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On the geodesics of deformed Sasaki metric

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Abstract: We define in this note a natural metric over the tangent bundle TM by using a vertical deformation of Sasaki metric. First we present the geometric result concerning the Levi-Civita connection and all forms of Riemannian curvature tensors of this metric. Secondly, we study the geodesics on the tangent bundle TM and unit tangent bundle T_1M. Finally, we characterize the geodesic curvatures on T_1M.

Key words: Deformed Sasaki metric, geodesics, tangent bundle, unit tangent bundle

1. Introduction

In 1958 Sasaki has discovered a new canonical almost Hermitian metric on the tangent bundle TM, since then many authors have largely devoted their studies to this topic. Among them we can mention Dombrowski [6], Opriou [13], Yano and Ishihara [9, 20], etc.
The stiffness of the Sasaki metric has incited a number of authors to deform the Sasaki metric in order to achieve a kind of flexibility of its properties (see [5, 7, 11, 12, 14, 15] and others). In recent years Yampolsky [18, 19], A. Gezer and all [2, 3, 7], L.Bilen [4] (resp. M. Djaa and all [21, 22]) are introduced and studied a new deformation of the Sasaki metric over the tangent bundle TM, called Berger type deformed Sasaki metric (resp. Mus-Sasaki metric).

In this paper, we consider the tangent bundle TM over a Riemannian manifold endowed with a deformed Sasaki metric g_{DS} like a new distorted metric on TM. Firstly, we obtain the formulas describing the Levi-Civita connection of this metric (Theorem 3.6), the curvature tensor (Theorem 4.3), the sectional curvature (Theorem 4.6, Proposition 4.8) and the scalar curvature (Theorem 4.9). Secondly, we study the geodesics on the tangent bundle TM and on the unit tangent bundle T_1M (Theorem 5.3, Theorem 5.4 and Theorem 5.16). Finally, we characterize the geodesic curvatures on the unit tangent bundle T_1M (Theorem 5.19).

2. Lifts to tangent bundles

Let M be an $m-$dimensional Riemannian manifold with a Riemannian metric g and TM be its tangent bundle denoted by $\pi : TM \rightarrow M$. A system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = w^i\right)$, $\bar{i} = n + i = n+1, ..., 2n$, where (w^i) is the cartesian coordinates in

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each tangent space T_PM at $P \in M$ with respect to the natural base $\left\{\frac{\partial}{\partial x^i}|_P\right\}$, P being an arbitrary point in U whose coordinates are (x^i).

Given a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M, the vertical lift X^V and the horizontal lift X^H of X are given, with respect to the induced coordinates, by

$$X^V = X^i \partial_{\bar{i}}, \tag{2.1}$$

$$X^H = X^i \partial_i - w^s \Gamma^i_{sk} X^k \partial_{\bar{i}}, \tag{2.2}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial w^i}$ and Γ^i_{sk} are the coefficients of the Levi-Civita connection ∇ of g [20]

In particular, we have the vertical spray w^V and the horizontal spray w^H on TM defined by

$$w^V = w^i (\partial_i)^V = w^i \partial_{\bar{i}}, \quad w^H = w^i (\partial_i)^H = w^i \overline{\partial}_i.$$

${}^V w$ is also called the canonical or Liouville vector field on TM.

Lemma 2.1 *[1] Let (M,g) be a Riemannian manifold, then for all $x \in M$. If $w = w^i \frac{\partial}{\partial x^i} \in T_xM$, then we have the following*

1. $X^H(g(w,w))_{(x,w)} = X^H(r)_{(x,w)} = 0$
2. $X^H(g(Y,w))_{(x,w)} = g(\nabla_X Y, w)_x$
3. $X^V(g(w,w))_{(x,w)} = X^V(r)_{(x,w)} = 2g(X,w)_x$
4. $X^V(g(Y,w))_{(x,w)} = g(X,Y)_x$
5. $X^H(f(r)) = 0$
6. $X^V(f(r)) = 2f'(r)g(X,w)_x$,

where $r = g(w,w) = |w|^2$ and $f : \mathbb{R} \to]0,+\infty[$ is smooth positive function.

For all vectors fields $X, Y \in \Gamma(TM)$, we have the followings formulas [6, 20]:

$$\begin{aligned}
\left[X^H, Y^H\right]_{(x,w)} &= [X,Y]^H_{(x,w)} - (R_x(X,Y)w)^V \\
\left[X^H, Y^V\right]_{(x,w)} &= (\nabla_X Y)^V_{(x,w)} \\
\left[X^V, Y^V\right]_{(x,w)} &= 0,
\end{aligned} \tag{2.3}$$

where $(x,w) \in TM$ and R is the curvature tensor of g defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

3. Deformed Sasaki metric

Definition 3.1 *Let $\alpha,\beta:\mathbb{R}^+\to\mathbb{R}$ be smooth functions. We define a deformed Sasaki metric g_{DS} on the tangent bundle TM of a Riemannian manifold (M,g) denoted by:*

1. $g_{DS}(X^H,Y^H)_p = g_x(X,Y)$,
2. $g_{DS}(X^H,Y^V)_p = 0$,
3. $g_{DS}(X^V,Y^V)_p = \alpha(r)g_x(X,Y)+\beta(r)g_x(X,w)g_x(Y,w)$,

where $X,Y\in\Gamma(TM)$, $\quad p=(x,w)\in TM$, $\quad r=g(w,w)$, $\quad \alpha>0$ *and* $\alpha+\beta r>0$.

Remark 3.2
1) If $\alpha=1$ and $\beta=0$, then g_{DS} is the Sasaki metric [16],
2 If $\beta=0$, then g_{DS} is one case of the Mus-Sasaki metric [22],
2) If $\alpha=\beta=\dfrac{1}{r+1}$, then g_{DS} is the Cheeger-Gromoll metric [5, 8].

Using Definition 3.1 and Lemma 2.1, we get the following lemma,

Lemma 3.3 *For all $X,Y,Z\in\Gamma(TM)$, we have:*

$$\begin{aligned}
1)\ X^H(g_{DS}(Y^V,Z^V)) &= g_{DS}((\nabla_XY)^V,Z^V)+g_{DS}(Y^V,(\nabla_XZ)^V),\\
2)\ X^V(g_{DS}(Y^V,Z^V)) &= 2\alpha' g(X,w)g(Y,Z)+2\beta' g(X,w)g(Y,w)g(Z,w)\\
&\quad +\beta[g(Z,w)g(X,Y)+g(Y,w)g(X,Z)].
\end{aligned}$$

Lemma 3.4

$$g(Z,w) = \frac{1}{\alpha(r)+r\beta(r)}g_{DS}(Z^V,(w)^V),$$

where $Z\in\Gamma(TM)$ and $w\in TM$.

Proof

$$\begin{aligned}
g_{DS}(Z^V,(w)^V) &= \alpha(r)g(Z,w)+\beta(r)g(Z,w)g(w,w)\\
&= [\alpha(r)+r\,\beta(r)]g(Z,w)
\end{aligned}$$

□

Lemma 3.5 *Let (TM,g_{DS}) be the tangent bundle of the Riemannian manifold (M,g), equipped with the deformed sasaki metric g_{DS}. If ∇ (resp. ${}^{DS}\nabla$) denote the Levi-Civita connection of (M,g) (resp. (TM,g_{DS})), then we have:*

1) $g_{DS}({}^{DS}\nabla_{X^H}Y^H,Z^H) = g_{DS}((\nabla_XY)^H,Z^H)$,

2) $g_{DS}({}^{DS}\nabla_{X^H}Y^H, Z^V) = -\frac{1}{2}g_{DS}((R(X,Y)w)^V, Z^V),$

3) $g_{DS}({}^{DS}\nabla_{X^H}Y^V, Z^H) = \frac{\alpha(r)}{2}g_{DS}((R(w,Y)X)^H, Z^H),$

4) $g_{DS}({}^{DS}\nabla_{X^H}Y^V, Z^V) = g_{DS}((\nabla_X Y)^V, Z^V),$

5) $g_{DS}({}^{DS}\nabla_{X^V}Y^H, Z^H) = \frac{\alpha(r)}{2}g_{DS}((R(w,X)Y)^H, Z^H),$

6) $g_{DS}({}^{DS}\nabla_{X^V}Y^H, Z^V) = 0,$

7) $g_{DS}({}^{DS}\nabla_{X^V}Y^V, Z^H) = 0,$

8) $g_{DS}({}^{DS}\nabla_{X^V}Y^V, Z^V) = g_{DS}\big(\frac{\alpha'}{\alpha}\big[g(X,w)Y^V + g(Y,w)X^V\big]$
$+\big[\frac{\beta-\alpha'}{\alpha+r\beta}g(X,Y) + \frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+r\beta)}g(X,w)g(Y,w)\big]W, Z^V\big).$

where $X,Y,W \in \Gamma(TM)$, $p=(x,w)\in TM$ *and* $W_p = w^i\frac{\partial}{\partial w^i} \in T_p(TM)$.

Proof Using Lemma 2.1, Lemma 3.3 and Koszul formula, we obtain:

$$\begin{aligned}
1)\ 2g_{DS}({}^{DS}\nabla_{X^H}Y^H, Z^H) =& X^H g_{DS}(Y^H,Z^H) + Y^H g_{DS}(Z^H,X^H) - Z^H g_{DS}(X^H,Y^H)\\
&+ g_{DS}(Z^H,[X^H,Y^H]) + g_{DS}(Y^H,[Z^H,X^H]) - g_{DS}(X^H,[Y^H,Z^H])\\
=& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y])\\
&+ g(Y,[Z,X]) - g(X,[Y,Z])\\
=& 2g(\nabla_X Y.Z)\\
=& 2g_{DS}((\nabla_X Y)^H, Z^H).
\end{aligned}$$

$$\begin{aligned}
2)\ 2g_{DS}({}^{DS}\nabla_{X^H}Y^H, Z^V) =& X^H g_{DS}(Y^H,Z^V) + Y^H g_{DS}(Z^V,X^H) - Z^V g_{DS}(X^H,Y^H)\\
&+ g_{DS}(Z^V,[X^H,Y^H]) + g_{DS}(Y^H,[Z^V,X^H]) - g_{DS}(X^H,[Y^H,Z^V])\\
=& g_{DS}(Z^V,[X^H,Y^H])\\
=& - g_{DS}((R(X,Y)w)^V, Z^V).
\end{aligned}$$

$$\begin{aligned}
3)\ 2g_{DS}({}^{DS}\nabla_{X^H}Y^V, Z^H) =& X^H g_{DS}(Y^V, Z^H) + Y^V g_{DS}(Z^H, X^H) - Z^H g_{DS}(X^H, Y^V) \\
& + g_{DS}(Z^H, [X^H, Y^V]) + g_{DS}(Y^V, [Z^H, X^H]) - g_{DS}(X^H, [Y^V, Z^H]) \\
=& - g_{DS}((R(Z,X)w)^V, Y^V) \\
=& - \alpha g(R(Z,X)w, Y) - \beta g(Y,w) g(R(Z,X)w, w)] \\
=& \alpha g(R(w,Y)X, Z) \\
=& \alpha g_{DS}((R(w,Y)X)^H, Z^H).
\end{aligned}$$

$$\begin{aligned}
4)\ 2g_{DS}({}^{DS}\nabla_{X^H}Y^V, Z^V) =& X^H g_{DS}(Y^V, Z^V) + Y^V g_{DS}(Z^V, X^H) - Z^V g_{DS}(X^H, Y^V) \\
& + g_{DS}(Z^V, [X^H, Y^V]) + g_{DS}(Y^V, [Z^V, X^H]) - g_{DS}(X^H, [Y^V, Z^V]) \\
=& X^H g_{DS}(Y^V, Z^V) + g_{DS}(Z^V, [X^H, Y^V]) + g_{DS}(Y^V, [Z^V, X^H]) \\
=& g_{DS}((\nabla_X Y)^V, Z^V) + g_{DS}(Y^V, (\nabla_X Z)^V) \\
& + g_{DS}(Z^V, (\nabla_X Y)^V) - g_{DS}(Y^V, (\nabla_X Z)^V) \\
=& 2g_{DS}((\nabla_X Y)^V, Z^V).
\end{aligned}$$

By a similar calculation we obtain the other formulas.

□

Let us introduce the notations:

$$\lambda = \alpha + \beta\, r, \qquad \overline{\alpha} = \frac{\alpha'}{\alpha}, \qquad \overline{\beta} = \frac{\beta - \alpha'}{\alpha + r\beta}, \qquad \overline{\delta} = \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha + r\beta)}, \tag{3.1}$$

and

$$\eta = \frac{1}{\lambda\alpha}[\lambda\alpha' + \alpha(\beta - \alpha') + (\alpha\beta' - 2\beta\alpha')\, r] = [\overline{\alpha} + \overline{\beta} + \overline{\delta}\, r]. \tag{3.2}$$

From Lemma 3.5, we obtain the following theorem

Theorem 3.6 *If ∇ (resp. ${}^{DS}\nabla$) is the Levi-Civita connection of (M,g) (resp. (TM, g_{DS})) and R denote the curvature tensor of (M,g), then we have:*

$$\begin{aligned}
(1)\ ({}^{DS}\nabla_{X^H}Y^H)_p &= (\nabla_X Y)^H_p - \frac{1}{2}(R_x(X,Y)w)^V, \\
(2)\ ({}^{DS}\nabla_{X^H}Y^V)_p &= (\nabla_X Y)^V_p + \frac{\alpha}{2}(R_x(w,Y)X)^H, \\
(3)\ ({}^{DS}\nabla_{X^V}Y^H)_p &= \frac{\alpha}{2}(R_x(w,X)Y)^H, \\
(4)\ ({}^{DS}\nabla_{X^V}Y^V)_p &= \overline{\alpha}\big[g_x(X,w)Y^V_p + g_x(Y,w)X^V_p\big] + \big[\overline{\beta} g_x(X,Y) + \overline{\delta} g_x(X,w) g_x(Y,w)\big] W_p,
\end{aligned}$$

where $X, Y \in \Gamma(TM)$, $p = (x,w) \in TM$, W is the canonical vertical vector at p defined by $W_p = w^i \frac{\partial}{\partial w^i} \in T_p(TM)$.

4. Curvatures of the deformed Sasaki metric on tangent bundle

Definition 4.1 *Let $K : TM \longrightarrow TM$ be an endomorphism of TM. The horizontal and vertical vector fields HK and VK are given by*

$$\begin{aligned} HK : TM &\to TTM \\ (x,w) &\mapsto (K(w))^H, \end{aligned}$$

$$\begin{aligned} VK : TM &\to TTM \\ (x,w) &\mapsto (K(w))^V. \end{aligned}$$

If $K = Id$ is the identity endomorphism, then we set $W = VK = VId$.

Locally we have

$$VK = w^i K_i^j \frac{\partial}{\partial w^j} = w^i (K(\frac{\partial}{\partial x^i}))^V \tag{4.1}$$

$$HK = w^i K_i^j \frac{\partial}{\partial x^j} - w^i w^k K_i^j \Gamma_{jk}^s \frac{\partial}{\partial w^s} = w^i (K(\frac{\partial}{\partial x^i}))^H \tag{4.2}$$

$$W = w^i (\frac{\partial}{\partial x^i})^V. \tag{4.3}$$

From Definition 4.1 and Theorem 3.6 we have:

Proposition 4.2 *Let (TM, g_{DS}) be the tangent bundle of a Riemannian manifold (M_m, g), endowed with the deformed Sasaki metric g_{DS} and K be an $(1,1)$-tensor field on M, then we have*

$$\begin{aligned}
(1) \quad ({}^{DS}\nabla_{X^H} HK)_p &= \big((\nabla_X K)(w)\big)^H - \frac{1}{2}(R_x(X_x, K_x(w))w)^V, \\
(2) \quad ({}^{DS}\nabla_{X^H} VK)_p &= \big((\nabla_X K))(w)\big)^V + \frac{\alpha}{2}\,(R_x(w, K_x(w))X_x)^H, \\
(3) \quad ({}^{DS}\nabla_{X^V} HK)_p &= (K(X))_p^H + \frac{\alpha}{2}\,(R_x(w, X_x)K_x(w))^H,
\end{aligned}$$

$$\begin{aligned}
(4) \quad ({}^{DS}\nabla_{X^V} VK)_p &= (K(X))_p^V + \overline{\alpha}\left[g(X,w)(K(w))^V + g(K(w),w)X^V\right] \\
&\quad + \overline{\beta}\, g(X, K(w))W_p + \overline{\delta}\, g(X,w)g(K(w),w)W_p, \\
(5) \quad ({}^{DS}\nabla_{X^H} W)_p &= 0, \\
(6) \quad ({}^{DS}\nabla_{X^V} W)_p &= [1 + \overline{\alpha}\, r]X^V + \eta\, g(X,w)W_p,
\end{aligned}$$

where $p = (x,w) \in TM$, $X \in \Gamma(TM)$, $r = |w|^2$, $W_p = w^i \frac{\partial}{\partial w^i} \in T_p(TM)$.

Using Theorem 3.6, Proposition 4.2 and Koszul's formula, we obtain the following theorem.

Theorem 4.3 *If $\widetilde{R}$ denotes the Riemannian curvature tensor of the tangent bundle (TM, g_{DS}), then we have*

$$\begin{aligned}(1)\quad \widetilde{R}(X^H,Y^H)Z^H &= \frac{\alpha}{4}\big[R(w,R(X,Z)w)Y - R(w,R(Y,Z)w)X + 2R(w,R(X,Y)w)Z\big]^H \\ &+\big[R(X,Y)Z\big]^H + \frac{1}{2}\big[(\nabla_Z R)(X,Y)w\big]^V,\end{aligned}$$

$$\begin{aligned}(2)\quad \widetilde{R}(X^H,Y^V)Z^V &= \frac{\alpha}{2}\big[R(Z,Y)X\big]^H - \frac{\alpha^2}{4}\big[R(w,Y)R(w,Z)X\big]^H \\ &+\frac{\alpha'}{2}\big[g(Z,w)R(w,Y)X - g(Y,w)R(w,Z)X\big]^H,\end{aligned}$$

$$\begin{aligned}(3)\quad \widetilde{R}(X^V,Y^V)Z^V &= 2\overline{\alpha}\,'g(Z,w)[g(X,w)Y^V - g(Y,w)X^V] \\ &+\overline{\alpha}[g(X,Z)Y^V - g(Y,Z)X^V] \\ &+\overline{\alpha}^2 g(Z,w)[g(Y,w)X^V - g(X,w)Y^V] \\ &+[1+\overline{\alpha}\,r][\overline{\beta}g(Y,Z) + \overline{\delta}g(Y,w)g(Z,w)]X^V \\ &-[1+\overline{\alpha}\,r][\overline{\beta}g(X,Z) + \overline{\delta}g(X,w)g(Z,w)]Y^V \\ &+\overline{\delta}\,[g(X,Z)g(Y,w) - g(Y,Z)g(X,w)]W \\ &+2\overline{\beta}\,'\,[g(X,w)g(Y,Z) - g(Y,w)g(X,Z)]W \\ &+\overline{\alpha}\,\overline{\beta}\,[g(Y,w)g(X,Z) - g(X,w)g(Y,Z)]W \\ &+\eta\,\overline{\beta}\,[g(Y,Z)g(X,w) - g(X,Z)g(Y,w)]W,\end{aligned}$$

$$\begin{aligned}(4)\quad \widetilde{R}(X^V,Y^V)Y^V &= \Big[[2\overline{\alpha}\,' - \overline{\alpha}^2 - \overline{\delta}(1+\overline{\alpha}\,r)]g(Y,w)g(X,w) \\ &\qquad +[\overline{\alpha} - \overline{\beta}(1+\overline{\alpha}\,r)]g(X,Y)\Big]Y^V \\ &+\Big[\,[\,\overline{\beta}\,(1+\overline{\alpha}\,r) - \overline{\alpha}\,]|Y|^2 + [\,\overline{\delta}\,(1+\overline{\alpha}\,r) + \overline{\alpha}^2 - 2\overline{\alpha}\,'\,]g(Y,w)^2\Big]X^V \\ &+\Big[\,[\,\overline{\delta}\, - 2\overline{\beta}\,' + \overline{\alpha}\,\overline{\beta}\, - \eta\,\overline{\beta}\,]\big[\,g(X,Y)g(Y,w) - |Y|^2 g(X,w)\big]\Big]W,\end{aligned}$$

where $p = (x,w) \in TM$ *and* $X, Y, Z \in \Gamma(TM)$.

From the Theorem 4.3, we get the following propostion.

Proposition 4.4 *The curvature tensor $\widetilde{R}$ with respect to (TM, g_{DS}) verifies the following formulas*

$$\begin{aligned}
(1)\quad g_{DS}(\widetilde{R}(X^H,Y^H)Y^H,X^H)_p &= g(R(X,Y)Y,X)_x-\frac{3\,\alpha}{4}|R(X,Y)w|^2,\\
(2)\quad g_{DS}(\widetilde{R}(X^H,Y^V)Y^V,X^H)_p &= \frac{\alpha^2}{4}|R(w,Y)X|^2,\\
(3)\quad g_{DS}(\widetilde{R}(X^V,Y^V)Y^V,X^V)_p &= \Big[[2\overline{\alpha}\,'-\overline{\alpha}^2-\overline{\delta}(1+\overline{\alpha}\,r)]g(Y,w)g(X,w)\\
&+[\overline{\alpha}-\overline{\beta}(1+\overline{\alpha}\,r)]g(X,Y)\Big]\Big(\alpha\,g(X,Y)+\beta\,g(X,w)g(Y,w)\Big)\\
&+\Big[\,[\,\overline{\beta}\,(1+\overline{\alpha}\,r)-\overline{\alpha}\,]|Y|^2+[\,\overline{\alpha}^2-2\overline{\alpha}\,'\,]g(Y,w)^2\\
&+\overline{\delta}\,(1+\overline{\alpha}\,r)\,\Big]\Big(\alpha\,|X|^2+\beta\,g(X,w)^2\Big)\\
&+\lambda\,[\,\overline{\delta}\,-2\overline{\beta}\,'+\overline{\alpha}\,\overline{\beta}\,-\eta\,\overline{\beta}\,]\\
&\qquad\big(\,g(X,Y)g(Y,w)-|Y|^2g(X,w)\big)g(X,w),
\end{aligned}$$

where $p=(x,w)\in TM$.

Now, we consider the sectional curvature $\widetilde{K}$ on (TM,g_{DS}) for P is given by

$$\widetilde{K}(\widetilde{X},\widetilde{Y})=\frac{g_{DS}(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Y},\widetilde{Y})}{g_{DS}(\widetilde{X},\widetilde{X})g_{DS}(\widetilde{Y},\widetilde{Y})-g_{DS}(\widetilde{X},\widetilde{Y})^2},\tag{4.4}$$

where $P=P(\widetilde{X},\widetilde{Y})$ denotes the plane spanned by $\{\widetilde{X},\widetilde{Y}\}$, for all linearly independent vector fields $\widetilde{X},\widetilde{Y}$ on TM.

Lemma 4.5 *If* $X,Y\in\Gamma(TM)$ *are orthonormal vector fields on* (M_m,g)*, then we have*

$$\begin{aligned}
g_{DS}(X^H,X^H) &= \|X^H\|^2=|X|^2=g(X,X),\\
g_{DS}(X^H,Y^H) &= g(X,Y)=0=g_{DS}(X^H,Y^V),\\
g_{DS}(X^V,Y^V)_p &= \beta\,g(X,u)g(Y,w)\\
g_{DS}(Y^V,Y^V)_p &= \alpha+\beta\,g(Y,w)^2,
\end{aligned}$$

where $p=(x,w)\in TM$.

Let $\widetilde{K}(X^H,Y^H)$, $\widetilde{K}(X^H,Y^V)$ and $\widetilde{K}(X^V,Y^V)$ denote the sectional curvature of the plane spanned respectively by $\{X^H,Y^H\}$, $\{X^H,Y^V\}$ and $\{X^V,Y^V\}$ on (TM,g_{DS}) , where X,Y are orthonormal vector fields on (M_m,g). From Proposition 4.8 and Lemma 4.5, we obtain the following theorem.

Theorem 4.6 *For the sectional curvature* $\widetilde{K}$ *of* (TM, g_{DS}) *we have the followings*

$$\begin{aligned}
(1)\quad \widetilde{K}(X^H,Y^H)_p &= K(X,Y)_x - \frac{3\,\alpha}{4}|R(X,Y)w|^2,\\
(2)\quad \widetilde{K}(X^H,Y^V)_p &= \frac{\alpha^2}{4(\alpha+\beta\; g(Y,w)^2}|R(w,Y)X|^2,\\
(3)\quad \widetilde{K}(X^V,Y^V)_p &= \frac{1}{\alpha^2+\alpha\,\beta\,[g(X,w)^2+g(Y,w)^2]}\Big\{\\
&\Big[[2\overline{\alpha}\,'-\overline{\alpha}^2-\overline{\delta}(1+\overline{\alpha}\,r)]g(Y,w)g(X,w)\\
&+[\overline{\alpha}-\overline{\beta}(1+\overline{\alpha}\,r)]g(X,Y)\Big]\Big(\alpha\; g(X,Y)+\beta\; g(X,w)g(Y,w)\Big)\\
&+\Big[\;[\;\overline{\beta}\;(1+\overline{\alpha}\,r)-\overline{\alpha}\;]|Y|^2+[\;\overline{\alpha}^2-2\overline{\alpha}\,'\;]g(Y,w)^2\\
&+\overline{\delta}\;(1+\overline{\alpha}\,r)\;\Big]\Big(\alpha\;|X|^2+\beta\;g(X,w)^2\Big)\\
&+\lambda\;[\;\overline{\delta}\;-2\overline{\beta}\,'\;+\overline{\alpha}\,\overline{\beta}\;-\eta\,\overline{\beta}\;]\\
&\qquad\Big(\;g(X,Y)g(Y,w)-|Y|^2g(X,w)\Big)g(X,w)\Big\},
\end{aligned}$$

where $p=(x,w)\in TM$ *and* K *is the sectional curvature of* (M,g).

Remark 4.7 *Let* $p=(x,w)\in TM$ *such as* $w\in T_xM\backslash\{0\}$ *and* $\{E_i\}_{i=1,m}$ *be an orthonormal frame of* T_xM *such that* $E_1=\frac{w}{|w|}$, *then*

$$\{\widetilde{E}_i=E_i^H,\;\widetilde{E}_{m+1}=\frac{1}{\sqrt{\lambda}}E_1^V,\;\widetilde{E}_{m+j}=\frac{1}{\sqrt{\alpha}}E_j^V\} \tag{4.5}$$

are orthonormal basis of T_pTM.

Proposition 4.8 *The sectional curvature* $\widetilde{K}$ *of* (TM,g_{DS}) *satisfies the following equations*

$$\begin{aligned}
\widetilde{K}(\widetilde{E}_i,\widetilde{E}_j) &= K(E_i,E_j)-\frac{3\,\alpha}{4}|R(E_i,E_j)w|^2,\\
\widetilde{K}(\widetilde{E}_i,\widetilde{E}_{m+1}) &= \frac{\alpha^2}{4\lambda}|R(w,E_1)E_i|^2\\
&= 0,\\
\widetilde{K}(\widetilde{E}_i,\widetilde{E}_{m+k}) &= \frac{\alpha}{4}|R(w,E_k)E_i|^2,\\
\widetilde{K}(\widetilde{E}_{m+k},\widetilde{E}_{m+1}) &= \frac{1}{\lambda}\big[(\;\overline{\beta}+\overline{\delta}\;)(1+\overline{\alpha}\;r)-\overline{\alpha}+(\;\overline{\alpha}^{\;2}-2\;\overline{\alpha}\,'\;)r\;\big],\\
\widetilde{K}(\widetilde{E}_{m+k},\widetilde{E}_{m+t}) &= \frac{1}{\alpha}\big[(\;\overline{\beta}+\overline{\delta}\;)(1+\overline{\alpha}\;r)-\overline{\alpha}\;\big]
\end{aligned}$$

for $i,j=1,m$ *and* $k,t=2,m$, *where* K *is a sectional curvature of* (M,g).

Theorem 4.9 *The scalar curvature* $\widetilde{\sigma}$ *with respect to* (TM, g_{DS}) *is given by*

$$\begin{aligned} \widetilde{\sigma} &= \sigma - \frac{\alpha}{4}\sum_{\substack{i,j=1\\ i\neq j}}^{m}|R(E_i,E_j)w|^2 \\ &+\frac{2(m-1)}{\lambda}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}+(\,\overline{\alpha}^{\,2}-2\,\overline{\alpha}^{\,\prime}\,)r\,\big] \\ &+\frac{(m-1)(m-2)}{\alpha}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}\,\big], \end{aligned} \tag{4.6}$$

where σ *is a scalar curvature of* (M, g).

Proof Respect to the definition of scalar curvature, we have

$$\begin{aligned} \widetilde{\sigma} &= \sum_{i,j=1}^{m}\widetilde{K}(\widetilde{E}_i,\widetilde{E}_j)+2\sum_{i=1}^{m}\widetilde{K}(\widetilde{E}_i,\widetilde{E}_{m+1})+2\sum_{i=1,k=2}^{m}\widetilde{K}(\widetilde{E}_i,\widetilde{E}_{m+k}) \\ &+2\sum_{k=2}^{m}\widetilde{K}(\widetilde{E}_{m+k},\widetilde{E}_{m+1})+\sum_{k,t=2}^{m}\widetilde{K}(\widetilde{E}_{m+k},\widetilde{E}_{m+t}). \end{aligned}$$

Using Proposition 4.8, we have

$$\begin{aligned} \widetilde{\sigma} &= \sigma-\frac{3\,\alpha}{4}\sum_{i,j}^{m}|R(E_i,E_j)w|^2+\frac{2\,\alpha}{4}\sum_{i=1,k=2}^{m}|R(w,E_k)E_i|^2 \\ &+2\sum_{k\geq 2}^{m}\frac{1}{\lambda}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}+(\,\overline{\alpha}^{\,2}-2\,\overline{\alpha}^{\,\prime}\,)r\,\big] \\ &+\frac{1}{\alpha}\sum_{\substack{k,t=2\\ k\neq t}}^{m}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}\,\big]. \end{aligned}$$

Let us take into account that $R(w,E_1)E_i = R(w,\frac{w}{|w|})E_i = 0$ and

$$\begin{aligned} \sum_{i=1,k=2}^{m}|R(w,E_k)E_i|^2 &= \sum_{i,j=1}^{m}|R(w,E_j)E_i|^2 \\ &= \sum_{i,j=1}^{m}|R(E_i,E_j)w|^2 \end{aligned}$$

(see [22]), then

$$\begin{aligned}\widetilde{\sigma} &= \sigma - \frac{\alpha}{4}\sum_{i,j}^{m}|R(E_i,E_j)w|^2 \\ &+\frac{2(m-1)}{\lambda}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}+(\,\overline{\alpha}^{\,2}-2\,\overline{\alpha}\,'\,)r\,\big] \\ &+\frac{(m-1)(m-2)}{\alpha}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}\,\big].\end{aligned}$$

□

Corollary 4.10 *Let (M_m,g) be a locally flat manifold and (TM,g_{DS}) be its tangent bundle equipped with the deformed Sasaki metric. Then the scalar curvature $\widetilde{\sigma}$ with respect to g_{DS} is given by*

$$\begin{aligned}\widetilde{\sigma} &= \frac{2(m-1)}{\lambda}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}+(\,\overline{\alpha}^{\,2}-2\,\overline{\alpha}\,'\,)r\,\big] \\ &+\frac{(m-1)(m-2)}{\alpha}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}\,\big].\end{aligned}$$

Corollary 4.11 *If (M_m,g) has a constant sectional curvature b, then scalar curvature $\widetilde{\sigma}$ with respect to g_{DS} is given by*

$$\begin{aligned}\widetilde{\sigma} &= m\,(m-1)\,b-\frac{b^2\,\alpha\,(m-1)\,r}{2} \\ &+\frac{2(m-1)}{\lambda}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}+(\,\overline{\alpha}^{\,2}-2\,\overline{\alpha}\,'\,)r\,\big] \\ &+\frac{(m-1)(m-2)}{\alpha}\big[(\,\overline{\beta}+\overline{\delta}\,)(1+\overline{\alpha}\,r)-\overline{\alpha}\,\big].\end{aligned} \tag{4.7}$$

Result is obtained in [17] by another formula.

Proof Using the property of constant sectional curvature, for $X,Y,Z\in\Gamma(TM)$, we have

$$R(X,Y)Z = b\,\big(g(Y,Z)X-g(X,Z)Y\big)$$

then

$$\begin{aligned}
\sigma &= \sum_{\substack{i,j=1\\ i\neq j}}^{m} g(R(E_i,E_j)E_j,E_i)\\
&= b\sum_{\substack{i,j=1\\ i\neq j}}^{m} g(E_j,E_j)g(E_i,E_i)\\
\sigma &= m(m-1)b, \qquad (4.8)\\
\sum_{\substack{i,j=1\\ i\neq j}}^{m} |R(E_i,E_j)w|^2 &= b^2\sum_{\substack{i,j=1\\ i\neq j}}^{m} |g(E_j,w)E_i-g(E_i,w)E_j|^2\\
&= b^2\sum_{\substack{i,j=1\\ i\neq j}}^{m} \left(|g(E_j,w)|^2+g(E_i,w)|^2\right)\\
&= 2\,b^2\sum_{\substack{i,j=1\\ i\neq j}}^{m} |g(E_j,w)|^2\\
\sum_{\substack{i,j=1\\ i\neq j}}^{m} |R(E_i,E_j)w|^2 &= 2b^2(m-1)|w|^2.
\end{aligned}$$

Substituting formulas (4.8) and (4.9) in (4.6), we obtain formula (4.7). □

5. Geodesics of deformed Sasaki metric

Lemma 5.1 *[10] Let M_m be a smooth manifold and X,Y be a vector fields on M. If $x\in M$ and $w\in T_xM$ such that $Y_x=w$, then we have:*

$$d_xY(X_x)=X^H_{(x,w)}+(\nabla_XY)^V_{(x,w)}$$

.

Lemma 5.2 *Let $x(t)$ be a smooth curve on Riemannian manifold (M,g). Then any curve in the shape of $C(t)=(x(t),z(t))$ on TM verifies the following formula*

$$\dot C=\dot x^H+(\nabla_{\dot x}z)^V. \qquad (5.1)$$

Proof
Locally, for $Z\in\Gamma(TM)$ such that $Z(x(t))=z(t)$, then we have

$$\dot C(t)=dC(t)=dZ(x(t))$$

. Using Lemma 5.1 we obtain

$$\dot C(t)=dZ(x(t))=\dot x^H+[\nabla_{\dot x}z]^V$$

. □

Theorem 5.3 *Let (TM, g_{DS}) be a tangent bundle of the manifold (M, g) endowed with the deformed Sasaki metric. If $C(t) = (x(t), z(t))$ is a curve over TM such that $z(t)$ is a vector field along the curve $x(t)$, then:*

$$\begin{aligned} {}^{DS}\nabla_{\dot{C}}\dot{C} &= (\nabla_{\dot{x}}\dot{x})^H + \alpha(R(z, \nabla_{\dot{x}}z)\dot{x})^H + (\nabla_{\dot{x}}\nabla_{\dot{x}}z)^V + \frac{2\alpha'}{\alpha}g(\nabla_{\dot{x}}z, z)(\nabla_{\dot{x}}z)^V \\ &\quad + \big[\frac{\beta-\alpha'}{\alpha+r\beta}g(\nabla_{\dot{x}}z, \nabla_{\dot{x}}z) + \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha+r\beta)}g(\nabla_{\dot{x}}z, z)^2\big]z^V. \end{aligned} \tag{5.2}$$

Proof
Using Lemma 5.2 we obtain:

$$\begin{aligned} {}^{DS}\nabla_{\dot{C}}\dot{C} &= {}^{DS}\nabla_{[\dot{x}^H + (\nabla_{\dot{x}}z)^V]}[\dot{x}^H + (\nabla_{\dot{x}}z)^V] \\ &= {}^{DS}\nabla_{\dot{x}^H}\dot{x}^H + {}^{DS}\nabla_{\dot{x}^H}(\nabla_{\dot{x}}z)^V + {}^{DS}\nabla_{(\nabla_{\dot{x}}z)^V}\dot{x}^H + {}^{DS}\nabla_{(\nabla_{\dot{x}}z)^V}(\nabla_{\dot{x}}z)^V \\ &= (\nabla_{\dot{x}}\dot{x})^H - \frac{1}{2}(R(\dot{x}, \dot{x})z)^V + (\nabla_{\dot{x}}\nabla_{\dot{x}}z)^V + \frac{\alpha}{2}(R(z, \nabla_{\dot{x}}z)\dot{x})^H \\ &\quad + \frac{\alpha}{2}(R(z, \nabla_{\dot{x}}z)\dot{x})^H + \frac{\alpha'}{\alpha}\big[g(\nabla_{\dot{x}}z, z)(\nabla_{\dot{x}}z)^V + g(\nabla_{\dot{x}}z, z)(\nabla_{\dot{x}}z)^V\big] \\ &\quad \big[\frac{\beta-\alpha'}{\alpha+r\beta}g(\nabla_{\dot{x}}z, \nabla_{\dot{x}}z) + \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha+r\beta)}g(\nabla_{\dot{x}}z, z)g(\nabla_{\dot{x}}z, z)\big]z^V \end{aligned}$$

then,

$$\begin{aligned} {}^{DS}\nabla_{\dot{C}}\dot{C} &= (\nabla_{\dot{x}}\dot{x})^H + \alpha(R(z, \nabla_{\dot{x}}z)\dot{x})^H + (\nabla_{\dot{x}}\nabla_{\dot{x}}z)^V + \frac{2\alpha'}{\alpha}g(\nabla_{\dot{x}}z, z)(\nabla_{\dot{x}}z)^V \\ &\quad + \big[\frac{\beta-\alpha'}{\alpha+r\beta}g(\nabla_{\dot{x}}z, \nabla_{\dot{x}}z) + \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha+r\beta)}g(\nabla_{\dot{x}}z, z)^2\big]z^V. \end{aligned}$$

□

From the Theorem 5.3, we get the following theorem.

Theorem 5.4 *Let (TM, g_{DS}) be a tangent bundle of the manifold (M, g) endowed with the deformed Sasaki metric. If $C(t) = (x(t), z(t))$ is a curve over TM such that $z(t)$ is a vector field along the curve $x(t)$. Then $C(t)$ is a geodesic curve if and only if*

$$\left\{ \begin{aligned} \nabla_{\dot{x}}\dot{x} &= -\alpha R(z, \nabla_{\dot{x}}z)\dot{x} \\ \nabla_{\dot{x}}\nabla_{\dot{x}}z &= -\frac{2\alpha'}{\alpha}g(\nabla_{\dot{x}}z, z)\nabla_{\dot{x}}z \\ &\quad -\big[\frac{\beta-\alpha'}{\alpha+r\beta}g(\nabla_{\dot{x}}z, \nabla_{\dot{x}}z) + \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha+r\beta)}g(\nabla_{\dot{x}}z, z)^2\big]z. \end{aligned} \right. \tag{5.3}$$

The curve $C(t) = (x(t), z(t))$ on TM is called a horizontal lift of the curve $x(t)$ if $\nabla_{\dot{x}}z = 0$. Thus, we have the following corollary.

Corollary 5.5 *Let (TM, g_{DS}) be a tangent bundle of the manifold (M, g) endowed with the deformed Sasaki metric and $C(t) = (x(t), z(t))$ is a horizontal lift of the curve $x(t)$. Then $C(t)$ is a geodesic on (TM, g_{DS}) if and only if $x(t)$ is a geodesic on (M_m, g).*

A curve $C(t) = (x(t), \dot{x}(t))$ over (TM, g_{DS}) is said to be a natural lift of the curve $x(t)$. From Corollary 5.5, we deduce the following corollary.

Corollary 5.6 *Let (TM, g_{DS}) be a tangent bundle of the manifold (M, g) endowed with the deformed Sasaki metric. If $x(t)$ is a geodesic on (M, g) then the natural lift $C(t) = (x(t), \dot{x}(t))$ is a geodesic on (TM, g_{DS}).*

Remark 5.7 *For an horizontal lift $C(t) = (x(t), z(t))$ of the curve $x(t)$, we have:*

$$\begin{aligned} \nabla_{\dot{x}} z = 0 \quad &\Leftrightarrow \quad \frac{dz^k}{dt} + \Gamma_{ij}^k z^i \frac{dx^j}{dt} = 0 \\ &\Leftrightarrow \quad z(t) = \exp\big(-\int A(t)dt\big).K \end{aligned}$$

where $K \in \mathbb{R}^n,\ A(t) = [a_{kj}]\ ,\ a_{kj} = \sum_{i=1}^{n} \Gamma_{ij}^k \frac{dx^j}{dt}$.

The Remark 5.7 allows us to build a several geodesics examples over (TM, g_{DS}).

Example 5.8 *Let $M = \mathbb{R}$ equipped with the Riemannian metric $g = e^x dx^2$. The Christoffel symbols of the Levi-Civita connection are given by:*

$$\Gamma_{11}^1 = \frac{1}{2} g^{11} \Big(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\Big) = \frac{1}{2}$$

. So the geodesics $x(t)$ with respect to g, checking the equation

$$x'' + \frac{1}{2}(x')^2 = 0. \tag{5.4}$$

The solutions of differential equation (5.4) are given by

$$x'(t) = \frac{2b}{b\,t + c}, \qquad x(t) = a + 2\ln(b\,t + c), \qquad a, b, c \in \mathbb{R}.$$

From the Remark 5.7, we deduce that $C(t) = \big(a + 2\ln(b\,t + c),\ K(b\,t + c)^{-1}\big)$ is a geodesic on TM.

Example 5.9 *Let $M = \mathbb{R}^3_*$ endowed with the metric h defined by:*

$$h_{11} = x^2\ ,\ h_{22} = y^2\ ,\ h_{33} = z^2\ ,\ h_{ij} = 0, \forall i \neq j$$

Then, the symbols of Christoffel with respect to the Levi-Civita connection are given by:

$$\Gamma_{11}^1 = \frac{1}{x}\ ,\ \Gamma_{22}^2 = \frac{1}{y}\ ,\ \Gamma_{33}^3 = \frac{1}{z}\ ,\ \Gamma_{ij}^k = 0\ \forall (i, j, k) \in \{1, 2, 3\}^3 \setminus \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$$

Let $\eta(t)=(x_1(t),x_2(t),x_3(t))$ *be a curve on* (M,g) *and* $\delta(t)=(z_1(t),z_2(t),z_3(t))$ *be a curve on* TM *along* $\eta(t)$*. If* $C(t)=(\eta(t),\delta(t))$ *is an horizontal lift of the curve* $\eta(t)$*, then from the Remark 5.7, we obtain*

$$A(t)=\begin{pmatrix} \frac{x_1'(t)}{x_1(t)} & 0 & 0 \\ 0 & \frac{x_2'(t)}{x_2(t)} & 0 \\ 0 & 0 & \frac{x_3'(t)}{x_3(t)} \end{pmatrix},$$

and

$$\delta(t)=\exp\big(-\int A(t)dt\big).K=\ \big(\frac{k_1}{x_1(t)},\ \frac{k_2}{x_2(t)},\ \frac{k_3}{x_3(t)}\big)\ ,\ k_1,k_2,k_3\in\mathbb{R}.$$

Moreover, $\eta(t)=(x_1(t),x_2(t),x_3(t))$ *is a geodesic if and only if the functions* $x_1(t),x_2(t),x_3(t)$ *are solutions of the following equation*

$$f''+f^{-1}(f')^2=0$$

. Then we get

$$\eta(t)=(2(a_1t+b_1)^{\frac{1}{2}},\ 2(a_2t+b_2)^{\frac{1}{2}},\ 2(a_3t+b_3)^{\frac{1}{2}})$$

$$\delta(t)=\big(k_1(a_1t+b_1)^{-\frac{1}{2}},\ k_2(a_2t+b_2)^{-\frac{1}{2}},\ k_3(a_3t+b_3)^{-\frac{1}{2}}\big)\ ,\ k_1,k_2,k_3\in\mathbb{R}.$$

Theorem 5.10 *Let* (TM,g_{DS}) *be a tangent bundle of a flat manifold* (M,g) *endowed with the deformed Sasaki metric. If* $x(t)$ *is a geodesic on* M*, Then* $C(t)=(x(t),z(t))$ *is a geodesic over* (TM,g_{DS}) *if and only if*

$$\begin{aligned}\nabla_{\dot{x}}\nabla_{\dot{x}}z &= -\frac{2\alpha'}{\alpha}g(\nabla_{\dot{x}}z,z)\nabla_{\dot{x}}z \\ &\quad -\big[\frac{\beta-\alpha'}{\alpha+r\beta}g(\nabla_{\dot{x}}z,\nabla_{\dot{x}}z)+\frac{\alpha\beta'-2\alpha'\beta}{\alpha(\alpha+r\beta)}g(\nabla_{\dot{x}}z,z)^2\big]z.\end{aligned} \tag{5.5}$$

Proof While $x(t)$ is a geodesic on M and $R=0$ then we have $\nabla_{\dot{x}}\dot{x}=0$ and $R(z,\nabla_{\dot{x}}z)\dot{x}=0$. From the formula (5.3) we obtain the result. □

Corollary 5.11 *Let* (TM,g_{DS}) *be a tangent bundle of a manifold* (M,g) *equipped with the deformed Sasaki metric and* $C(t)=(x(t),z(t))$ *be a geodesic on* TM*. If* $g(z,z)$ *is a constant then:*

$$\begin{cases}\nabla_{\dot{x}}\dot{x} = -\alpha R(z,\nabla_{\dot{x}}z)\dot{x} \\ \nabla_{\dot{x}}\nabla_{\dot{x}}z = -\frac{\beta-\alpha'}{\alpha+r\beta}|\nabla_{\dot{x}}z|^2 z.\end{cases} \tag{5.6}$$

5.1. Geodesics of the deformed Sasaki metric on unit tangent bundle T_1M

.

Let T_1M be the unit tangent bundle of TM. T_1M is an hypersurface in TM defined by

$$T_1M = \{(x,w) \in TM,\ g(w,w) = |w|^2 = 1\}. \tag{5.7}$$

The unit normal vector field to T_1M is given by

$$\begin{aligned} \mathcal{W} : TM_0 = TM\backslash M &\rightarrow T(TM) \\ (x,w) &\mapsto \mathcal{W}_{(x,w)} = \left(\frac{w}{\sqrt{\lambda}}\right)^V \end{aligned} \tag{5.8}$$

where $\lambda = \alpha + r\beta$. If we set

$$\begin{aligned} F : TM &\rightarrow \mathbb{R} \\ (x,w) &\mapsto g(w,w) \end{aligned}$$

Then F is a submersion and $T_1M = F^{-1}(\{1\})$. From (5.8), we obtain

$$\begin{aligned} g_{DS}(\mathcal{W},\mathcal{W})_{(x,w)} &= |w|^2 = r \\ g_{DS}(\mathcal{W},X^H) &= 0 = X^H(F) = g_{DS}(X^H, grad_{g_{DS}}F) \\ g_{DS}(X^V, grad_{g_{DS}}F) &= X^V(F) = 2g(X,w) \\ g_{DS}(X^V,\mathcal{W}) &= \sqrt{\lambda}\, g(X,w) = \frac{\sqrt{\lambda}}{2} g_{DS}(X^V, grad_{g_{DS}}F). \end{aligned}$$

Therefore, $\mathcal{W} = \frac{\sqrt{\lambda}}{2} grad_{g_{DS}}F$ is a canonical vector field normal to T_1M .

Given a vector field X on M, the tangential lift X^T of X is given by

$$X^T(x,w) = \left[X^V - g_{DS}(X^V,\mathcal{W})\mathcal{W}\right]_{(x,w)}. \tag{5.9}$$

If $\overline{\nabla}$ is the induced connection on T_1M, then we have

$$\overline{\nabla}_{\widetilde{X}}\widetilde{Y} = {}^{DS}\nabla_{\widetilde{X}}\widetilde{Y} - g_{DS}({}^{DS}\nabla_{\widetilde{X}}\widetilde{Y},\mathcal{W})\mathcal{W} \tag{5.10}$$

for all $\widetilde{X},\widetilde{Y} \in \Gamma(T(T_1M))$.

Subsequently, we denote $x' = \dot{x}$, $\quad x'' = \nabla_{\dot{x}}\dot{x}$, $\quad z' = \nabla_{\dot{x}}z$ and $\quad z'' = \nabla_{\dot{x}}\nabla_{\dot{x}}z$.

Lemma 5.12 *Let (TM, g_{DS}) be a tangent bundle of the manifold (M,g) endowed with the deformed Sasaki metric. and $C(t) = (x(t), z(t))$ be a curve on T_1M such $z(t)$ is a vector field along $x(t)$. Then we have*

$$|z| = 1, \quad and \quad g(z',z) = 0$$

.

From Theorem 5.4, formula (5.10) and Lemma 5.12, we obtain the following lemma

Lemma 5.13 *$C(t)=(x(t),z(t))$ is a geodesic on T_1M if and only if*

$$\begin{cases} (1) \quad x'' &= a\, R(z,z')x', \\ (2) \quad z'' &= b|z'|^2 z+\rho z=(\rho+b|z'|^2)z, \end{cases} \tag{5.11}$$

where $r=1$, $a=-\alpha(1)=Const$, $b=\frac{\alpha'(1)-\beta(1)}{\alpha(1)+\beta(1)}=Const$ and ρ is some function.

Remark 5.14 *:*

(1) As $|z|^2=1$ (i.e $z\in T_1M$), then $g(z',z)=0$,

(2) $0=\nabla_{\dot{x}}g(z',z)=g(z'',z)+|z'|^2$, so $g(z'',z)=-|z'|^2$.

Lemma 5.15 *Let $C(t)=(x(t),z(t))$ be a geodesic on T_1M. If we put $c=|z'|$, then we have*

$$\rho=-(1+b)c^2 \tag{5.12}$$

$$c'=0. \tag{5.13}$$

Proof

From formula (5.11) and Remark 5.14, we obtain

$$\begin{aligned} z'' &= (\rho+bc^2)z \\ g(z'',z) &= \rho+bc^2=-|z'|^2=-c^2 \end{aligned}$$

therefore $\rho=-(1+b)c^2$. In the other hand, we have

$$\begin{aligned} \frac{1}{2}(c^2)' &= g(z'',z') \\ &= (\rho+b)|z'|^2)g(z,z') \\ &= 0. \end{aligned}$$

□

Using Lemma 5.13 and Lemma 5.15, we obtain the following theorem.

Theorem 5.16 *Let (TM,g_{DS}) be a tangent bundle of the manifold (M,g) endowed with the deformed Sasaki metric g_{DS} and $C(t)=(x(t),z(t))$ be a curve on T_1M such $z(t)\in T_{x(t)}M$, . If we put $c=|z'|$, then $C(t)$ is a geodesic on T_1M if and only if*

$$\begin{aligned} c &= const, \quad \rho = -(1+b)c^2 = const, && (5.14)\\ x'' &= aR(z,z')x', && (5.15)\\ z'' &= -c^2 z, && (5.16)\end{aligned}$$

where $a = -\alpha(1) = Const$.

Theorem 5.17 *Let* (TM, g_{DS}) *be a tangent bundle of a locally symmetric manifold* (M,g) $(\nabla R = 0)$, *endowed with the deformed Sasaki metric,* $C(t) = (x(t), z(t))$ *be a geodesic on* T_1M *and* $\eta = \pi \circ C$. *Then* $R(z,z')$ *is parallel along* η.

Proof :

Using formula (5.16), we get

$$\begin{aligned} R'(z,z') &= \nabla_{\dot{x}} R(z,z')\\ &= (\nabla_{\dot{x}} R)(z,z') + R(z',z') + R(z,z'')\\ &= R(z,z'')\\ &= R(z,-c^2 z)\\ &= 0.\end{aligned}$$

□

Theorem 5.18 *Let* (TM, g_{DS}) *be a tangent bundle of a locally symmetric manifold* (M,g), *endowed with the deformed Sasaki metric, and* $C(t) = (x(t), z(t))$ *be a geodesic on* T_1M. *Then we have*

$$|x^{(p)}| = const \qquad \forall\, p \geq 1. \tag{5.17}$$

Proof

From Theorem 5.16 and Theorem 5.17, we have

$$x'' = -\alpha_1 R(z,z')x', \qquad and \qquad R'(z,z') = \nabla_{\dot{x}} R(z,z') = 0.$$

So

$$x^{(p+1)} = a\, R(z,z')x^{(p)} \qquad p \geq 1 \tag{5.18}$$

and

$$\frac{d}{dt}|x^{(p)}|^2 = 2g(x^{(p+1)}, x^{(p)}) = 2a\, g(R(z,z')x^{(p)}, x^{(p)}) = 0.$$

Then

$$|x^{(p)}| = const \qquad \forall\, p \geq 1.$$

□

Theorem 5.19 *Let (TM, g_{DS}) be a tangent bundle of a locally symmetric manifold (M, g), endowed with the deformed Sasaki metric and $C(t) = (x(t), z(t))$ be a geodesic on T_1M. Then all geodesic curvatures of $\eta = x(t)$ are constant.*

Proof

If s is an arc length parameter on η, then $\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$. Since C is a geodesic then $\|\dot{C}\| = \|\frac{d}{dt}C\| = K = const$ and

$$K^2 = \|\dot{C}\|^2 = |\frac{ds}{dt}|^2 + \alpha(1)|z'|^2 + \beta(1)g(z', z)^2 = |\frac{ds}{dt}|^2 - a\,c^2. \quad (5.19)$$

Hence

$$|\frac{ds}{dt}| = \sqrt{K^2 + a\,c^2} = \tau = const. \quad (5.20)$$

where $\tau^2 = K^2 + a\,c^2 = const$.

If $\nu_1, ..., \nu_{2n-1}$ denote the Frenet frame along η and by $k_1, ..., k_{2n-1}$ the geodesic curvatures of η. Then from (5.20), we obtain

$$\begin{aligned} x' &= \tau\,\nu_1 \\ x'' &= \tau^2 k_1 \nu_2 \\ x^{(3)} &= \tau^3 k_1(-k_1\nu_1 + k_2\nu_3) \\ &\vdots \end{aligned}$$

Using (5.17) we deduce $k_1 = const$, $k_2 = const$, ..., $k_{2n-1} = const$.

□

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