Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2096 – 2108
© TÜBİTAK
doi:10.55730/1300-0098.3256

Distortion bound and growth theorems for a subclass of analytic functions defined by q-derivative

Osman ALTINTAŞ[1], **Nizami MUSTAFA**[2,*]
[1]Department of Mathematics Education, Faculty of Education, Başkent University, Ankara, Turkey
[2]Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey

Received: 08.10.2021 • **Accepted/Published Online:** 15.04.2022 • **Final Version:** 04.07.2022

Abstract: In this study, we introduce and examine a subclasses of analytic and univalent functions defined by q-derivative. Here, we give necessary conditions for the functions to belong to these subclasses, and distortion bound and growth theorems for the functions belonging to these classes.

Key words: Coefficient bound, convex function, q- derivative, quasi close-to-q -convex function

1. Introduction and preliminaries

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane and $H(U)$ is the class of analytic functions in U. By A, we denote the class of functions $f \in H(U)$ given in the following form

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ... = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}. \tag{1.1}$$

Also, let T be the subclass of A with nonpositive coefficients

$$f(z) = z - a_2 z^2 - a_3 z^3 - ... - a_n z^n - ... = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{1.2}$$

We denote by S the subclass of A consisting of the functions univalent in U. For $\alpha \in [0, 1)$, one of the important subclass of S is $C(\alpha)$ - convex function class of order α (see for details [8, 10] , also [20]).

We define this class as in the following way:

$$C(\alpha) = \left\{ f \in S : \mathrm{Re}\left(\frac{(zf'(z))'}{f'(z)} \right) > \alpha, z \in U \right\}.$$

In special case, when $\alpha = 0$, $C = C(0)$ is well known convex function class in U.

As it is known that a function f is subordine to the function g and written as $f \prec g$ if there exist a (Schwartz) function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ (see [10]).

*Correspondence: nizamimustafa@gmail.com

2010 *AMS Mathematics Subject Classification:* 30C45, 30C50, 30C80

For $q \in (0,1)$, in the fundamental paper [12] by Jackson the q-derivative D_q of a function $f \in H(U)$ was introduced as follows:

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & \text{if} \quad z \neq 0, \\ f^{'}(0), & \text{if} \quad z = 0. \end{cases} \tag{1.3}$$

It follos from that $\lim\limits_{q\to 1^-} D_q f(z) = f'(z)$.

From (1.3), we can easily see that

$$D_q z^n = [n]_q z^{n-1}, n \in \mathbb{N}, \tag{1.4}$$

where $[n]_q = \sum\limits_{k=1}^{n} q^{k-1}$ is the q-analogue of the natural numbers n (which is called the basic number n). It can be easily shown that $[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}$, $[0]_q = 0$, $[1]_q = 1$, $\lim\limits_{q\to 1^-} [n]_q = n$, $D_q (zD_q f(z)) = D_q f(z) + zD_q^2 f(z)$. For more properties of D_q see [9, 11, 14].

Studies on the q-derivative were firstly conducted by Jackson [13], Carmichael [7], Mason [15], Adams [1] and Trjizinsky [21]. But, this topic has been forgotten for a long time. Later, some properties related with function theory involving q-theory were introduced by Ismail et al. in the paper [11]. The q-derivative has wide applications in the geometric function theory. This subject still continues to be the subject of study of many mathematicians today (see [2, 17, 18]). As the study [2] suggests, there are a lot that can be done for this research topic. For example, q-analogy of starlikeness and convexity of analytic functions in the open unit disk and in arbitrary simply connected domains would be interesting for researchers in this field.

In [3], by using applications of the q-derivative, it was shown that Szasz Mirakyan operators are convex when convex functions are taken such that their result generalizes well known results for $q = 1$. Also, in [3] the authors showed that q-derivatives of these operators approach to q-derivatives of approximated functions.

Very recently, by Uçar et al. [23] and Uçar [22] some properties of q-close-to-convex functions were studied. By Polatoğlu in [16], while q - starlike functions were investigated, growth and distortion theorems for this class were given. Quasi starlike and quasi convex functions were studied by Altıntaş and Kılıc in [5]. Altıntaş and Mustafa studied quasi q-starlike and quasi q-convex functions (see [6]).

Very recently, in [4] Altıntaş and Aydoğan studied quasi q-convex functions.

For $q \in (0,1)$ and $\alpha \in [0,1)$, we define by $C_q(\alpha)$ the subclass of A, which we will call q-convex function class of order α, as follows:

$$C_q(\alpha) = \left\{ f \in S : \operatorname{Re} \frac{D_q (zD_q f(z))}{D_q f(z)} > \alpha, z \in U \right\}.$$

Also, we will denote $TC_q(\alpha) = T \cap C_q(\alpha)$.

Let $g \in T$ be given as the following series:

$$g(z) = z - b_2 z^2 - b_3 z^3 - ... - b_n z^n - ... = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0. \tag{1.5}$$

Inspired by the studies mentioned above, we introduce the following subclasses of analytic functions.

Definition 1.1 *For $\lambda,\mu\in[0,1]$, $\alpha,\beta\in[0,1)$, a function $f\in T$ in the form (1.2) is said to be in the class quasi close-to-q-convex functions with respect to parameters μ and λ of order α and is shown as $QCC_q(\lambda,\mu,\beta,\alpha)$ if the following condition is satisfied*

$$\Re\left(\frac{zD_qf(z)+\lambda z^2D_q^2f(z)}{\mu zD_qg(z)+(1-\mu)\,g(z)}\right)>\alpha, z\in U, \tag{1.6}$$

where $g\in C_q(\beta)$.

Definition 1.2 *For $\lambda,\mu\in[0,1]$, $\beta\in[0,1)$, $-1\le B<A\le 1$, a function $f\in T$ in the form (1.2) is said to be in the class quasi close-to-q-convex functions with respect to parameters μ and λ with subordination and is shown as $QCC_qP(\lambda,\mu,\beta,A,B)$ if the following condition is satisfied*

$$\frac{zD_qf(z)+\lambda z^2D_q^2f(z)}{\mu zD_qg(z)+(1-\mu)\,g(z)}\prec\frac{1+Az}{1+Bz}, z\in U, \tag{1.7}$$

where $g\in C_q(\beta)$.

From Definitions 1.1 and 1.2, for the different values of parameters, we have the following subclasses of analytic functions in the open unit disk U.

Remark 1.3 *1.1. $QCC_q(\lambda,0,\beta,\alpha)$ is defined by*

$$QCC_q(\lambda,0,\beta,\alpha)=\left\{f\in T:\Re\left(\frac{zD_qf(z)+\lambda z^2D_q^2f(z)}{g(z)}\right)>\alpha, z\in U\right\}.$$

1.2. $QCC_q(\lambda,1,\beta,\alpha)$ is defined by

$$QCC_q(\lambda,1,\beta,\alpha)=\left\{f\in T:\Re\left(\frac{D_qf(z)+\lambda zD_q^2f(z)}{D_qg(z)}\right)>\alpha, z\in U\right\}.$$

.

1.3. $QCC_q(0,0,\beta,\alpha)$ is defined by

$$QCC_q(0,0,\beta,\alpha)=\left\{f\in T:\Re\left(\frac{zD_qf(z)}{g(z)}\right)>\alpha, z\in U\right\}.$$

1.4. $QCC_q(0,1,\beta,\alpha)=CC_q(\beta,\alpha)$-close-to-$q$-convex functions order α is defined by

$$CC_q(\beta,\alpha)=\left\{f\in T:\Re\left(\frac{D_qf(z)}{D_qg(z)}\right)>\alpha, z\in U\right\}.$$

1.5. $QCC_q(1,0,\beta,\alpha)$ is defined by

$$QCC_q(1,0,\beta,\alpha) = \left\{ f \in T : \Re\left(\frac{zD_q f(z) + z^2 D_q^2 f(z)}{g(z)} \right) > \alpha, z \in U \right\}.$$

1.6. $QCC_q(1,1,\beta,\alpha)$ *is defined by*

$$QCC_q(1,1,\beta,\alpha) = \left\{ f \in T : \Re\left(\frac{D_q f(z) + zD_q^2 f(z)}{D_q g(z)} \right) > \alpha, z \in U \right\}.$$

1.7. $QCC_q P(\lambda,0,\beta,A,B)$ *is defined by*

$$QCC_q P(\lambda,0,\beta,A,B) = \left\{ f \in T : \frac{zD_q f(z) + \lambda z^2 D_q^2 f(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, z \in U \right\}.$$

1.8. $QCC_q P(\lambda,1,\beta,A,B)$ *is defined by*

$$QCC_q P(\lambda,1,\beta,A,B) = \left\{ f \in T : \frac{D_q f(z) + \lambda zD_q^2 f(z)}{D_q g(z)} \prec \frac{1+Az}{1+Bz}, z \in U \right\}.$$

1.9. $QCC_q P(0,0,\beta,A,B)$ *is defined by*

$$QCC_q P(0,0,\beta,A,B) = \left\{ f \in T : \frac{zD_q f(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, z \in U \right\}.$$

1.10. $QCC_q P(1,0,\beta,A,B)$ *is defined by*

$$QCC_q P(1,0,\beta,A,B) = \left\{ f \in T : \frac{zD_q f(z) + z^2 D_q^2 f(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, z \in U \right\}.$$

1.11. $QCC_q P(0,1,\beta,A,B) = CC_q P(\beta,A,B)$ *is defined by*

$$CC_q P(\beta,A,B) = \left\{ f \in T : \frac{D_q f(z)}{D_q g(z)} \prec \frac{1+Az}{1+Bz}, z \in U \right\}.$$

1.12. $QCC_q P(1,1,\beta,A,B)$ *is defined by*

$$QCC_q P(1,1,\beta,A,B) = \left\{ f \in T : \frac{D_q f(z) + zD_q^2 f(z)}{D_q g(z)} \prec \frac{1+Az}{1+Bz}, z \in U \right\}.$$

1.13. $QCC(\lambda,\mu,\beta,\alpha)$ *- the class quasi close-to-convex functions with respect to parameters* μ *and* λ *of order* α *is defied by*

$$QCC(\lambda,\mu,\beta,\alpha) = \left\{ f \in T : \Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{\mu zg'(z) + (1-\mu)g(z)} \right) > \alpha, z \in U \right\}.$$

1.14.$QCC(0,1,\beta,\alpha) = CC(\beta,\alpha)$ *- close-to-convex functions of order* α *is defined by*

$$CC(\beta,\alpha)=\left\{f\in T:\Re\left(\frac{f'(z)}{g'(z)}\right)>\alpha, z\in U\right\}.$$

1.15. $QCCP(0,1,\beta,A,B)=CCP(\beta,A,B)$ *is defined by*

$$CCP(\beta,A,B)=\left\{f\in T:\frac{f'(z)}{g'(z)}\prec\frac{1+Az}{1+Bz}, z\in U\right\}.$$

The results obtained in our study are valid for all the classes given in the above remarks in the special values of the parameters. So this work covers the broad class of analytical functions.

In this study, we give necessary conditions for the functions to belong to classes $QCC_q(\lambda,\mu,\beta,\alpha)$ and $QCC_qP(\lambda,\mu,\beta,A,B)$. Also, here distortion bound and growth theorems for the functions belonging to these classes are given.

To prove our main results, we shall require the following lemma.

Lemma 1.4 *([6])If* $f\in T$ *belongs to the class* $C_q(\alpha)$ *for* $q\in(0,1)$, $\alpha\in[0,1)$, *then the following condition is satisfied*

$$\sum_{n=2}^{\infty}\left([n]_q-\alpha\right)[n]_q a_n\leq 1-\alpha.$$

The result obtained here is sharp.

2. Main results

In this section, we give the following theorems, which states the necessary conditions for a function to belong to these classes $QCC_q(\lambda,\mu,\beta,\alpha)$ and $QCC_qP(\lambda,\mu,\beta,A,B)$, respectively.

Theorem 2.1 *If* $f\in QCC_q(\lambda,\mu,\beta,\alpha)$, *then the following condition is satisfied*

$$\sum_{n=2}^{\infty}\left(1+\lambda[n-1]_q\right)[n]_q a_n\leq 1-\alpha+\frac{\alpha(1-\beta)\left[1+\left([2]_q-1\right)\mu\right]}{\left([2]_q-\beta\right)[2]_q}. \tag{2.1}$$

The result obtained here is sharp.

Proof Assume that $f\in QCC_q(\lambda,\mu,\beta,\alpha)$ for $\lambda,\mu\in[0,1]$, $\alpha,\beta\in[0,1)$. Then,

$$\Re\left(\frac{zD_qf(z)+\lambda z^2D_q^2f(z)}{\mu zD_qg(z)+(1-\mu)g(z)}\right)>\alpha, \tag{2.2}$$

where the function $g\in C_q(\beta)$ is defined by (1.5).

From the definition of $q-$derivative, by simple computation, we can write

$$\frac{zD_qf(z)+\lambda z^2D_q^2f(z)}{\mu zD_qg(z)+(1-\mu)\,g\,(z)}=\frac{z-\sum\limits_{n=2}^{\infty}\left(1+\lambda\,[n-1]_q\right)[n]_q\,a_nz^n}{z-\sum\limits_{n=2}^{\infty}\left[1+\left([n]_q-1\right)\mu\right]b_nz^n}.$$

In that case, the inequality (2.2) is written as follows:

$$\Re\left(\frac{z-\sum\limits_{n=2}^{\infty}\left(1+\lambda\,[n-1]_q\right)[n]_q\,a_nz^n}{z-\sum\limits_{n=2}^{\infty}\left[1+\left([n]_q-1\right)\mu\right]b_nz^n}\right)>\alpha.$$

It is clear that the fraction in the parentheses in the last inequality is real if z is chosen real. Therefore, letting $z\to 1^-$ through real values, we have

$$\frac{1-\sum\limits_{n=2}^{\infty}\left(1+\lambda\,[n-1]_q\right)[n]_q\,a_n}{1-\sum\limits_{n=2}^{\infty}\left[1+\left([n]_q-1\right)\mu\right]b_n}\geq\alpha;$$

that is,

$$1-\sum_{n=2}^{\infty}\left(1+\lambda\,[n-1]_q\right)[n]_q\,a_n\geq\alpha\left\{1-\sum_{n=2}^{\infty}\left[1+\left([n]_q-1\right)\mu\right]b_n\right\}.$$

The last inequality is equivalent to

$$\sum_{n=2}^{\infty}\left(1+\lambda\,[n-1]_q\right)[n]_q\,a_n\leq 1-\alpha+\alpha\sum_{n=2}^{\infty}\left(1-\mu+\mu\,[n]_q\right)b_n. \tag{2.3}$$

According to Lemma 1.4, if $g\in C_q\,(\beta)\,,\ q\in(0,1)\,,\ \beta\in[0,1)\,,$ then

$$\sum_{n=2}^{\infty}\left([n]_q-\beta\right)[n]_q\,b_n\leq 1-\beta. \tag{2.4}$$

From the inequality (2.4), we obtain

$$\sum_{n=2}^{\infty}[n]_q\,b_n\leq\frac{1-\beta}{[2]_q-\beta}. \tag{2.5}$$

Similarly

$$\sum_{n=2}^{\infty}b_n\leq\frac{1-\beta}{\left([2]_q-\beta\right)[2]_q}. \tag{2.6}$$

Using (2.5) and (2.6), we can write

$$\begin{aligned}\sum_{n=2}^{\infty}\left(1-\mu+\mu\,[n]_q\right)b_n &= (1-\mu)\sum_{n=2}^{\infty}b_n+\mu\sum_{n=2}^{\infty}[n]_q\,b_n\leq\\ \frac{(1-\mu)(1-\beta)}{\left([2]_q-\beta\right)[2]_q}+\frac{\mu(1-\beta)}{[2]_q-\beta} &= \frac{(1-\beta)\left[1+\left([2]_q-1\right)\mu\right]}{\left([2]_q-\beta\right)[2]_q}.\end{aligned}$$

Considering the last inequality in (2.3), we obtain the inequality (2.1).

To see that result obtained in the theorem is sharp, we note that equality is attained in the inequality when f is chosen as follows:

$$f(z)=f_n(z)=z-\frac{(1-\alpha)\left([2]_q-\beta\right)[2]_q+\alpha(1-\beta)\left[1+\left([2]_q-1\right)\mu\right]}{\left(1+\lambda\,[n-1]_q\right)\left([2]_q-\beta\right)[2]_q\,[n]_q}z^n$$

for each $n=2,3,\ldots$.

Thus, the proof of Theorem 2.1 is completed. $\square$

Since $\lim\limits_{q\to1^-}QCC_q(\lambda,\mu,\beta,\alpha)=QCC(\lambda,\mu,\beta,\alpha)$ and $\lim\limits_{q\to1^-}[n]_q=n$, from Theorem 2.1, we obtain the following result.

Theorem 2.2 *If $f\in QCC(\lambda,\mu,\beta,\alpha)$, then we have*

$$\sum_{n=2}^{\infty}n\,(1+\lambda(n-1))\,a_n\leq 1-\alpha+\frac{\alpha(1-\beta(1+\mu))}{2(2-\beta)}.$$

Theorem 2.3 *Let $f\in QCC_qP(\lambda,\mu,\beta,A,B)$. Then, the following inequality is satisfied*

$$\sum_{n=2}^{\infty}\left(1+\lambda\,[n-1]_q\right)[n]_q\,a_n\leq\frac{A-B}{1-B}+\frac{(1-A)(1-\beta)\left[1+\left([2]_q-1\right)\mu\right]}{(1-B)\left([2]_q-\beta\right)[2]_q}. \tag{2.7}$$

The result obtained here is sharp.

Proof Assume that $f\in QCC_qP(\lambda,\mu,A,B)$ for $\lambda,\mu\in[0,1]$, $\beta\in[0,1),-1\leq B<A\leq1$. That is,

$$\frac{zD_qf(z)+\lambda z^2D_q^2f(z)}{\mu zD_qg(z)+(1-\mu)\,g(z)}\prec\frac{1+Az}{1+Bz},z\in U.$$

Then, using the inequality (2.6) from [6], we have

$$\Re\left(\frac{zD_qf(z)+\lambda z^2D_q^2f(z)}{\mu zD_qg(z)+(1-\mu)\,g(z)}\right)>v,$$

with $\frac{1-A}{1-B} = v \in [0,1)$ (see, also [19]). This means that $f \in QCC_q(\lambda, \mu, \beta, v)$ with $\nu = \frac{1-A}{1-B}$. In this case, from Theorem 2.1, we have

$$\sum_{n=2}^{\infty} \left(1 + \lambda [n-1]_q\right) [n]_q a_n \le 1 - v + \frac{v(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{\left([2]_q - \beta\right)[2]_q},$$

which is the same of the inequality (2.7).

Since equality is attained in the inequality when the function f is chosen as follows:

$$f(z) = f_n(z) = z - \frac{(A-B)\left([2]_q - \beta\right)[2]_q + (1-A)(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{(1-B)\left(1 + \lambda[n-1]_q\right)\left([2]_q - \beta\right)[2]_q [n]_q} z^n$$

for each $n = 2, 3, \ldots$, the result obtained in the theorem is sharp.

Thus, the proof of Theorem 2.3 is completed. □

For different values of the parameters, from Theorems 2.1 and 2.3, numerous results can be obtained. We give some of them below.

Corollary 2.4 *([4]) If $f \in QCC_qP(\lambda, 1, \beta, A, B)$, then we have*

$$\sum_{n=2}^{\infty} \left(1 + \lambda [n-1]_q\right) [n]_q a_n \le 1 - \frac{(1-A)\left([2]_q - 1\right)}{(1-B)\left([2]_q - \beta\right)}$$

The result is sharp.

Corollary 2.5 *If $f \in QCC_q(\lambda, 1, \beta, \alpha)$, then we have*

$$\sum_{n=2}^{\infty} \left(1 + \lambda [n-1]_q\right) [n]_q a_n \le 1 - \frac{\left([2]_q - 1\right)\alpha}{[2]_q - \beta}.$$

The result is sharp.

Corollary 2.6 *If $f \in CC_q(\beta, \alpha)$, then we have*

$$\sum_{n=2}^{\infty} [n]_q a_n \le 1 - \frac{\left([2]_q - 1\right)\alpha}{[2]_q - \beta}.$$

The result is sharp.

Corollary 2.7 *If $f \in QCC_q(1, 1, \beta, \alpha)$, then we have*

$$\sum_{n=2}^{\infty} \left(1 + [n-1]_q\right) [n]_q a_n \le 1 - \frac{\left([2]_q - 1\right)\alpha}{[2]_q - \beta}.$$

The result is sharp.

Corollary 2.8 *If* $f \in QCC_q P(\lambda, 1, \beta, A, B)$*, then we have*

$$\sum_{n=2}^{\infty} \left(1 + \lambda [n-1]_q\right) [n]_q a_n \le \frac{\left([2]_q - 1\right)(A-B) + (1-\beta)(1-B)}{(1-B)\left([2]_q - \beta\right)}.$$

The result is sharp.

Corollary 2.9 *If* $f \in CC_q P(\beta, A, B)$*, then we have*

$$\sum_{n=2}^{\infty} [n]_q a_n \le \frac{\left([2]_q - 1\right)(A-B) + (1-\beta)(1-B)}{(1-B)\left([2]_q - \beta\right)}.$$

The result is sharp.

Corollary 2.10 *If* $f \in QCC_q P(1, 1, \beta, A, B)$*, then we have*

$$\sum_{n=2}^{\infty} \left(1 + [n-1]_q\right) [n]_q a_n \le \frac{\left([2]_q - 1\right)(A-B) + (1-\beta)(1-B)}{(1-B)\left([2]_q - \beta\right)}.$$

The result is sharp.

Corollary 2.11 *If* $f \in CC(\beta, \alpha)$*, then we have*

$$\sum_{n=2}^{\infty} n a_n \le 1 - \frac{\alpha}{2-\beta}.$$

The result is sharp.

Corollary 2.12 *If* $f \in CCP(\beta, A, B)$*, then we have*

$$\sum_{n=2}^{\infty} n a_n \le \frac{(A-B) + (1-\beta)(1-B)}{(1-B)(2-\beta)}.$$

The result is sharp.

Also, from Theorem 2.1, we obtain the following results on the coefficient bounds.

Theorem 2.13 *Let* $f \in QCC_q(\lambda, \mu, \beta, \alpha)$*. Then, the following inequalities are satisfied*

$$\sum_{n=2}^{\infty} a_n \le \frac{1}{(1+\lambda)[2]_q} \left\{ 1 - \alpha + \frac{\alpha(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{\left([2]_q - \beta\right)[2]_q} \right\}, \tag{2.8}$$

$$\sum_{n=2}^{\infty} [n]_q a_n \le \frac{1}{1+\lambda} \left\{ 1 - \alpha + \frac{\alpha(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{\left([2]_q - \beta\right)[2]_q} \right\}. \tag{2.9}$$

Proof Suppose that $f \in QCC_q(\lambda, \mu, \beta, \alpha)$ for $\lambda, \mu \in [0,1]$, $\alpha, \beta \in [0,1)$. In this case, from Theorem 2.1, we have

$$\sum_{n=2}^{\infty} \left(1 + \lambda [n-1]_q\right) [n]_q a_n \leq 1 - \alpha + \frac{\alpha(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{\left([2]_q - \beta\right)[2]_q}.$$

It follows from that

$$(1+\lambda)[2]_q \sum_{n=2}^{\infty} a_n \leq 1 - \alpha + \frac{\alpha(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{\left([2]_q - \beta\right)[2]_q}.$$

From this inequality, (2.8) is obtained immediately.

Similarly, from Theorem 2.1, we have

$$(1+\lambda) \sum_{n=2}^{\infty} [n]_q a_n \leq 1 - \alpha + \frac{\alpha(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{\left([2]_q - \beta\right)[2]_q}.$$

From this, the inequality (2.9) is obtained.

Thus, the proof of Theorem 2.13 is completed. □

The following theorem is a direct result of Theorem 2.1.

Theorem 2.14 *If $f \in QCC_q(\lambda, \mu, \beta, \alpha)$, then the following inequality is satisfied*

$$a_n \leq \frac{1}{(1+\lambda)[n]_q} \left\{ 1 - \alpha + \frac{\alpha(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{\left([2]_q - \beta\right)[2]_q} \right\}, n = 2, 3, \ldots .$$

Remark 2.15 *Numerous consequences of the results obtained in Theorems 2.3 and 2.13 can be given for different values of the parameters.*

The following theorems can be proved similarly to the proof of Theorems 2.13 and 2.14.

Theorem 2.16 *If $f \in QCC_qP(\lambda, \mu, \beta, A, B)$, then the following inequalities are satisfied*

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{(1+\lambda)[2]_q} \left\{ \frac{A-B}{1-B} + \frac{(1-A)(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{(1-B)\left([2]_q - \beta\right)[2]_q} \right\}, \tag{2.10}$$

$$\sum_{n=2}^{\infty} [n]_q a_n \leq \frac{1}{1+\lambda} \left\{ \frac{A-B}{1-B} + \frac{(1-A)(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{(1-B)\left([2]_q - \beta\right)[2]_q} \right\}. \tag{2.11}$$

Theorem 2.17 *If $f \in QCC_qP(\lambda, \mu, \beta, A, B)$, then we have*

$$a_n \leq \frac{1}{(1+\lambda)[n]_q} \left\{ \frac{A-B}{1-B} + \frac{(1-A)(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{(1-B)\left([2]_q - \beta\right)[2]_q} \right\}, \; n = 2, 3, \ldots .$$

Remark 2.18 *From the above Theorems 2.16 and 2.17, numerous consequences for different values of the parameters can be given.*

3. Distortion bound and growth theorems

In this section, we give distortion bound and growth theorems for the functions belonging to the classes $QCC_q(\lambda, \mu, \beta, \alpha)$ and $QCC_qP(\lambda, \mu, \beta, A, B)$.

Theorem 3.1 *Let $f \in QCC_q(\lambda, \mu, \beta, \alpha)$. Then, we have*

$$r - r^2\theta(\lambda, \mu, \beta, \alpha, q) \leq |f(z)| \leq r + r^2\theta(\lambda, \mu, \beta, \alpha, q), \quad |z| = r \leq 1,$$

where

$$\theta(\lambda, \mu, \beta, \alpha, q) = \frac{1}{(1+\lambda)[2]_q}\left\{1 - \alpha + \frac{\alpha(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{\left([2]_q - \beta\right)[2]_q}\right\}. \tag{3.1}$$

Proof Assume that $f \in QCC_q(\lambda, \mu, \beta, \alpha)$ for $\lambda, \mu \in [0,1]$, $\alpha, \beta \in [0,1)$. Then, using the first inequality of Theorem 2.13, we can easily show that

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} a_n |z^n| \leq r + r^2 \sum_{n=2}^{\infty} a_n \\ &\leq r + \frac{r^2}{(1+\lambda)[2]_q}\left\{1 - \alpha + \frac{\alpha(1-\beta)}{\left([2]_q - \beta\right)[2]_q}\left[1 + \left([2]_q - 1\right)\mu\right]\right\} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} a_n |z^n| \geq r - r^2 \sum_{n=2}^{\infty} a_n \\ &\geq r - \frac{r^2}{(1+\lambda)[2]_q}\left\{1 - \alpha + \frac{\alpha(1-\beta)}{\left([2]_q - \beta\right)[2]_q}\left[1 + \left([2]_q - 1\right)\mu\right]\right\}. \end{aligned}$$

Combination of these inequalities gives us the results of the theorem.
Thus, the proof of Theorem 3.1 is competed. □

Theorem 3.2 *If $f \in QCC_qP(\lambda, \mu, \beta, A, B)$, then we have*

$$r - r^2\varphi(\lambda, \mu, \beta, A, B, q) \leq |f(z)| \leq r + r^2\varphi(\lambda, \mu, \beta, A, B, q), \quad |z| = r \leq 1,$$

where

$$\varphi(\lambda, \mu, \beta, A, B, q) = \frac{1}{(1+\lambda)[2]_q}\left\{\frac{A-B}{1-B} + \frac{(1-A)(1-\beta)\left[1 + \left([2]_q - 1\right)\mu\right]}{(1-B)\left([2]_q - \beta\right)[2]_q}\right\}. \tag{3.2}$$

Proof The proof of Theorem 3.2 can be proved similarly to the proof of Theorem 3.1. For this reason, we do not give the proof of Theorem 3.2 in detail in order not to increase the volume of the study. □

The following theorems can be proved similarly to the proof of Theorem 3.1.

Theorem 3.3 *If $f \in QCC_q(\lambda, \mu, \beta, \alpha)$, then the following inequality is satisfied*

$$1 - [2]_q r\theta(\lambda, \mu, \beta, \alpha, q) \leq |D_q f(z)| \leq 1 + [2]_q r\theta(\lambda, \mu, \beta, \alpha, q), |z| = r \leq 1,$$

where the function $\theta(\lambda, \mu, \beta, \alpha, q)$ is defined by (3.1).

Theorem 3.4 *If $f \in QCC_q P(\lambda, \mu, \beta, A, B)$, then we have*

$$1 - [2]_q r\varphi(\lambda, \mu, \beta, A, B, q) \leq |D_q f(z)| \leq 1 + [2]_q r\varphi(\lambda, \mu, \beta, A, B, q), |z| = r \leq 1,$$

where the function $\varphi(\lambda, \mu, \beta, \alpha, q)$ is defined by (3.2).

Remark 3.5 *Numerous consequences of the results obtained in Theorems 3.1–3.4 can be given for the subclasses defined in Remark 1.3.*

References

[1] Adams CR. On the linear ordinary q-difference equation. Annals of Mathematics 1929; 30: 195-205.

[2] Agrawal S, Sahoo SK. Geometric properties of basic hypergeometric functions. Journal of Difference Equations and Applications 2014; 20 (11): 1502-1522.

[3] Ali A, Viyaj G. The q-derivative and applications to q-Szasz Mirakyan operator. Calcolo 2006; 43 (3): 151-170.

[4] Altıntaş O, Aydoğan M. Coefficient estimates for the class of quasi q - convex functions. Turkish Journal of Mathematics 2020; 44 (1): 342-347.

[5] Altıntaş O, Kılıc ÖÖ. Coefficient estimates for a class containing quasi-convex functions. Turkish Journal of Mathematics 2018; 42 (5): 2819-2825.

[6] Altıntaş O, Mustafa N. Coefficient bounds and distortion theorems for the certain analytic functions. Turkish Journal of Mathematics 2019; 43 (2): 985-997.

[7] Carmichael RD. The general theory of linear q-difference equations. American Journal of Mathematics 1912; 34: 147-168.

[8] Darus M and Thomas DK. Logaritmically convex functions. Indian Journal of Pure and Applied Mathematics 1998; 29 (10): 1049-1059.

[9] Exton H. q-Hypergeomtric Functions and Applications. Ellis Horwood Limited, Chichester, 1983

[10] Goodman AW. Univalent Functions. Volume I. Washington, USA: Polygonal, 1983.

[11] Ismail MEH, Merkes E, Styer D. A generalization of starlike functions. Complex Variables 1990; 14: 77-84.

[12] Jackson FH. On q-functions and a certain difference operator. Transactions of the Royal Society of Edinburgh 1908; 46: 253-281.

[13] Jackson FH. On q-definite integrals. The Quarterly Journal of Pure and Apllied Mathematics 1910; 41: 193-203.

[14] Jackson FH. On basic double hypergeometric functions. The Quarterly Journal of Mathematics 1942; 13: 69-82.

[15] Mason TE. On properties of the solution of linear q - difference equations with entire function coefficients. American Journal of Mathematics 1915; 37: 439-444.

[16] Polatoğlu Y. Growth and distortion theorems for generalised q- starlike functions. Advances in Mathematics Scientific Journal 2016; 5 (1): 7-12.

[17] Raghavender K, Swaminathan A. Close-to-convexity of basic hypergeometric functions using their Taylor coefficients. Journal of Mathematical Analysis and Applications 2012; 35: 111-125.

[18] Sahoo SK, Sharma NL. On a generalization of close-to-convex functions. Annales Polonici Mathematicia 2015; 113 (1): 93-108.

[19] Srivastava HM, Altıntaş O, Serenbay SK. Coefficient bounds for certain subclasses of starlike functions of complex order. Applied Mathematics Letters 2011; 24: 1359-1363.

[20] Srivastava HM, Owa S. Editors, Current Topics in Analytic Function Theory. World Scientific, Singapure, 1992.

[21] Trjizinsky WJ. Analytic theory of linear q- difference equations. Acta Mathematica 1933; 61 (1): 1-38.

[22] Uçar HE. Coefficient inequality for q - starlike functions. Applied Mathematics and Computation 2016; 276: 122-126.

[23] Uçar HE, Mert O, Polatoğlu Y. Some properties of q - close-to-convex functions. Hacettepe Journal of Mathematics and Statistics 2017; 46 (6): 1105-1112.