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Introducing selective d-separability in bitopological spaces

Dedicated to the memory of Professor Hans-Peter Künzi

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Abstract: We introduce D-separability and its game-theoretic version, D^+-separability in bitopological spaces, and investigate their relationships with d-separability and a weaker form of H-separability which will be called DH-separability. Further we give the connection of these notions with the selective versions of separability-types properties under the bitopological context. We also obtain some results about the d-separability properties of bitopological spaces which are slightly different from those one expects for the classical case

Key words: Selection principles, bitopological space, D-separability, selective separability

1. Introduction

The selective version of separability in bitopological spaces was studied by Kočinac and Özçağ in [12]. In this paper, we introduce a selective (and stronger) version of d-separability in bitopological context. On the first side, one expects to get as the similar results about the d-separability properties for bitopological spaces as obtained for the classical case, however as being seen the selective version of d-separability of bitopological spaces is slightly different from the classical case.

A space is called d-separable if it has a dense subset representable as the union of countably many discrete subsets. In [2] Bella, Matveev and Spadaro presented the notion called D-separability which is a selective version of the d-separability, based on the foundation laid by Kurepa in [15]. In the same paper, the authors introduced D^+-separability, a game theoretic version of D-separability. The relationships between these concepts were considered in detail by Aurichi, Dias, and Jungueira in [1]. These selective versions and some ZFC examples separating these properties have been studied by several authors [1, 2, 24]

Definition 1.1 [2] A topological space X is called:

(a) D-*separable* if for every sequence $(D_n : n \in \mathbb{N})$ of dense subsets of X there is a sequence $(E_n : n \in \mathbb{N})$ of discrete sets of X such that for all $n \in \mathbb{N}$ $E_n \subset D_n$ and $\bigcup_{n\in\mathbb{N}} E_n$ is dense in X.

(b) D^+-*separable* if TWO has the winning strategy in the game $\mathsf{G}_{\mathsf{dis}}(\mathcal{D},\mathcal{D})$, defined by the following rules: In each inning $n \in \mathbb{N}$, ONE chooses a dense subset D_n of X and player TWO chooses a discrete $E_n \subset D_n$; TWO wins the play if $\bigcup E_n$ is dense in X, otherwise ONE is the winner.

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Clearly, every D^+-separable space is D-separable and every D-separable space is d-separable. In [1] the authors investigated relationships between d-separability and D-separability and presented conditions so that these concepts are equivalent.

The principal aim of this paper is to adapt the above concepts to bitopological spaces.

Kelly defined a bitopological space (shortly bispace) (X, τ_1, τ_2) to be a set X with two topologies τ_1 and τ_2 (in general, unrelated) on it [10]. While studying various types of bitopologies, this article focuses on the cases where one of the topologies is finer than the other, which are typical applications of the theory of bitopological spaces. In fact, the question of how a property provided by one of the topologies, such as the d-property, the D-property, is moved to bitopological spaces when one of the topologies is thinner than the other is our focus.

The systematic study on selection principles mainly selective version of separability in bitopological spaces was started by Kočinac and Özçağ in [12] where M-separability, R-separability, H and GN-separability were introduced. Some results on selection principles in the bitopological context related to function spaces and hyperspaces appear in [13], while in [18] the tightness properties and selective versions of separability are given in bitopological function spaces endowed with set-open topologies. Selective properties and the corresponding topological games in the space $(\mathsf{C}(X), \tau_k, \tau_p)$ were studied in [16], and topological games related to weak forms of the covering properties in bitopological spaces were discussed in [7]. A new version of separability by using θ-closure and θ-density was investigated in [19] and recently the selective versions of the ccc-property and the star ccc-property in bitopological spaces appeared in [14]. However, there are still few papers that deal with bitopological spaces and selection principles.

The layout of the paper is as follows. Sections 2 introduces a selective version of d-separability called $\mathsf{D}_{(i,j)}$-separability in bitopological spaces and presents relationships among D-separability and $\mathsf{D}_{(i,j)}$-separability. In Section 3 we begin by defining the game-theoretic version of $\mathsf{D}_{(i,j)}$-separability. This section contains several important results to find the conditions which make the bitopological space $\mathsf{D^+}_{(i,j)}$-separable. Section 4 is devoted to the relationships between $\mathsf{D^+}_{(i,j)}$-separability and other selective versions of separability in bitopological spaces.

The remainder of this introduction is given over to some background material. Our aim is to give just enough material to enable a casual reader to gain a general idea of the contents of the paper but some other notions will be defined throughout the sections as will be needed.

Selection principles

Since Scheepers in [22] began a systematic study of selection principles in topology and their relations to game theory and Ramsey theory, many papers on this topic appeared in the literature. There is now considerable literature on this subject, and an adequate introduction to the theory and the motivation for its study may be obtained from [11, 21, 22].

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$\mathsf{S}_{\mathsf{fin}}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_1, A_2, \ldots)$ of elements of $\mathcal{A}$ there is a sequence $(B_1, B_2, \ldots)$ such that for each $n \in \mathbb{N}$, B_n is a finite subset of A_n, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$ (see [22]);

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(A_1, A_2, \ldots)$ of elements of $\mathcal{A}$ there is a sequence $(b_1, b_2, \ldots)$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$, and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$ (see [22]).

If $\mathcal{O}$ is the collection of open covers of a topological space X, then:

- $\mathsf{S}_{\mathsf{fin}}(\mathcal{O},\mathcal{O})$ is the Menger covering property [17].
- $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ is the Rothberger covering property [20].

Topological games

There is a deep relationship between selection principles theory and game theory.

The symbol $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play a round for each positive integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \cdots; A_n, b_n; \cdots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

$\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ denotes the game where in the n-th round ONE chooses a set $A_n \in \mathcal{A}$, while TWO responds by choosing a finite set $B_n \subset A_n$. A play $(A_1, B_1, \cdots, A_n, B_n, \cdots)$ is won by TWO if and only if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

It is evident that if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ (resp. $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$) then the selection hypothesis $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ (resp. $\mathsf{S}_{\mathsf{fin}}(\mathcal{A},\mathcal{B})$) is true. The converse implication need not be always true (see, [23, Example 3]). When this converse implication is true, the game characterizes the selection principle, and is a powerful tool to extract additional information about $\mathcal{A}$ and $\mathcal{B}$.

The reader is referred to [6] for standard topological notations and terminology, while we will follow [5, 10] for bitopological spaces. By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of natural numbers and the set of real numbers, respectively. X is a σ-space if X has a σ-discrete network. X is monotonically normal if one can assign to every point $x \in X$ and open set $U \subseteq X$ an open set $H(x,U) \subseteq U$ such that $x \in H(x,U)$ and if $H(x,U) \cap H(y,V) \neq \emptyset$ then either $x \in V$ or $y \in U$. The function H is called a monotone normality operator. X is stratifiable if one can assign to every $n \in \mathbb{N}$ and every closed set $H \subseteq X$ an open set $G(n,H) \supset H$ so that $H = \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(G(n,H))$ and $G(n,H) \subseteq G(n,K)$ whenever $H \subseteq K$. Every stratifiable space is both monotonically normal and σ-space [8]. The closure and interior of a subset A of a space (X,τ) are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. When (X,τ_1,τ_2) is a bitopological space and $A \subset X$, then $\mathrm{Cl}_i(A)$ and $\mathrm{Int}_i(A)$, $i = 1,2$, denote the closure and interior in the space (X,τ_i). If (X,τ_1,τ_2) is a bitopological space and $\mathcal{P}$ is some topological property then (i,j)-$\mathcal{P}$ denotes an analogue of this property for τ_i with respect to τ_j and p-$\mathcal{P}$ denotes the conjuction $(1,2)$-$\mathcal{P} \wedge (2,1)$-$\mathcal{P}$ where "p" is the abbreviation for "pairwise". Also note that (X,τ_i) has a property $\mathcal{P}$ if and only if (X,τ_1,τ_2) has a property i-$\mathcal{P}$. By a bitopological space $(X,\tau_1 \leq \tau_2)$ is always meant a bitopological space (X,τ_1,τ_2) with $\tau_1 \subseteq \tau_2$.

2. Bitopological D-separability

In this section, we introduce $\mathsf{D}_{(i,j)}$-separability which is a counterpart of D-separability in bitopological spaces, and discuss the conditions that express $\mathsf{D}_{(i,j)}$-separability when one of the topologies is d-separable.

Let (X,τ_1,τ_2) be a bitopological space. We denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the collection of dense subsets of (X,τ_1) and (X,τ_2), respectively.

Definition 2.1 (X,τ_1,τ_2) is $\mathsf{D}_{(i,j)}$*-separable* ($i,j = 1,2$; $i \neq j$), if for every sequence $(D_n : n \in \mathbb{N})$ of elements of $\mathcal{D}_i$ there is a sequence $(E_n : n \in \mathbb{N})$ of discrete sets of (X,τ_i) such that $E_n \subset D_n$, $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{D}_j$.

Now we investigate how the relation between $\mathsf{D}_{(i,j)}$-separability and d-separability appears in the context of bitopological spaces.

Proposition 2.2 *Let (X,τ_1,τ_2) be a bitopological space. If $\tau_2 \leq \tau_1$ then:*

(i) $\mathsf{D}_{(2,1)}$-separability implies (X,τ_2) is D-separable (and d-separable)

(ii) (X,τ_2) is D-separable $\Rightarrow$ (X,τ_1,τ_2) is $\mathsf{D}_{(1,2)}$-separable.

Proof (i) Let $(D_n : n \in \mathbb{N})$ be a sequence of dense subsets of (X,τ_2). Since (X,τ_1,τ_2) is $\mathsf{D}_{(2,1)}$-separable, there exists a sequence $(E_n : n \in \mathbb{N})$ of discrete subsets of (X,τ_2), such that $\forall n \in \mathbb{N}$, $E_n \subset D_n$ and $\bigcup_{n\in\mathbb{N}} E_n$ is dense in (X,τ_1), consequently it is also dense in (X,τ_2).

(ii) Let $(D_n : n \in \mathbb{N})$ be a sequence of dense subsets of (X,τ_1) so also of (X,τ_2) because $\tau_2 \leq \tau_1$. Since (X,τ_2) is D-separable there exists a sequence $(E_n : n \in \mathbb{N})$ of discrete subsets of (X,τ_2) such that $\forall n \in \mathbb{N}$, $E_n \subset D_n$ and $\bigcup_{n\in\mathbb{N}} E_n$ is dense in (X,τ_2). Since $\tau_2 \leq \tau_1$ the sequence $(E_n : n \in \mathbb{N})$ is discrete in (X,τ_1). This means that (X,τ_1,τ_2) is $\mathsf{D}_{(1,2)}$-separable. □

In view of the above result, we conclude that one of the topologies being d-separable is not enough for the bitopological space to be $\mathsf{D}_{(i,j)}$-separable. The question then naturally arises of what kind of additional conditions can be added to one of the d-separable topologies that makes a bitopological space $\mathsf{D}_{(i,j)}$-separable.

Now our aim is to find a condition such that the bitopological space will be $\mathsf{D}_{(1,2)}$-separable. In [1] the authors established conditions under which d-separability implies D-separability. One of these conditions given in [1] and denoted by $(*)$ is as follows:

$(*)$ For every discrete $D \subseteq X$ and every sequence $(E_n : n \in \mathbb{N})$ of dense subsets of X, there is a sequence $(D_n : n \in \mathbb{N})$ of discrete subsets of X such that $\forall n \in \mathbb{N}$ $(D_n \subseteq E_n)$ and $D \subseteq \mathrm{Cl}(\bigcup D_n)$.

Proposition 2.3 *Let (X,τ_1,τ_2) be a bitopological space with $\tau_1 \geq \tau_2$. If (X,τ_2) is d-separable and satisfies $(*)$, then (X,τ_1,τ_2) is $\mathsf{D}_{(1,2)}$-separable.*

Proof Let $(C_m)_{m\in\mathbb{N}}$ be a sequence of τ_2-discrete subsets of X with $C = \bigcup_{m\in\mathbb{N}} C_m$ that is dense in (X,τ_2). Let $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection. Let $(D_k : k \in \mathbb{N})$ be a sequence of dense subsets in (X,τ_1) so also in (X,τ_2) since $\tau_1 \geq \tau_2$. For each $m \in \mathbb{N}$, it follows from $(*)$ that there is a sequence $(B_n^m : n \in \mathbb{N})$ of τ_2-discrete subsets of X also τ_1-discrete, such that $(B_n^m) \subseteq D_{f(m,n)}$ for all $n \in \mathbb{N}$ and $C_m \subseteq \mathrm{Cl}_2(\bigcup_{n\in\mathbb{N}} B_n^m)$. Now for each $(m,n) \in \mathbb{N}\times\mathbb{N}$ we define $E_{f(m,n)} = B_n^m$ and

$$C = \bigcup_{m\in\mathbb{N}} C_m \subseteq \bigcup_{m\in\mathbb{N}} \mathrm{Cl}_2(\bigcup_{n\in\mathbb{N}} B_n^m) \subseteq \mathrm{Cl}_2(\bigcup_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}} B_n^m) \subseteq \mathrm{Cl}_2(\bigcup_{k\in\mathbb{N}} E_k).$$

Then $X = \mathrm{Cl}_2(C) \subseteq \mathrm{Cl}_2(\bigcup_{k\in\mathbb{N}} E_k)$. This means that $\bigcup_{k\in\mathbb{N}} E_k$ is dense in (X,τ_2). □

Proposition 2.4 *Let (X,τ_1,τ_2) be a bitopological space with $\tau_1 \geq \tau_2$. If (X,τ_1,τ_2) is $\mathsf{D}_{(2,1)}$-separable, then (X,τ_2) is d-separable and satisfies $(*)$.*

Proof Clearly $\mathsf{D}_{(2,1)}$-separability implies (X,τ_2) has the condition $(*)$. □

The following property was introduced in [1] when investigating the conditions under which d-separability implies D-separability.

Definition 2.5 [1] A topological space X satisfies property P if for every discrete $D \subseteq X$, every open neighbourhood assignment $(V_d)_{d\in D}$ such that $\forall d \in D$ $(V_d \cap D = \{d\})$ and every sequence $(E_n : n \in \mathbb{N})$ of dense subsets of X, there is a sequence $(D_n : n \in \mathbb{N})$ of discrete subsets of X such that $\forall n \in \mathbb{N}$ $(D_n \subseteq E_n)$ and $\forall d \in D$ $(V_d \cap \bigcup_{n\in\mathbb{N}} D_n \neq \emptyset)$.

In [1] the authors considered how the condition $(*)$ is related with D-separability and property P and they proved the following:

Proposition 2.6 *[1] The following holds:*

1. *A topological space is* D*-separable if and only if it is* d*-separable and satisfies* $(*)$*;*
2. $(*)$ *implies property* P*;*
3. *for first-countable spaces,* $(*)$ *and property* P *are equivalent;*
4. *a first-countable space is* D*-separable if and only if it is* d*-separable and satisfies property* P*.*

In view of Proposition 2.6 it is appropriate to state the following theorem for the bitopological spaces.

Theorem 2.7 *Let* (X, τ_1, τ_2) *be a bitopological space with* (X, τ_2) *first countable. If* (X, τ_2) *is* d*-separable, satisfies property* P *and* $\tau_1 \geq \tau_2$ *then* (X, τ_1, τ_2) *is* $\mathsf{D}_{(1,2)}$*-separable.*

Proof By Proposition 2.6 (4), (X, τ_2) is D-separable. Since $\tau_1 \geq \tau_2$ we obtained that (X, τ_1, τ_2) is $\mathsf{D}_{(1,2)}$-separable by Proposition 2.2. □

We recall that a topological space X is screenable if every open cover of X has an open refinement that is a countable union of cellular families. Theorem 2.7 has the following consequences.

Corollary 2.8 *If a* d*-separable space* (X, τ_2) *is first countable and hereditary screenable, then* $(X, \tau_1 \geq \tau_2)$ *is* $\mathsf{D}_{(1,2)}$*-separable.*

Proof By [1, Proposition 2.8] every hereditary screenable space satisfies property P so the results follow from Theorem 2.7 and Proposition 2.6. □

In [1] the authors weakened the hypothesis that every topological space with a σ-point-finite base is d-separable. We recall that a topological space X is quasi-developable [3] if there is a quasi-development for X, i.e. a sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of families of open subsets of X such that, whenever $x \in U$ with U open in X, there is $n \in \mathbb{N}$ such that $\emptyset \neq \mathrm{St}(x, \mathcal{G}_n) \subset U$, where $\mathrm{St}(x, \mathcal{G}_n) = \{V \in \mathcal{G}_n : x \in V\}$. In [1] the authors obtained that a topological space has a σ-point-finite base $\Rightarrow$ X is quasi-developable $\Rightarrow$ X is d-separable. This leads to the following:

Corollary 2.9 *If a first countable, quasi-developable space* (X, τ_2) *satisfies property* P*, then* $(X, \tau_1 \geq \tau_2)$ *is* $\mathsf{D}_{(1,2)}$*-separable.*

3. Characterizing $\mathsf{D}_{(i,j)}$-separability

In this section we characterize $\mathsf{D}_{(i,j)}$-separability game-theoretically.

Definition 3.1 A bitopological space (X, τ_1, τ_2) is $\mathsf{D^+}_{(i,j)}$-separable if TWO has a winning strategy in the game $\mathsf{G_{dis}}(\mathcal{D}_\mathsf{i}, \mathcal{D}_\mathsf{j})$, defined by the following rules: In each inning $n \in \mathbb{N}$, ONE chooses a τ_i-dense subset D_n of X and TWO chooses a τ_i-discrete $E_n \subset D_n$; TWO wins the play if $\bigcup E_n$ is dense in (X, τ_j), otherwise ONE is the winner.

Clearly every $\mathsf{D^+}_{(i,j)}$-separable bitopological space is $\mathsf{D}_{(i,j)}$-separable and if $\tau_1 \geq \tau_2$, every $\mathsf{D}_{(2,1)}$-separable bitopological space implies d-separability of (X, τ_2).

Now our question is to find the topological conditions that satisfy $\mathsf{D^+}_{(i,j)}$-separability of (X, τ_1, τ_2) when one of the topologies of (X, τ_1, τ_2) is d-separable.

Now recall that a topological space X is discretely generated [4] if whenever $A \subset X$ and $x \in \mathrm{Cl}(A)$ there is a discrete $D \subset A$ with $x \in \mathrm{Cl}(D)$.

Proposition 3.2 *Let (X, τ_1, τ_2) be a bitopological space such that $\tau_1 \geq \tau_2$. If (X, τ_2) is separable and discretely generated, then (X, τ_1, τ_2) is $\mathsf{D^+}_{(1,2)}$-separable.*

Proof Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of (X, τ_2). We will show that $(X, \tau_1 \geq \tau_2)$ is $\mathsf{D^+}_{(1,2)}$-separable. In the n.th inning of the game $\mathsf{G_{dis}}(\mathcal{D}_1, \mathcal{D}_2)$ ONE chooses a τ_1-dense subset D_n of X, so the sets D_n are also dense in (X, τ_2). Since (X, τ_2) is discretely generated TWO chooses a τ_2-discrete hence τ_1-discrete $E_n \subset D_n$ with $x_n \in \mathrm{Cl}_2(E_n)$. Therefore $\bigcup E_n$ is dense in (X, τ_2), hence this is a winning strategy for TWO in this game. □

The authors generalized the concept of discretely generated in [1] by considering discrete subsets instead of points. This led to the following concept.

Definition 3.3 [1] A topological space X is said to be DDG (for discretely-(discretely generated)) if for every discrete $D \subset X$ and every $A \subset X$ such that $D \subset \mathrm{Cl}(A)$ there is a discrete $D_0 \subseteq A$ such that $D \subseteq \mathrm{Cl}(D_0)$.

We may state at once:

Proposition 3.4 *Let (X, τ_1, τ_2) be a bitopological space such that $\tau_1 \geq \tau_2$. If (X, τ_2) is d-separable and DDG, then (X, τ_1, τ_2) is $\mathsf{D^+}_{(1,2)}$-separable.*

Proof Let $C = \bigcup C_n$ be a dense set in (X, τ_2) with C_n τ_2-discrete for all $n \in \mathbb{N}$. In the n.th inning of the game $\mathsf{G_{dis}}(\mathcal{D}_1, \mathcal{D}_2)$ ONE chooses a τ_1-dense subset D_n of X, also dense in (X, τ_2). Since (X, τ_2) is DDG, the player TWO chooses a τ_2-discrete subset E_n of X, which are discrete in (X, τ_1) with $E_n \subset D_n$ and we have $C_n \subset \mathrm{Cl}_2(E_n)$. Since C is dense in (X, τ_2), the set $\bigcup E_n$ is dense in (X, τ_2), which defines a winning strategy for TWO. □

Now we define the bitopological versions of the notions discretely generated and discretely-(discretely generated).

Definition 3.5 A bitopological space (X, τ_1, τ_2) is said to be:

(a) (i, j)-disceretely generated whenever $A \subset X$ and $x \in \mathrm{Cl}_i(A)$ there is a τ_i-discrete $D \subset A$ with $x \in \mathrm{Cl}_j(D)$.

(b) (i, j)-DDG (for discretely-(discretely generated)) if for every τ_j-discrete $D \subset X$ and every $A \subset X$ such that $D \subset \mathrm{Cl}_i(A)$ there is a τ_i-discrete $D_0 \subseteq A$ such that $D \subseteq \mathrm{Cl}_j(D_0)$.

Recall that a subset A of (X, τ_1, τ_2) is double dense in X if A is dense in both (X, τ_1) and (X, τ_2). X is *double-separable* if there is a countable set A which is double dense in X.

Proposition 3.6 *Let (X, τ_1, τ_2) be a double separable bitopological space with $(1,2)$-discretely generated, then (X, τ_1, τ_2) is $\mathsf{D}^+{}_{(1,2)}$-separable.*

Proof Let $\{x_n : n \in \mathbb{N}\}$ be a countable set dense in (X, τ_1) and (X, τ_2). Since $\{x_n : n \in \mathbb{N}\}$ is dense in (X, τ_1) and $x_n \in \mathrm{Cl}_1(\{x_n : n \in \mathbb{N}\})$, in the n.th inning of the game $\mathsf{G}_{\mathsf{dis}}(\mathcal{D}_1, \mathcal{D}_2)$, the player TWO answers a τ_1-dense subset D_n of X played by ONE with τ_1-discrete $E_n \subset D_n$ such that $x_n \in \mathrm{Cl}_2(E_n)$ since (X, τ_1, τ_2) is $(1,2)$-discretely generated. We note that $\{x_n : n \in \mathbb{N}\}$ is a dense set in (X, τ_2) therefore $\bigcup E_n$ is dense in (X, τ_2), hence this is a winning strategy for TWO in this game. □

Proposition 3.7 *Let (X, τ_1, τ_2) be an $(1,2)$-DDG bitopological space. If (X, τ_2) is d-separable, then (X, τ_1, τ_2) is $\mathsf{D}^+{}_{(1,2)}$-separable.*

Proof Let $C = \bigcup C_n$ be a dense set in (X, τ_2) with C_n τ_2-discrete for all $n \in \mathbb{N}$. In the n.th inning of the game $\mathsf{G}_{\mathsf{dis}}(\mathcal{D}_1, \mathcal{D}_2)$, ONE chooses a τ_1-dense subset D_n of X. Since (X, τ_1, τ_2) is $(1,2)$-DDG there exists a τ_1-discrete $E_n \subset D_n$ with $C_n \subset \mathrm{Cl}_2(E_n)$. Since C is dense in (X, τ_2), the set $\bigcup E_n$ is dense in (X, τ_2), which defines a winning strategy for TWO. □

The following result is clear from the definitions and we omit the proof.

Proposition 3.8 *Let (X, τ_1, τ_2) be a bitopological space. If (X, τ_2) is D^+-separable, then $(X, \tau_1 \geq \tau_2)$ is $\mathsf{D}^+{}_{(1,2)}$-separable.*

Remark 3.9 *Proposition 3.8 shows that if we provide the conditions under which (X, τ_2) is D^+-separable then the bitopological space $(X, \tau_1 \geq \tau_2)$ will be $\mathsf{D}^+{}_{(1,2)}$-separable. For example in the Proposition 3.2 we may easily obtain the $\mathsf{D}^+{}_{(1,2)}$-separability of $(X, \tau_1 \geq \tau_2)$ from the Proposition 3.8 by observing that (X, τ_2) is D^+-separable by [1, Proposition 4.1]. Similarly Proposition 3.4 is a consequence of [1, Proposition 4.3] and Proposition 3.8.*

Proposition 3.10 *[1] Consider the following statements about a topological space X :*

1. *X is metrizable;*
2. *X is monotonically normal;*
3. *X is discretely generated and hereditarily collectionwise Hausdorff;*

4. *X is DDG;*
5. *X satisfies property P.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$.

In view of Proposition 3.10 the following results are immediate consequences of Proposition 3.4.

Corollary 3.11 *Let $(X, \tau_1 \geq \tau_2)$ be a bitopological space with (X, τ_2) is monotonically normal. Then (X, τ_2) is d-separable if and only if (X, τ_1, τ_2) is $\mathsf{D}^+{}_{(1,2)}$-separable.*

Corollary 3.12 *Let (X, τ_1, τ_2) be a bitopological space such that $\tau_1 \geq \tau_2$, (X, τ_2) be DG and hereditarily collectionwise Hausdorff. (X, τ_2) is d-separable if and only if (X, τ_1, τ_2) is $\mathsf{D}^+{}_{(1,2)}$-separable.*

4. Versions of separability related to $\mathsf{D}_{(i,j)}$-separability

The selective version of separability in bitopological spaces were first studied in [12] and $\mathsf{M}_{(i,j)}$-separability, $\mathsf{R}_{(i,j)}$-separability and $\mathsf{H}_{(i,j)}$, $\mathsf{GN}_{(i,j)}$-separabilities were introduced. Also in [12] these properties were studied in connection with function spaces. In this section we introduce DH-separability in bitopological spaces and investigate the relationships between $\mathsf{D}_{(i,j)}$-separability and some selective versions of separability mentioned above. We begin by defining the game corresponds to $\mathsf{M}_{(i,j)}$-separability.

Definition 4.1 Let (X, τ_1, τ_2) be a bitopological space. Then X is:

- $\mathsf{M}_{(i,j)}$*-separable* (or selectively-separable) ($i, j = 1, 2;\ i \neq j$), if for each sequence $(D_n : n \in \mathbb{N})$ of elements of $\mathcal{D}_i$ there are finite sets $F_n \subset D_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{D}_j$, i.e. if $\mathsf{S}_{\mathsf{fin}}(\mathcal{D}_i, \mathcal{D}_j)$ holds [12];
- $\mathsf{M}^+{}_{(i,j)}$ separable if TWO has the winning strategy for the following game: ONE picks a τ_i-dense set D_n; TWO picks a finite $F_n \subset D_n$, TWO wins the play if $\bigcup F_n$ is dense in (X, τ_j).

It is clear that if TWO has a winning strategy in the game $\mathsf{M}^+{}_{(i,j)}$, then (X, τ_1, τ_2) is $\mathsf{M}_{(i,j)}$-separable. In the definition of $\mathsf{M}_{(i,j)}$-separability, the sets F_n are supposed to be finite rather than discrete. It is a simple fact that for a bitopological space (X, τ_1, τ_2) with (X, τ_1) is Hausdorff, every $\mathsf{M}_{(i,j)}$-separable bitopological space is $\mathsf{D}_{(i,j)}$-separable.

Recall that (X, τ_1, τ_2) has countable (τ_i, τ_j)*-fan tightness* ($i \neq j;\ i, j = 1, 2$) if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of subsets of X such that $x \in \mathrm{Cl}_i(A_n)$ for each $n \in \mathbb{N}$, there are finite sets $F_n \subset A_n$, $n \in \mathbb{N}$, with $x \in \mathrm{Cl}_j\left(\bigcup_{n\in\mathbb{N}} F_n\right)$. (X, τ_1, τ_2) has countable (τ_i, τ_j)*-strong fan tightness* ($i \neq j;\ i, j = 1, 2$), if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of subsets of X such that $x \in \mathrm{Cl}_i(A_n)$ for each $n \in \mathbb{N}$, there are points $x_n \in A_n$, $n \in \mathbb{N}$, with $x \in \mathrm{Cl}_j(\{x_n : n \in \mathbb{N}\})$. We recall from [12] that (X, τ_1, τ_2) is $\mathsf{R}_{(i,j)}$*-separable* ($i, j = 1, 2;\ i \neq j$), if for each sequence $(D_n : n \in \mathbb{N})$ of elements of $\mathcal{D}_i$ there is a sequence $\{b_n : n \in \mathbb{N}\}$, such that for each $n \in \mathbb{N}$ $b_n \in D_n$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{D}_j$ i.e. if $\mathsf{S}_1(\mathcal{D}_i, \mathcal{D}_j)$ holds. As shown in detail in [12] $\mathsf{R}_{(i,j)}$*-separability* implies $\mathsf{M}_{(i,j)}$*-separability.* The following results are immediate consequences of the relationships between $\mathsf{M}_{(i,j)}$, $\mathsf{R}_{(i,j)}$, and $\mathsf{D}_{(i,j)}$-separability.

Corollary 4.2 *Let $(X, \tau_1 \leq \tau_2)$ be a bitopological space with (X, τ_1) is Hausdorff. If (X, τ_2) is separable and (X, τ_1, τ_2) has a countable $(1,2)$-fan tightness, then (X, τ_1, τ_2) is $\mathsf{D}_{(1,2)}$-separable.*

Proof Immediate from [12, Corollary 6]. □

Corollary 4.3 *Let (X, τ_1, τ_2) be a double separable bitopological space such that (X, τ_1) is Hausdorff. If (X, τ_1, τ_2) has a countable $(1,2)$-strong fan tightnes, then (X, τ_1, τ_2) is $\mathsf{D}_{(1,2)}$-separable.*

Proof Clear from [12, Theorem 2]. □

Corollary 4.4 *Let (X, τ_1, τ_2) be a bitopological space with $\tau_1 \leq \tau_2$ and (X, τ_1) is Hausdorff. If (X, τ_2) is separable and (X, τ_1, τ_2) has a countable $(1,2)$-strong fan tightness, then (X, τ_1, τ_2) is $\mathsf{D}_{(1,2)}$-separable.*

Proof Immediate by [12, Corollary 3]. □

Recall that Ω and $\mathcal{K}$ denote the families of ω-covers and k-covers, respectively. An open cover $\mathcal{U}$ of a topological space X is an *ω-cover* (a *k-cover*) of X if $X \notin \mathcal{U}$ and each finite (compact) subset of X belongs to a member of $\mathcal{U}$.

Example 4.5 If a Tychonoff space X with a countable base belongs to the class $\mathsf{S}_{\mathsf{fin}}(\mathcal{K}, \Omega)$, then, by [12, Theorem 7], the function bispace $(\mathsf{C}(X), \tau_p, \tau_k)$ is $\mathsf{M}_{(\tau_k, \tau_p)}$-separable. Therefore this function bispace is $\mathsf{D}_{(\tau_k \tau_p)}$-separable (Here τ_p denotes the pointwise topology, and τ_k denotes the compact-open topology on the set $\mathsf{C}(X)$ of all continuous real-valued functions on a Tychonoff space X).

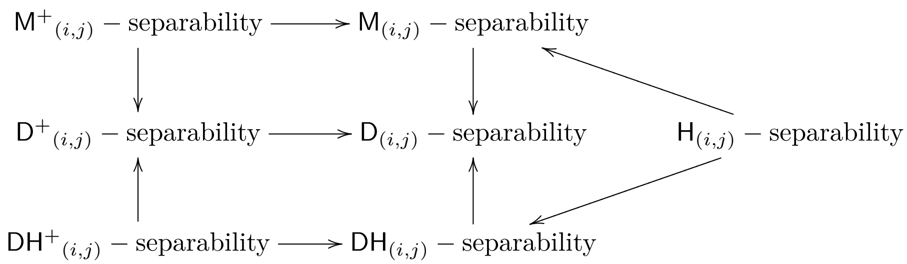
We recall that [12] (X, τ_1, τ_2) is $\mathsf{H}_{(i,j)}$-separable if for each sequence $(D_n : n \in \mathbb{N})$ of elements of $\mathcal{D}_i$ there are finite sets $F_n \subset D_n$ $n \in \mathbb{N}$, such that each τ_j-open subset of X intersects F_n for all but finitely many n. In [12] it is pointed out that $\mathsf{H}_{(i,j)}$-separability implies $\mathsf{M}_{(i,j)}$-separability. Now we consider a weaker form of $\mathsf{H}_{(i,j)}$-separability: the sets F_n are supposed to be discrete rather than finite; we call this property $\mathsf{DH}_{(i,j)}$-separability.

Definition 4.6 A bitopological space (X, τ_1, τ_2) is called:

- $\mathsf{DH}_{(i,j)}$-separable if for each sequence $(D_n : n \in \mathbb{N})$ of elements of $\mathcal{D}_i$ there is a sequence $F_n \subset D_n$ of τ_i-discrete sets such that each τ_j-open subset of X intersects F_n for all but finitely many n.

- $\mathsf{DH}^{+}{}_{(i,j)}$-separable if TWO has the winning strategy in the game $\mathsf{G}_{\mathsf{dis,H}}(\mathcal{D}_\mathsf{i}, \mathcal{D}_\mathsf{j})$, defined by the following rules: In each inning $n \in \mathbb{N}$, ONE chooses a τ_i-dense subset D_n of X and TWO chooses a τ_i-discrete $F_n \subset D_n$; TWO wins the play if every nonempty τ_j-open set in X intersects all but finitely many F_n's.

The following diagram gives the relationships among the properties we have discussed so far. We assume (X, τ_i) is Hausdorff throughout the diagram especially for the arrows pointing from a selective separability type properties to a selective d-separability-type properties.



We end this section by giving topological conditions that imply $\mathsf{DH^+}_{(i,j)}$-separability of a bitopological space. The following result is immediate from Definition 4.6.

Proposition 4.7 *Let (X,τ_1,τ_2) be a bitopological space. If (X,τ_2) is $\mathsf{DH^+}$-separable, then $(X,\tau_1\geq\tau_2)$ is $\mathsf{DH^+}_{(1,2)}$-separable.*

In view of [2, Proposition 18] the following is an immediate consequence of Propositon 4.7.

Theorem 4.8 *Let (X,τ_1,τ_2) be a bitopological space.*

1. *If (X,τ_2) has a σ-disjoint π-base, then $(X,\tau_1\geq\tau_2)$ is $\mathsf{DH^+}_{(1,2)}$-separable.*
2. *If (X,τ_2) has a σ-locally finite π-base, then $(X,\tau_1\geq\tau_2)$ is $\mathsf{DH^+}_{(1,2)}$-separable.*
3. *If (X,τ_2) is T_1 and has a σ-closure preserving π-base, then $(X,\tau_1\geq\tau_2)$ is $\mathsf{DH^+}_{(1,2)}$-separable.*

Corollary 4.9 *Let (X,τ_1,τ_2) be a bitopological space with $\tau_1\geq\tau_2$. If (X,τ_2) is collectionwise Hausdorff discretely generated space with a σ-closed discrete dense set, then (X,τ_1,τ_2) is $\mathsf{DH^+}_{(i,j)}$-separable.*

Proof Immediate by [2, Proposition 22] and Propositon 4.7. □

Recall that X is a σ-space if X has a σ-discrete network.

Corollary 4.10 *Let (X,τ_1,τ_2) be a bitopological space with $\tau_1\geq\tau_2$. If (X,τ_2) is monotonically normal σ-space, then (X,τ_1,τ_2) is $\mathsf{DH^+}_{(i,j)}$-separable.*

Proof Straightforward by [2, Corollary 23] and Propositon 4.7. □

Finally since stratifiable spaces are both monotonically normal σ-space we have the following result by Corollary 4.10 and Propositon 4.7.

Corollary 4.11 *If (X,τ_2) is a stratifiable space, then $(X,\tau_1\geq\tau_2)$ is $\mathsf{DH^+}_{(1,2)}$-separable.*

Example 4.12 *There exists a countable $\mathsf{DH^+}_{(1,2)}$-separable (hence $\mathsf{D}_{(1,2)}$-separable) bitopological space (X,τ_1,τ_2) which is not $\mathsf{M}_{(1,2)}$-separable.*

Proof We consider the space Seq($\mathcal{F}$). Let us recall [25] for each natural number $n \in \omega$, let ${}^n\omega = \{t : t$ is a function and $t : n \to \omega\}$ then Seq $= \bigcup\{{}^n\omega : n < \omega\}$. If $t \in {}^n\omega$ and $k \in \omega$, we denote the function $t \frown k = t \cup \{(n,k)\} \in {}^{n+1}\omega$. Let $\mathcal{F}$ be an ultrafilter on ω. By Seq($\mathcal{F}$) we denote the topological space having Seq as the underlying set and the topology $\tau_{\text{seq}}(\mathcal{F})$ defined by declaring a set $U \subseteq$ Seq to be open if and only if for any $t \in U$ $\{n : t \frown n \in U\} \in \mathcal{F}$. $Seq(\mathcal{F})$ is always Hausdorff and zero-dimensional dense-in-itself space ([2], see [25] for more information). Moreover the space Seq($\mathcal{F}$) is countable, clearly σ-space and monotonically normal by Theorem 3.2 of [9], hence Seq($\mathcal{F}$) is DH^+-separable by [2, Corollary 23]. Now take any Hausdorff topology υ finer than $\tau_{\text{seq}}(\mathcal{F})$. The bitopological space (Seq, $\upsilon \geq \tau_{\text{seq}}(\mathcal{F})$) is $\mathsf{DH^+}_{(\upsilon,\tau_{\text{seq}}(\mathcal{F})}$-separable by Proposition 4.7. Clearly the bitopological space is $\mathsf{D}_{(\upsilon,\tau_{\text{seq}}(\mathcal{F}))}$-separable. However the space $Seq(\mathcal{F})$ is never M-separable (see [2]) and by [12, Fact 2] the bitopological space (Seq, $\upsilon \geq \tau_{\text{seq}}(\mathcal{F})$) is not $\mathsf{M}_{(\tau_{\text{seq}}(\mathcal{F}),\upsilon)}$-separable. It is obvious that for any topologies $\tau_1 \geq \tau_2$, $\mathsf{M}_{(2,1)}$-separability implies $\mathsf{M}_{(1,2)}$-separability and the revised implication does not hold in general. Thus we must show that (Seq, $\upsilon \geq \tau_{\text{seq}}(\mathcal{F})$) is not $\mathsf{M}_{(\upsilon,\tau_{\text{seq}}(\mathcal{F}))}$-separable. Indeed let $(D_n : n \in \mathbb{N})$ be a sequence of dense subsets of υ so dense in $\tau_{\text{seq}}(\mathcal{F})$. If K is an infinite subset of ω take the set $D_K = \cup\{{}^k\omega : k \in K\}$ as a dense set in $\tau_{\text{seq}}(\mathcal{F})$. For any choice of finite set $F_n \subseteq {}^n\omega$ the set $\cup\{F_n : n < \omega\}$ is closed and nowhere dense in $\tau_{\text{seq}}(\mathcal{F})$. If $H_n = \cup\{{}^k\omega : n \leq k < \omega\}$ then the sequence of $\tau_{\text{seq}}(\mathcal{F})$-dense sets $\{H_n : n < \omega\}$ witnesses that the bitopological space is not $\mathsf{M}_{(\upsilon,\tau_{\text{seq}}(\mathcal{F}))}$-separable. □

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