} = R_{pqrs} - \frac{1}{3}(\rho_{ps}g_{qr} + \rho_{qr}g_{ps} - \rho_{pr}g_{qs} - \rho_{qs}g_{pr}) + \frac{\tau}{12}(g_{ps}g_{qr} - g_{pr}g_{qs}),$$

we obtain the explicit formula of the curvature identity on 5-dimensional Einstein manifolds.

**Lemma 3.1** *Let $M = (M, g)$ be a 5-dimensional Einstein manifold. The following curvature identity holds on $M$:*

$$\begin{aligned} &\Big(||R||^2 + \frac{\tau^2}{5}\Big)(g_{ik}g_{jl} - g_{il}g_{jk}) - 4(\overline{R}_{ik}g_{jl} + \overline{R}_{jl}g_{ik} - \overline{R}_{il}g_{jk} - \overline{R}_{jk}g_{il}) \\ &+ 8(R_{iabl}R_{kabj} - R_{iabk}R_{labj}) + 4R_{abij}R_{abkl} + \frac{12}{5}\tau R_{ijkl} = 0. \end{aligned} \tag{3.2}$$

**Proof** Since $\rho_{ps} = \frac{\tau}{5}g_{ps}$, we obtain $W_{pqrs} = R_{pqrs} - \frac{\tau}{20}(g_{ps}g_{qr} - g_{pr}g_{qs})$. Now we apply the Weyl tensor to the identity (3.1). Then we have

$$\begin{aligned} ||W||^2 = ||R||^2 - \frac{\tau^2}{10}, \quad W_{abcj}W_{abcl} = R_{abcj}R_{abcl} - \frac{\tau^2}{50}g_{jl}, \\ W_{iabl}W_{kabj} = R_{iabl}R_{kabj} + \frac{\tau}{10}R_{ijkl} - \frac{\tau^2}{80}g_{il}g_{jk} + \frac{\tau^2}{400}g_{ik}g_{jl}, \\ W_{abij}W_{abkl} = R_{abij}R_{abkl} + \frac{\tau}{5}R_{ijkl} + \frac{\tau^2}{200}(g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned}$$

Similarly, we can obtain remaining terms. Then, by rearranging all terms, we complete the proof. □

We note that there are useful formulae as follows:

$$R_{ibcd}R_{jbuv}R_{cudv} = \frac{1}{2}\hat{R}_{ij}, \quad R_{ibcd}R_{jubv}R_{cudv} = \frac{1}{4}\hat{R}_{ij}, \quad R_{ibcd}R_{jucv}R_{bvdu} = \mathring{R}_{ij} - \frac{1}{4}\hat{R}_{ij}. \tag{3.3}$$

Now, we transvect each term of (3.2) with $R_{pjkl}$ and use $\rho_{ip}=\frac{\tau}{5}g_{ip}$. Then, we obtain

$$\begin{aligned}
&\Big(||R||^2+\frac{\tau^2}{5}\Big)(g_{ik}g_{jl}-g_{il}g_{jk})R_{pjkl}=-\frac{2}{5}\tau\Big(||R||^2+\frac{\tau^2}{5}\Big)g_{ip},\\
&-4(\overline{R}_{ik}g_{jl}+\overline{R}_{jl}g_{ik}-\overline{R}_{il}g_{jk}-\overline{R}_{jk}g_{il})R_{pjkl}=-4\Big(-\frac{2}{5}\tau\overline{R}_{ip}-2\check{R}_{ip}\Big),\\
&8(R_{iabl}R_{kabj}-R_{iabk}R_{labj})R_{pjkl}=-16\mathring{R}_{ip}+4\hat{R}_{ip},\\
&4R_{abij}R_{abkl}R_{pjkl}=4\hat{R}_{ip},\\
&\frac{12}{5}\tau R_{ijkl}R_{pjkl}=\frac{12}{5}\tau\overline{R}_{ip}.
\end{aligned}$$

For the third equation above, we use (3.3). Hence, we complete the proof of Theorem A-(a).

Now we prove Theorem A-(b). Applying the condition (1.1) to (3.2), we obtain

$$R_{ijab}R_{abkl}+2R_{iabl}R_{kabj}-2R_{iabk}R_{labj}+\frac{3}{5}\tau R_{ijkl}=\Big(\frac{3}{20}||R||^2-\frac{1}{20}\tau^2\Big)(g_{ik}g_{jl}-g_{il}g_{jk}). \tag{3.4}$$

Transvecting each term of (3.4) with $R_{pjkl}$ and using (1.1), we obtain

$$\begin{aligned}
&R_{ijab}R_{abkl}R_{pjkl}=\hat{R}_{ip},\\
&2R_{iabl}R_{kabj}R_{pjkl}=-2\Big(\mathring{R}_{ip}-\frac{1}{4}\hat{R}_{ip}\Big),\\
&-2R_{iabk}R_{labj}R_{pjkl}=-2\Big(\mathring{R}_{ip}-\frac{1}{4}\hat{R}_{ip}\Big),\\
&\frac{3}{5}\tau R_{ijkl}R_{pjkl}=\frac{3}{25}\tau||R||^2g_{ip},\\
&(g_{ik}g_{jl}-g_{il}g_{jk})R_{pjkl}=-\frac{2}{5}\tau g_{ip}.
\end{aligned}$$

For the second and third equations, we use (3.3). Therefore, we complete the proof of Theorem A-(b).

For a 5-dimensional 2-stein manifold, there is the following orthonormal basis introduced by Nikolayevsky [12].

**Proposition 3.2** ([12, Proposition 4]) *Let $M=(M,g)$ be a 2-stein manifold of dimension 5. Then there exists an orthonormal basis $\{e_i\}$ at each point $p\in M$ such that*

$$\begin{aligned}
&R_{1212}=R_{1313}=R_{2323}=R_{2424}=R_{3434}=\alpha-\beta,\ R_{1414}=\alpha-4\beta,\\
&R_{1515}=R_{4545}=\alpha,\ R_{2525}=R_{3535}=\alpha-3\beta,\\
&R_{1234}=\beta,\ R_{1235}=\sqrt{3}\beta,\ R_{1324}=-\beta,\ R_{1325}=\sqrt{3}\beta,\\
&R_{1423}=-2\beta,\ R_{2425}=\sqrt{3}\beta,\ R_{3435}=-\sqrt{3}\beta,
\end{aligned}$$

*and all the other components of $R$ vanish.*

By using Nikolayevsky's basis, we show the following theorem.

**Theorem 3.3** *Let $M=(M,g)$ be a 5-dimensional 2-stein manifold. The identity (3.4) holds on $M$.*

**Proof** Each term of the left hand side of (3.4) in the case of $i=1$, $j=2$, $k=3$, $l=4$ is as follows:

$$\begin{aligned} R_{12ab}R_{ab34} &= 2(2\alpha-5\beta)\beta, \\ 2R_{1ab4}R_{3ab2} &= 2(2\alpha-5\beta)\beta, \\ -2R_{1ab3}R_{4ab2} &= 2(2\alpha+\beta)\beta, \\ \frac{3}{5}\tau R_{1234} &= -6(2\alpha-3\beta)\beta. \end{aligned}$$

Since $R_{12ab}R_{ab34}+2R_{1ab4}R_{3ab2}-2R_{1ab3}R_{4ab2}+\frac{3}{5}\tau R_{1234}=0$ and $(g_{13}g_{24}-g_{14}g_{23})=0$ in (3.4), Equation (3.4) holds for $i=1$, $j=2$, $k=3$, $l=4$. For other choice of $i,j,k,l$, similar processes can be done for the proof of Theorem 3.3. □

Now we give the examples for Theorem A.

**Example 3.4** *Let $M$ be the Riemannian product manifold of a 3-dimensional Riemannian manifold $M_1$ of constant sectional curvature $k$ and a surface $M_2$ of constant sectional curvature $2k$ $(k\neq 0)$. Then the manifold $M$ is Einstein, but not super-Einstein. Let $\{e_i\}$, $i=1,\dots,5$ be an orthonormal basis of $T_pM$ at any point $p=(p_1,p_2)\in M$, where $\{e_1,e_2,e_3\}$ and $\{e_4,e_5\}$ are bases for $T_{p_1}M_1$ and $T_{p_2}M_2$, respectively. Then we have*

$$R_{1221}=R_{1331}=R_{2332}=k, \quad R_{4554}=2k, \tag{3.5}$$

*and all the other components of $R$ vanish. From (3.5) we have $\tau=10k$, $||R||^2=28k^2$, and*

$$\overline{R}_{ij}=\begin{cases}4k^2 & \text{if } i=j=1,2,3\\ 8k^2 & \text{if } i=j=4,5\\ 0 & \text{otherwise}\end{cases}, \qquad \check{R}_{ij}=\begin{cases}8k^3 & \text{if } i=j=1,2,3\\ 16k^3 & \text{if } i=j=4,5\\ 0 & \text{otherwise}\end{cases},$$

$$\hat{R}_{ij}=\begin{cases}-8k^3 & \text{if } i=j=1,2,3\\ -32k^3 & \text{if } i=j=4,5\\ 0 & \text{otherwise}\end{cases}, \qquad \mathring{R}_{ij}=\begin{cases}-2k^3 & \text{if } i=j=1,2,3\\ 0 & \text{otherwise}\end{cases}.$$

*Therefore, we find that the curvature identity of Theorem A-(b) does not hold, but that of Theorem A-(a) holds on $M$.*

**Example 3.5** *Let $M=SL(3)/SO(3)$. Then $M$ is a 5-dimensional 2-stein manifold. The inner product and the curvature tensor are given by*

$$<X,Y>=\operatorname{Tr}XY, \quad R(X,Y)Z=-[[X,Y],Z]$$

*for* $X$, $Y$, $Z \in \mathfrak{X}(M)$. *Then, with orthonormal basis of* $SL(3)/SO(3)$

$$X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \quad X_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$$
$$X_4 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&0\end{pmatrix}, \quad X_5 = \frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},$$

*we have the components of the curvature tensor:*

$$\begin{aligned}
&R_{1221} = R_{1331} = R_{2332} = R_{2442} = R_{3443} = -\frac{1}{2},\ R_{1441} = -2,\\
&R_{1551} = R_{4554} = 0,\ R_{2552} = R_{3553} = -\frac{3}{2},\\
&R_{1234} = -\frac{1}{2},\ R_{1235} = -\frac{\sqrt{3}}{2},\ R_{1324} = \frac{1}{2},\ R_{1325} = -\frac{\sqrt{3}}{2},\\
&R_{1423} = 1,\ R_{2425} = -\frac{\sqrt{3}}{2},\ R_{3435} = \frac{\sqrt{3}}{2},
\end{aligned} \tag{3.6}$$

*and all the other components vanish. From* (3.6), *we compute the Ricci tensor* $\rho_{ij}$. *For* $i \neq j$, $\rho_{ij} = 0$ *and* $\rho_{11} = \sum_{a=1}^{5} R_{1aa1} = -3$. *Similarly,* $\rho_{22} = \rho_{33} = \rho_{44} = \rho_{55} = -3$ *and so we get the scalar curvature* $\tau$ *is* $-15$. *Making use of* (3.6), $||R||^2 = 75$, $\hat{R}_{ij} = 75\delta_{ij}$, *and* $\mathring{R}_{ij} = \frac{15}{4}\delta_{ij}$, *where* $\delta_{ij}$ *denotes the Kronecker delta. Therefore, we find that the curvature identity of Theorem A-*(b) *holds on* $M$.

## 4. Proof of Theorem B

We obtain the explicit formula of the curvature identity on 6-dimensional Einstein manifolds for the case when $m = 6$ and $r = 2$ of the identity (2.1). To prove Theorem B, we need Lemma 4.1.

**Lemma 4.1** *Let* $M = (M, g)$ *be a* 6*-dimensional Einstein manifold. The following curvature identity holds on* $M$:

$$\begin{aligned}
&\Big\{\frac{1}{8}\Big(||R||^2 + \frac{\tau^2}{3}\Big)\Big(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj}\Big)\Big\}\\
&+ \Big\{-\frac{1}{2}\Big(\overline{R}_{ij}(g_{hk}g_{lm} - g_{hm}g_{lk}) + \overline{R}_{ik}(g_{hm}g_{lj} - g_{hj}g_{lm}) + \overline{R}_{im}(g_{hj}g_{lk} - g_{hk}g_{lj})\\
&\qquad + \overline{R}_{hj}(g_{im}g_{lk} - g_{ik}g_{lm}) + \overline{R}_{hk}(g_{ij}g_{lm} - g_{im}g_{lj}) + \overline{R}_{hm}(g_{ik}g_{lj} - g_{ij}g_{lk})\\
&\qquad + \overline{R}_{lj}(g_{ik}g_{hm} - g_{im}g_{hk}) + \overline{R}_{lk}(g_{im}g_{hj} - g_{ij}g_{hm}) + \overline{R}_{lm}(g_{ij}g_{hk} - g_{ik}g_{hj})\Big)\Big\}
\end{aligned} \tag{4.1}$$

$$+\Big\{\Big(-T_{ijkh}+T_{ikjh}+\frac{1}{2}S_{ihjk}+\frac{\tau}{3}R_{ihjk}\Big)g_{lm}-\Big(-T_{ijkl}+T_{ikjl}+\frac{1}{2}S_{iljk}+\frac{\tau}{3}R_{iljk}\Big)g_{hm}$$
$$-\Big(-T_{ijmh}+T_{imjh}+\frac{1}{2}S_{ihjm}+\frac{\tau}{3}R_{ihjm}\Big)g_{lk}+\Big(-T_{ijml}+T_{imjl}+\frac{1}{2}S_{iljm}+\frac{\tau}{3}R_{iljm}\Big)g_{hk}$$
$$+\Big(-T_{ikmh}+T_{imkh}+\frac{1}{2}S_{ihkm}+\frac{\tau}{3}R_{ihkm}\Big)g_{lj}-\Big(-T_{ikml}+T_{imkl}+\frac{1}{2}S_{ilkm}+\frac{\tau}{3}R_{ilkm}\Big)g_{hj}$$
$$-\Big(-T_{hjml}+T_{hmjl}+\frac{1}{2}S_{hljm}+\frac{\tau}{3}R_{hljm}\Big)g_{ik}+\Big(-T_{hjkl}+T_{hkjl}+\frac{1}{2}S_{hljk}+\frac{\tau}{3}R_{hljk}\Big)g_{im}$$
$$+\Big(-T_{hkml}+T_{hmkl}+\frac{1}{2}S_{hlkm}+\frac{\tau}{3}R_{hlkm}\Big)g_{ij}\Big\}$$
$$+A_{hjkmil}-A_{ljkmih}-A_{ijkmhl}+A_{ijmkhl}-A_{hjmkil}+A_{ljmkih}-A_{ikmjhl}+A_{hkmjil}-A_{lkmjih}$$
$$=0,$$

*where* $T_{pqrs}=R_{pabq}R_{rabs}$, $S_{pqrs}=R_{abpq}R_{abrs}$, *and* $A_{pqrstu}=R_{apqr}R_{astu}$.

**Proof** The proof is similar to that of Lemma 3.1 for the case when $m=6$ and $r=2$. Here, the Weyl tensor W is given by

$$W_{pqrs}=R_{pqrs}-\frac{1}{4}(\rho_{ps}g_{qr}+\rho_{qr}g_{ps}-\rho_{pr}g_{qs}-\rho_{qs}g_{pr})+\frac{\tau}{20}(g_{ps}g_{qr}-g_{pr}g_{qs}).$$

Next we expand (2.1) for $W$ and use the Einstein condition $\rho_{ij}=\frac{\tau}{6}g_{ij}$. Then we can obtain (4.1). For more details, we refer to Appendix. □

Now, we transvect each term of (4.1) with $R_{ihjk}$. For the first term of (4.1), we have

$$\frac{1}{8}\Big(||R||^2+\frac{\tau^2}{3}\Big)\Big(g_{ij}g_{hk}g_{lm}-g_{ij}g_{hm}g_{lk}-g_{ik}g_{hj}g_{lm}+g_{ik}g_{hm}g_{lj}+g_{im}g_{hj}g_{lk}-g_{im}g_{hk}g_{lj}\Big)R_{ihjk}$$
$$=-\frac{1}{6}\Big(\tau||R||^2+\frac{\tau^3}{3}\Big)g_{lm}.$$

For the second term, we have

$$-\frac{1}{2}\overline{R}_{ij}(g_{hk}g_{lm}-g_{hm}g_{lk})R_{ihjk}=\frac{\tau}{12}||R||^2g_{lm}-\frac{1}{2}\check{R}_{lm},$$

$$-\frac{1}{2}\overline{R}_{ik}(g_{hm}g_{lj}-g_{hj}g_{lm})R_{ihjk}=\frac{\tau}{12}||R||^2g_{lm}-\frac{1}{2}\check{R}_{lm},\quad -\frac{1}{2}\overline{R}_{im}(g_{hj}g_{lk}-g_{hk}g_{lj})R_{ihjk}=-\frac{\tau}{6}\overline{R}_{lm},$$

$$-\frac{1}{2}\overline{R}_{hj}(g_{im}g_{lk}-g_{ik}g_{lm})R_{ihjk}=\frac{\tau}{12}||R||^2g_{lm}-\frac{1}{2}\check{R}_{lm},$$

$$-\frac{1}{2}\overline{R}_{hk}(g_{ij}g_{lm}-g_{im}g_{lj})R_{ihjk}=\frac{\tau}{12}||R||^2g_{lm}-\frac{1}{2}\check{R}_{lm},$$

$$-\frac{1}{2}\overline{R}_{hm}(g_{ik}g_{lj}-g_{ij}g_{lk})R_{ihjk}=-\frac{\tau}{6}\overline{R}_{lm},\quad -\frac{1}{2}\overline{R}_{lj}(g_{ik}g_{hm}-g_{im}g_{hk})R_{ihjk}=-\frac{\tau}{6}\overline{R}_{lm},$$

$$-\frac{1}{2}\overline{R}_{lk}(g_{im}g_{hj}-g_{ij}g_{hm})R_{ihjk}=-\frac{\tau}{6}\overline{R}_{lm},\quad -\frac{1}{2}\overline{R}_{lm}(g_{ij}g_{hk}-g_{ik}g_{hj})R_{ihjk}=\tau\overline{R}_{lm}.$$

Thus, the second term of (4.1) transvecting with $R_{ihjk}$ becomes $\frac{\tau}{3}||R||^2 g_{lm} - 2\check{R}_{lm} + \frac{\tau}{3}\overline{R}_{lm}$.
For the third term, we have

$$\begin{aligned}
&\Big( - T_{ijkh} + T_{ikjh} + \frac{1}{2}S_{ihjk} + \frac{\tau}{3}R_{ihjk}\Big) g_{lm} R_{ihjk} \\
&= \left( -R_{iabj}R_{kabh}R_{ihjk} + R_{iabk}R_{jabh}R_{ihjk} + \frac{1}{2}R_{abih}R_{abjk}R_{ihjk} + \frac{1}{3}\tau R_{ihjk}R_{ihjk} \right) g_{lm} \\
&= \left( -2\mathring{R} + \hat{R} + \frac{1}{3}\tau||R||^2 \right) g_{lm}, \\
-&\Big( - T_{ijkl} + T_{ikjl} + \frac{1}{2}S_{iljk} + \frac{\tau}{3}R_{iljk}\Big) g_{hm} R_{ihjk} \\
&= \left( R_{iabj}R_{kabl}R_{imjk} - R_{iabk}R_{jabl}R_{imjk} - \frac{1}{2}R_{abil}R_{abjk}R_{imjk} - \frac{\tau}{3}R_{iljk}R_{imjk} \right) \\
&= 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{\tau}{3}\overline{R}_{lm}, \\
-&\Big( - T_{ijmh} + T_{imjh} + \frac{1}{2}S_{ihjm} + \frac{\tau}{3}R_{ihjm}\Big) g_{lk} R_{ihjk} = 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{\tau}{3}\overline{R}_{lm}, \\
&\Big( - T_{ijml} + T_{imjl} + \frac{1}{2}S_{iljm} + \frac{\tau}{3}R_{iljm}\Big) g_{hk} R_{ihjk} \\
&= \left( -R_{iabj}R_{mabl} + R_{iabm}R_{jabl} + \frac{1}{2}R_{abil}R_{abjm} + \frac{\tau}{3}R_{iljm} \right) (-\rho_{ij}) \\
&= -\frac{\tau}{6} \left( -\frac{\tau^2}{36}g_{lm} + \overline{R}_{lm} + \frac{1}{2}\overline{R}_{lm} - \frac{\tau^2}{18}g_{lm} \right) \\
&= \frac{\tau^3}{72}g_{lm} - \frac{\tau}{4}\overline{R}_{lm}, \\
&\Big( - T_{ikmh} + T_{imkh} + \frac{1}{2}S_{ihkm} + \frac{\tau}{3}R_{ihkm}\Big) g_{lj} R_{ihjk} = 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{\tau}{3}\overline{R}_{lm}, \\
-&\Big( - T_{ikml} + T_{imkl} + \frac{1}{2}S_{ilkm} + \frac{\tau}{3}R_{ilkm}\Big) g_{hj} R_{ihjk} = \frac{\tau^3}{72}g_{lm} - \frac{\tau}{4}\overline{R}_{lm}, \\
-&\Big( - T_{hjml} + T_{hmjl} + \frac{1}{2}S_{hljm} + \frac{\tau}{3}R_{hljm}\Big) g_{ik} R_{ihjk} = \frac{\tau^3}{72}g_{lm} - \frac{\tau}{4}\overline{R}_{lm}, \\
&\Big( - T_{hjkl} + T_{hkjl} + \frac{1}{2}S_{hljk} + \frac{\tau}{3}R_{hljk}\Big) g_{im} R_{ihjk} = 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{\tau}{3}\overline{R}_{lm}, \\
&\Big( - T_{hkml} + T_{hmkl} + \frac{1}{2}S_{hlkm} + \frac{\tau}{3}R_{hlkm}\Big) g_{ij} R_{ihjk} = \frac{\tau^3}{72}g_{lm} - \frac{\tau}{4}\overline{R}_{lm}.
\end{aligned}$$

Thus, the third term of (4.1) transvecting with $R_{ihjk}$ becomes

$$\left( -2\mathring{R} + \hat{R} + \frac{\tau}{3}||R||^2 + \frac{\tau^3}{18} \right) g_{lm} + 8\mathring{R}_{lm} - 4\hat{R}_{lm} - \frac{7}{3}\tau\overline{R}_{lm}.$$

Making use of (3.3), the remaining terms in (4.1) transvecting with $R_{ihjk}$, we have

$$
\begin{aligned}
&(A_{hjkmil} - A_{ljkmih} - A_{ijkmhl} + A_{ijmkhl} - A_{hjmkil} + A_{ljmkih} - A_{ikmjhl} + A_{hkmjil} - A_{lkmjih})R_{ihjk} \\
&= R_{ahjk}R_{amil}R_{ihjk} - R_{aljk}R_{amih}R_{ihjk} - R_{aijk}R_{amhl}R_{ihjk} + R_{aijm}R_{akhl}R_{ihjk} - R_{ahjm}R_{akil}R_{ihjk} \\
&\quad + R_{aljm}R_{akih}R_{ihjk} - R_{aikm}R_{ajhl}R_{ihjk} + R_{ahkm}R_{ajil}R_{ihjk} - R_{alkm}R_{ajih}R_{ihjk} \\
&= -4\check{R}_{lm} - 2\hat{R}_{lm} + 4\mathring{R}_{lm}.
\end{aligned}
$$

Summing up all terms, then we complete the proof of Theorem B-(a).

Now, we prove Theorem B-(b). Applying (1.1) to (4.1), we have

$$
\begin{aligned}
&-\frac{1}{8}\Big(||R||^2 - \frac{\tau^2}{3}\Big)\Big(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj}\Big) \\
&+\Big(-T_{ijkh} + T_{ikjh} + \frac{1}{2}S_{ihjk} + \frac{\tau}{3}R_{ihjk}\Big)g_{lm} - \Big(-T_{ijkl} + T_{ikjl} + \frac{1}{2}S_{iljk} + \frac{\tau}{3}R_{iljk}\Big)g_{hm} \\
&-\Big(-T_{ijmh} + T_{imjh} + \frac{1}{2}S_{ihjm} + \frac{\tau}{3}R_{ihjm}\Big)g_{lk} + \Big(-T_{ijml} + T_{imjl} + \frac{1}{2}S_{iljm} + \frac{\tau}{3}R_{iljm}\Big)g_{hk} \\
&+\Big(-T_{ikmh} + T_{imkh} + \frac{1}{2}S_{ihkm} + \frac{\tau}{3}R_{ihkm}\Big)g_{lj} - \Big(-T_{ikml} + T_{imkl} + \frac{1}{2}S_{ilkm} + \frac{\tau}{3}R_{ilkm}\Big)g_{hj} \\
&-\Big(-T_{hjml} + T_{hmjl} + \frac{1}{2}S_{hljm} + \frac{\tau}{3}R_{hljm}\Big)g_{ik} + \Big(-T_{hjkl} + T_{hkjl} + \frac{1}{2}S_{hljk} + \frac{\tau}{3}R_{hljk}\Big)g_{im} \\
&+\Big(-T_{hkml} + T_{hmkl} + \frac{1}{2}S_{hlkm} + \frac{\tau}{3}R_{hlkm}\Big)g_{ij} + A_{hjkmil} - A_{ljkmih} - A_{ijkmhl} + A_{ijmkhl} \\
&- A_{hjmkil} + A_{ljmkih} - A_{ikmjhl} + A_{hkmjil} - A_{lkmjih} = 0.
\end{aligned}
\tag{4.2}
$$

We transvect each term of (4.2) with $R_{ihjk}$. Here, we give a few representative terms as follows:

$$
\begin{aligned}
&-\frac{1}{8}\Big(||R||^2 - \frac{\tau^2}{3}\Big)\Big(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj}\Big)R_{ihjk} \\
&= \frac{1}{6}\tau\Big(||R||^2 - \frac{\tau^2}{3}\Big)g_{lm}, \\
&-\Big(-T_{ijkl} + T_{ikjl} + \frac{1}{2}S_{iljk} + \frac{\tau}{3}R_{iljk}\Big)g_{hm}R_{ihjk} = 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{1}{18}\tau||R||^2 g_{lm}, \\
&\Big(-T_{ijml} + T_{imjl} + \frac{1}{2}S_{iljm} + \frac{\tau}{3}R_{iljm}\Big)g_{hk}R_{ihjk} = \Big(\frac{\tau^3}{72} - \frac{\tau}{24}||R||^2\Big)g_{lm}, \\
&(A_{hjkmil} - A_{ljkmih} - A_{ijkmhl} + A_{ijmkhl} - A_{hjmkil} + A_{ljmkih} - A_{ikmjhl} + A_{hkmjil} - A_{lkmjih})R_{ihjk} \\
&= -4\check{R}_{lm} - 2\hat{R}_{lm} + 4\mathring{R}_{lm} \\
&= -\frac{\tau}{9}||R||^2 g_{lm} - 2\hat{R}_{lm} + 4\mathring{R}_{lm}.
\end{aligned}
$$

Similarly, we can obtain remaining terms by transvecting with $R_{ihjk}$. Then, by rearranging all terms, we have Theorem B-(b).

Now we give an example of Theorem B.

**Example 4.2** *Let $M$ be the Riemannian product manifold of 3-dimensional Riemannian manifolds $M_1(k)$ and $M_2(k)$ of constant sectional curvature $k$ ($k \neq 0$). Then we can easily check that $M$ is an Einstein manifold. But $M$ can never be 2-stein: Let $X = (X_1, X_2)$ is a tangent vector with $X_1$, $X_2$ its components tangent to $M_1(k)$ and $M_2(k)$, respectively. Then, $\mathrm{Tr}(R_X^2) = 2k^2(||X_1||^4 + ||X_2||^4)$ which cannot be equal to $\mu_2(||X_1||^2 + ||X_2||^2)^2$. Let $\{e_i\}$, $i = 1, 2, \ldots, 6$ be an orthonormal basis of $T_pM$ at any point $p = (p_1, p_2) \in M$, where $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6\}$ are bases for $T_{p_1}M_1(k)$ and $T_{p_2}M_2(k)$, respectively. Then, we have*

$$R_{1221} = R_{1331} = R_{2332} = R_{4554} = R_{4664} = R_{5665} = k, \tag{4.3}$$

*and all the other components of $R$ vanish. From (4.3) we have*

$$\tau = 12k, \quad ||R||^2 = 24k^2, \quad \mathring{R} = -12k^3, \quad \hat{R} = -48k^3,$$
$$\check{R}_{ij} = 8k^3\delta_{ij}, \quad \hat{R}_{ij} = -8k^3\delta_{ij}, \quad \mathring{R}_{ij} = -2k^3\delta_{ij}, \quad \overline{R}_{ij} = 4k^2\delta_{ij}.$$

*Therefore, we find that the curvature identities of Theorem B hold on $M$. Here we note that $M$ is a super-Einstein manifold.*

## 5. Appendix

In this appendix, we give the proof of Lemma 4.1.

By replacing the curvature tensor $R$ by the Weyl curvature tensor $W$ in (2.1), we can also obtain the curvature identity of $W$. In the case of $m = 6$ and $r = 2$, making use of the fact that the Weyl curvature tensor is traceless, we have the following curvature identity.

**Proposition 5.1** *The Weyl curvature tensor $W$ of any 6-dimensional Riemannian manifold satisfies the following identity*:

$$\begin{aligned}
0 =&||W||^2(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj})\\
&-4\{W_{iabc}W_{jabc}(g_{hk}g_{lm} - g_{hm}g_{lk}) + W_{iabc}W_{kabc}(g_{hm}g_{lj} - g_{hj}g_{lm}) + W_{iabc}W_{mabc}(g_{hj}g_{lk} - g_{hk}g_{lj})\\
&\quad + W_{habc}W_{jabc}(g_{im}g_{lk} - g_{ik}g_{lm}) + W_{habc}W_{kabc}(g_{ij}g_{lm} - g_{im}g_{lj}) + W_{habc}W_{mabc}(g_{ik}g_{lj} - g_{ij}g_{lk})\\
&\quad + W_{labc}W_{jabc}(g_{ik}g_{hm} - g_{im}g_{hk}) + W_{labc}W_{kabc}(g_{im}g_{hj} - g_{ij}g_{hm}) + W_{labc}W_{mabc}(g_{ij}g_{hk} - g_{ik}g_{hj})\}\\
&-8\{(W_{iabj}W_{kabh} - W_{iabk}W_{jabh})g_{lm} - (W_{iabj}W_{kabl} - W_{iabk}W_{jabl})g_{hm}\\
&\quad - (W_{iabj}W_{mabh} - W_{iabm}W_{jabh})g_{lk} + (W_{iabj}W_{mabl} - W_{iabm}W_{jabl})g_{hk}\\
&\quad + (W_{iabk}W_{mabh} - W_{iabm}W_{kabh})g_{lj} - (W_{iabk}W_{mabl} - W_{iabm}W_{kabl})g_{hj}\\
&\quad + (W_{habj}W_{kabl} - W_{habk}W_{jabl})g_{im} - (W_{habj}W_{mabl} - W_{habm}W_{jabl})g_{ik}\\
&\quad + (W_{habk}W_{mabl} - W_{habm}W_{kabl})g_{ij}\}\\
&+4\{W_{abih}W_{abjk}g_{lm} - W_{abih}W_{abjm}g_{lk} + W_{abih}W_{abkm}g_{lj} - W_{abil}W_{abjk}g_{hm}\\
&\quad + W_{abil}W_{abjm}g_{hk} - W_{abil}W_{abkm}g_{hj} + W_{abhl}W_{abjk}g_{im} - W_{abhl}W_{abjm}g_{ik} + W_{abhl}W_{abkm}g_{ij}\}\\
&+8\{W_{ahjk}W_{amil} - W_{aljk}W_{amih} - W_{ajhl}W_{aikm} - W_{aijk}W_{amhl} + W_{ajih}W_{almk}\\
&\quad + W_{ajil}W_{ahkm} - W_{ahjm}W_{akil} + W_{aljm}W_{akih} + W_{aijm}W_{akhl}\}.
\end{aligned} \tag{5.1}$$

*Proof of Lemma 4.1.* The Weyl tensor $W$ on a 6-dimensional Riemannian manifold is given by

$$W_{pqrs} = R_{pqrs} - \frac{1}{4}(\rho_{ps}g_{qr} + \rho_{qr}g_{ps} - \rho_{pr}g_{qs} - \rho_{qs}g_{pr}) + \frac{\tau}{20}(g_{ps}g_{qr} - g_{pr}g_{qs}).$$

Since $M$ is Einstein, we have $W_{pqrs} = R_{pqrs} - \frac{\tau}{30}(g_{ps}g_{qr} - g_{pr}g_{qs})$. We substitute the Weyl tensor into (5.1). Then, we have

$$\begin{aligned}
&||W||^2 = ||R||^2 - \frac{\tau^2}{15},\\
&W_{iabc}W_{jabc} = \overline{R}_{ij} - \frac{\tau^2}{90}g_{ij},\\
&W_{iabj}W_{kabh} - W_{iabk}W_{jabh} = R_{iabj}R_{kabh} - R_{iabk}R_{jabh} - \frac{2}{15}\tau R_{ihjk} - \frac{7}{900}\tau^2(g_{ij}g_{hk} - g_{ik}g_{hj}),\\
&W_{abih}W_{abjk} = R_{abih}R_{abjk} + \frac{2}{15}\tau R_{ihjk} + \frac{\tau^2}{450}(g_{ij}g_{hk} - g_{ik}g_{hj}),\\
&W_{ahjk}W_{amil} = R_{ahjk}R_{amil} - \frac{\tau}{30}(R_{kmil}g_{hj} - R_{jmil}g_{hk} + R_{lhjk}g_{im} - R_{ihjk}g_{lm})\\
&\qquad\qquad + \frac{\tau^2}{900}(g_{im}g_{hj}g_{lk} - g_{ik}g_{hj}g_{lm} - g_{im}g_{hk}g_{lj} + g_{ij}g_{hk}g_{lm}).
\end{aligned} \tag{5.2}$$

Making use of (5.2), the right-hand side of (5.1) is the following:

$$\begin{aligned}
&\left(||R||^2 - \frac{\tau^2}{15}\right)(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj})\\
&-4\Big\{\overline{R}_{ij}(g_{hk}g_{lm} - g_{hm}g_{lk}) + \overline{R}_{ik}(g_{hm}g_{lj} - g_{hj}g_{lm}) + \overline{R}_{im}(g_{hj}g_{lk} - g_{hk}g_{lj})\\
&\quad + \overline{R}_{hj}(g_{im}g_{lk} - g_{ik}g_{lm}) + \overline{R}_{hk}(g_{ij}g_{lm} - g_{im}g_{lj}) + \overline{R}_{hm}(g_{ik}g_{lj} - g_{ij}g_{lk})\\
&\quad + \overline{R}_{lj}(g_{ik}g_{hm} - g_{im}g_{hk}) + \overline{R}_{lk}(g_{im}g_{hj} - g_{ij}g_{hm}) + \overline{R}_{lm}(g_{ij}g_{hk} - g_{ik}g_{hj})\\
&\quad - \frac{\tau^2}{30}(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj})\Big\}\\
&-8\Big\{(R_{iabj}R_{kabh} - R_{iabk}R_{jabh})g_{lm} - (R_{iabj}R_{kabl} - R_{iabk}R_{jabl})g_{hm}\\
&\quad - (R_{iabj}R_{mabh} - R_{iabm}R_{jabh})g_{lk} + (R_{iabj}R_{mabl} - R_{iabm}R_{jabl})g_{hk}\\
&\quad + (R_{iabk}R_{mabh} - R_{iabm}R_{kabh})g_{lj} - (R_{iabk}R_{mabl} - R_{iabm}R_{kabl})g_{hj}\\
&\quad + (R_{habj}R_{kabl} - R_{habk}R_{jabl})g_{im} - (R_{habj}R_{mabl} - R_{habm}R_{jabl})g_{ik}\\
&\quad + (R_{habk}R_{mabl} - R_{habm}R_{kabl})g_{ij}\\
&\quad - \frac{2}{15}\tau(R_{ihjk}g_{lm} - R_{iljk}g_{hm} - R_{ihjm}g_{lk} + R_{iljm}g_{hk} + R_{ihkm}g_{lj}\\
&\qquad - R_{ilkm}g_{hj} + R_{hljk}g_{im} - R_{hljm}g_{ik} + R_{hlkm}g_{ij})\\
&\quad - \frac{7}{300}\tau^2(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj})\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+4\Big\{R_{abih}R_{abjk}g_{lm}-R_{abil}R_{abjk}g_{hm}-R_{abih}R_{abjm}g_{lk}+R_{abil}R_{abjm}g_{hk}+R_{abih}R_{abkm}g_{lj}\\
&\quad-R_{abil}R_{abkm}g_{hj}+R_{abhl}R_{abjk}g_{im}-R_{abhl}R_{abjm}g_{ik}+R_{abhl}R_{abkm}g_{ij}\\
&\quad+\frac{2}{15}\tau(R_{ihjk}g_{lm}-R_{iljk}g_{hm}-R_{ihjm}g_{lk}+R_{iljm}g_{hk}+R_{ihkm}g_{lj}\\
&\qquad-R_{ilkm}g_{hj}+R_{hljk}g_{im}-R_{hljm}g_{ik}+R_{hlkm}g_{ij})\\
&\quad+\frac{\tau^2}{150}(g_{ij}g_{hk}g_{lm}-g_{ij}g_{hm}g_{lk}-g_{ik}g_{hj}g_{lm}+g_{ik}g_{hm}g_{lj}+g_{im}g_{hj}g_{lk}-g_{im}g_{hk}g_{lj})\Big\}\\
&+8\Big\{R_{ahjk}R_{amil}-R_{aljk}R_{amih}-R_{aijk}R_{amhl}+R_{aijm}R_{akhl}-R_{ahjm}R_{akil}\\
&\quad+R_{aljm}R_{akih}-R_{aikm}R_{ajhl}+R_{ahkm}R_{ajil}-R_{alkm}R_{ajih}\\
&\quad-\frac{2}{15}\tau(R_{ilkm}g_{hj}-R_{iljm}g_{hk}-R_{hljk}g_{im}-R_{ihjk}g_{lm}-R_{ihkm}g_{lj}\\
&\qquad+R_{ihjm}g_{lk}+R_{iljk}g_{hm}-R_{hlkm}g_{ij}+R_{hljm}g_{ik})\\
&\quad+\frac{\tau^2}{150}(g_{ij}g_{hk}g_{lm}-g_{ij}g_{hm}g_{lk}-g_{ik}g_{hj}g_{lm}+g_{ik}g_{hm}g_{lj}+g_{im}g_{hj}g_{lk}-g_{im}g_{hk}g_{lj})\Big\}.
\end{aligned}
$$

Rearranging these terms, we have

$$
\begin{aligned}
0=&\Big\{\Big(||R||^2+\frac{\tau^2}{3}\Big)(g_{ij}g_{hk}g_{lm}-g_{ij}g_{hm}g_{lk}-g_{ik}g_{hj}g_{lm}+g_{ik}g_{hm}g_{lj}+g_{im}g_{hj}g_{lk}-g_{im}g_{hk}g_{lj})\Big\}\\
&-4\Big\{\overline{R}_{ij}(g_{hk}g_{lm}-g_{hm}g_{lk})+\overline{R}_{ik}(g_{hm}g_{lj}-g_{hj}g_{lm})+\overline{R}_{im}(g_{hj}g_{lk}-g_{hk}g_{lj})\\
&\quad+\overline{R}_{hj}(g_{im}g_{lk}-g_{ik}g_{lm})+\overline{R}_{hk}(g_{ij}g_{lm}-g_{im}g_{lj})+\overline{R}_{hm}(g_{ik}g_{lj}-g_{ij}g_{lk})\\
&\quad+\overline{R}_{lj}(g_{ik}g_{hm}-g_{im}g_{hk})+\overline{R}_{lk}(g_{im}g_{hj}-g_{ij}g_{hm})+\overline{R}_{lm}(g_{ij}g_{hk}-g_{ik}g_{hj})\Big\}\\
&+\Big(-8R_{iabj}R_{kabh}+8R_{iabk}R_{jabh}+4R_{abih}R_{abjk}+\frac{8}{3}\tau R_{ihjk}\Big)g_{lm}\\
&-\Big(-8R_{iabj}R_{kabl}+8R_{iabk}R_{jabl}+4R_{abil}R_{abjk}+\frac{8}{3}\tau R_{iljk}\Big)g_{hm}\\
&-\Big(-8R_{iabj}R_{mabh}+8R_{iabm}R_{jabh}+4R_{abih}R_{abjm}+\frac{8}{3}\tau R_{ihjm}\Big)g_{lk}\\
&+\Big(-8R_{iabj}R_{mabl}+8R_{iabm}R_{jabl}+4R_{abil}R_{abjm}+\frac{8}{3}\tau R_{iljm}\Big)g_{hk}\\
&+\Big(-8R_{iabk}R_{mabh}+8R_{iabm}R_{kabh}+4R_{abih}R_{abkm}+\frac{8}{3}\tau R_{ihkm}\Big)g_{lj}\\
&-\Big(-8R_{iabk}R_{mabl}+8R_{iabm}R_{kabl}+4R_{abil}R_{abkm}+\frac{8}{3}\tau R_{ilkm}\Big)g_{hj}\\
&+\Big(-8R_{habj}R_{kabl}+8R_{habk}R_{jabl}+4R_{abhl}R_{abjk}+\frac{8}{3}\tau R_{hljk}\Big)g_{im}\\
&-\Big(-8R_{habj}R_{mabl}+8R_{habm}R_{jabl}+4R_{abhl}R_{abjm}+\frac{8}{3}\tau R_{hljm}\Big)g_{ik}
\end{aligned}
$$

$$+\Big(-8R_{habk}R_{mabl}+8R_{habm}R_{kabl}+4R_{abhl}R_{abkm}+\frac{8}{3}\tau R_{hlkm}\Big)g_{ij}$$
$$+8(R_{ahjk}R_{amil}-R_{aljk}R_{amih}-R_{aijk}R_{amhl}+R_{aijm}R_{akhl}-R_{ahjm}R_{akil}+R_{aljm}R_{akih}$$
$$-R_{aikm}R_{ajhl}+R_{ahkm}R_{ajil}-R_{alkm}R_{ajih}).$$

This completes the proof of Lemma 4.1. □

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