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A new kind of F-contraction and some best proximity point results for such mappings with an application

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Abstract: In this paper, we aim to present a new and unified way, including the previously mentioned solution methods, to overcome the problem in [7] for closed and bounded valued F-contraction mappings. We also want to obtain a real generalization of fixed point results existing in the literature by using best proximity point theory. Further, considering the strong relationship between homotopy theory and various branches of mathematics such as category theory, topological spaces, and Hamiltonian manifolds in quantum mechanics, our objective is to present an application to homotopy theory of our best proximity point results obtained in the paper. In this sense, we first introduce a new family, which is larger than $\mathcal{F}^*$ that has often been used to give a positive answer to the problem. Then, we prove some best proximity point results for the new kind of F-contractions on quasi metric spaces via the new family. Additionally, we show that the note given by Almeida et al. [4] is not valid for our results. Therefore, our results are real generalizations of fixed point results in the literature. Moreover, we give comparative examples to demonstrate that our results unify and generalize some well-known results in the literature. As an application, we show that each homotopic mapping to φ satisfying all the hypotheses of our best proximity point result has also a best proximity point.

Key words: Quasi metric space, multivalued mappings, F-contraction, best proximity point, homotopy theory

1. Introduction and preliminaries

In the metric fixed point theory, the Banach contraction principle [12] is a too important tool that provides a constructive method for finding the fixed point. Due to its important applications to solve various problems in differential equations, nonlinear analysis, functional analysis, and approximation theory, this result has been generalized in different ways [6, 14, 17, 20, 24]. In this sense, Nadler [22] obtained one of the interesting generalizations of this result by taking into account multivalued mappings and showed that each multivalued contraction mapping on a complete metric space has a unique fixed point.

Another approach to these expansion efforts is to extend the underlying space. In this direction, considering quasi metric spaces many results have been generalized, including the Banach contraction principle obtained on ordinary metric spaces. However, since there is no symmetry condition on a quasi metric, the proofs of these generalizations are not clear as in the metric spaces. But, the study of asymmetric distance functions has still attracted the interest of a lot of authors due to the wide range of applications in many disciplines [11, 19]. Now, remind some important properties and notations related to quasi metric spaces:

Let $\Lambda \neq \emptyset$ and $\rho : \Lambda \times \Lambda \to [0, +\infty)$ be a function satisfying for all $\varkappa, \eta, \xi \in \Lambda$,

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(q1) $\rho(\varkappa,\eta)=\rho(\eta,\varkappa)=0$ if and only if $\varkappa=\eta$,

(q2) $\rho(\varkappa,\xi)\leq\rho(\varkappa,\eta)+\rho(\eta,\xi)$.

Then, the function ρ is said to be a quasi metric on Λ and the pair (Λ,ρ) is also said to be a quasi metric space. In this case, the mappings $\rho^{-1}:\Lambda\times\Lambda\to[0,+\infty)$ and $\rho^s:\Lambda\times\Lambda\to[0,+\infty)$ defined as

$$\rho^{-1}(\varkappa,\eta)=\rho(\eta,\varkappa) \text{ and } \rho^s(\varkappa,\eta)=\max\{\rho(\varkappa,\eta),\rho^{-1}(\varkappa,\eta)\}$$

for all $\varkappa,\eta\in\Lambda$ are quasi metric (called conjugate of ρ) and ordinary metric on Λ, respectively. Further, each quasi metric ρ generates a T_0-topology τ_ρ which has as a base the family ρ-open balls

$$\{B_\rho(\varkappa,r):\varkappa\in\Lambda,\ r>0\}$$

where $B_\rho(\varkappa,r)=\{\eta\in\Lambda:\rho(\varkappa,\eta)<r\}$. Also, if we take a sequence $\{\varkappa_n\}$ in Λ and $\varkappa\in\Lambda$, then it is clear that $\{\varkappa_n\}$ converges to $\varkappa$ w.r.t. τ_ρ iff $\lim_{n\to+\infty}\rho(\varkappa,\varkappa_n)=0$. In the rest of paper, we will denote the closure of $U\subseteq\Lambda$ w.r.t. τ_ρ by $\overline{U}^\rho$. Due to the asymmetric condition of a quasi metric, there are various definitions of the completeness on these spaces in the literature. According to classification constructed by Altun et al. [8], if each right (left) K-Cauchy sequence in a quasi metric space (Λ,ρ) converges to a point in Λ w.r.t. τ_ρ, then (Λ,ρ) is called right (left) K-complete. Also, if each right (left) K-Cauchy sequence in (Λ,ρ) converges to a point in Λ w.r.t. $\tau_{\rho^{-1}}$, then (Λ,ρ) is called right (left) M-complete quasi metric space. Remind that if for $\varepsilon>0$, there exists $k\geq0$ such that $\rho(\varkappa_n,\varkappa_m)<\varepsilon$ when $m>n\geq k$ $(n>m\geq k)$, then $\{\varkappa_n\}$ in Λ is called left (right) K-Cauchy sequence. Now, consider the following subclass

$$P(\Lambda)=\{U\subseteq\Lambda:U\neq\emptyset\},$$

$$C_\rho(\Lambda)=\{U\in P(\Lambda):U=\overline{U}^\rho\},$$

and

$$CB_\rho(\Lambda)=\{U\in C_\rho(\Lambda):U\text{ is bounded in }(\Lambda,\rho^s)\}.$$

Then, a mapping $H_\rho:CB_\rho(\Lambda)\times CB_\rho(\Lambda)\to\mathbb{R}$ defined by

$$H_\rho(U,V)=\max\left\{\sup_{\varkappa\in U}\rho(\varkappa,V),\sup_{\eta\in V}\rho(U,\eta)\right\},$$

for each $U,V\in CB_\rho(\Lambda)$ where $\rho(\varkappa,V)=\inf\{\rho(\varkappa,\eta):\eta\in V\}$ is a quasi metric on $CB_\rho(\Lambda)$.

On the other hand, considering the best proximity point theory, many results in the fixed point theory including Banach's result have been extended. Let $\emptyset\neq U,V$ be subsets of a quasi metric space (Λ,ρ). If $U\cap V=\emptyset$, the mapping $\Upsilon:U\to V$ cannot have a fixed point. Then, it is reasonable to investigate the existence of a point $\varkappa^*\in U$ such that $\rho(\varkappa^*,\Upsilon\varkappa^*)=\rho(U,V)$ that is called a best proximity point of Υ [13]. Also, the point $\varkappa^*$ is an optimal solution for the optimization problem $\min_{\varkappa\in U}\rho(\varkappa,\Upsilon\varkappa)$. Moreover, it is a fixed point of Υ when $U=V=\Lambda$. Therefore, this topic has attracted interest of many authors [2, 9, 18, 25, 26]. Now, we recall some concepts related to the best proximity point theory.

Definition 1.1 ([16]) *Let $\emptyset \neq U, V$ be subsets of a metric space (Λ, ρ). Then, if it is satisfied*

$$\left.\begin{array}{l}\rho(\varkappa_1, \eta_1) = \rho(U, V) \\ \rho(\varkappa_2, \eta_2) = \rho(U, V)\end{array}\right\} \textit{imply } \rho(\varkappa_1, \varkappa_2) \leq \rho(\eta_1, \eta_2)$$

for all $\varkappa_1, \varkappa_2 \in U$ and $\eta_1, \eta_2 \in V$, then the pair (U, V) is said to have the weak P-property.

Definition 1.2 ([3]) *Let $\emptyset \neq U, V$ be subsets of a metric space (Λ, ρ) and $\Upsilon : U \to P(V)$ be a mapping. Assume that $\alpha : U \times U \to [0, +\infty)$ is a function. If it is satisfied*

$$\left.\begin{array}{c}\alpha(\varkappa_1, \varkappa_2) \geq 1 \\ \rho(u_1, \eta_1) = \rho(U, V) \\ \rho(u_2, \eta_2) = \rho(U, V)\end{array}\right\} \textit{imply } \alpha(u_1, u_2) \geq 1$$

for all $\varkappa_1, \varkappa_2, u_1, u_2 \in U$ and $\eta_1 \in \Upsilon\varkappa_1, \eta_2 \in \Upsilon\varkappa_2$, then the mapping Υ is said to be α-proximal admissible.

Lately, a nice concept called F-contraction has been introduced by Wardowski [32], and so an interesting fixed point result for such mappings has been obtained. Thus, many results existing in the literature, including the Banach contraction principle have been unified and generalized. Before this result, remind the definition of F-contraction.

Let $\mathcal{F}$ be a class of all functions $F : (0, +\infty) \to \mathbb{R}$ with the following properties:

(F_1) F is strictly increasing,

(F_2) for all sequence $\{\beta_n\}$ in $(0, +\infty)$, $\lim_{n\to+\infty} F(\beta_n) = -\infty$ iff $\lim_{n\to+\infty} \beta_n = 0$,

(F_3) there exists k in $(0, 1)$ such that $\lim_{\beta\to0^+} \beta^k F(\beta) = 0$.

Definition 1.3 *Let $\Upsilon : \Lambda \to \Lambda$ be a mapping on a metric space (Λ, ρ) and $F \in \mathcal{F}$. Then, the mapping Υ is called F-contraction if there is $\tau > 0$ such that*

$$\tau + F(\rho(\Upsilon\varkappa, \Upsilon\eta)) \leq F(\rho(\varkappa, \eta))$$

for all $\varkappa, \eta \in \Lambda$ satisfying $\rho(\Upsilon\varkappa, \Upsilon\eta) > 0$.

Theorem 1.4 ([32]) *Let $\Upsilon : \Lambda \to \Lambda$ be F-contraction on a complete metric space (Λ, ρ). Then, Υ has a unique fixed point $\varkappa$. Also, each sequence $\{\varkappa_n\} = \{\Upsilon^n\varkappa_0\}$ for any initial point $\varkappa_0$ converges to $\varkappa$.*

Then, taking into account F-contractions, many best proximity point and fixed point results have been obtained [5, 10, 15, 27–30]. In this direction, Altun et al. [7] introduced multivalued F-contraction by considering the ideas of multivalued contraction and F-contraction. Hence, they proved the following result on a complete metric space for such mappings.

Theorem 1.5 ([7]) *Let $F \in \mathcal{F}$ and $\Upsilon : \Lambda \to K(\Lambda)$ (the class of all compact subsets of Λ) be a mapping on a complete metric space (Λ, ρ). Then, Υ has a fixed point $\varkappa$ if Υ is a multivalued F-contraction, that is, there is $\tau > 0$ satisfying*

$$\tau + F(H(\Upsilon\varkappa, \Upsilon\eta)) \leq F(\rho(\varkappa, \eta))$$

for all $\varkappa, \eta \in \Lambda$ with $H(\Upsilon\varkappa, \Upsilon\eta) > 0$ where H is a Pompeiu-Hausdorff metric.

In the same article, the authors asked that if we take $\Upsilon : \Lambda \to CB(\Lambda)$ (the family of all bounded and closed subsets of Λ), then is Theorem 1.5 valid? Then, they gave a favorable answer under the following assumption on F.

(F_4) $F(\inf U) = \inf F(U)$ for each $U \subseteq (0, +\infty)$ satisfying $\inf U > 0$.

In the rest of paper, we will indicate the class of all functions F with (F_1)-(F_4) by $\mathcal{F}^*$.

In the current study, we aim to find an affirmative answer including studies in the literature to the aforementioned question, which was arisen from [7] and has attracted the interest of many authors lately. In this direction, we first define a new family which is larger than $\mathcal{F}^*$ that has often been used to give a positive answer to the problem. Then, considering the note given by Almeida et al. [4], we also generalize the definition of weak P-property. Hence, we prove some best proximity point results for the new kind of F-contractions under generalized weak P-property on quasi metric spaces via the new family. Moreover, we give comparative examples to see that the results in the paper unify and generalize some nice and famous results in the literature. At the end of the paper, taking into account the strong relationship between homotopy theory and various branches of mathematics such as category theory, topological spaces, and Hamiltonian manifolds in quantum mechanics, we investigate the existence of a best proximity point for the homotopic mappings by using our main results.

2. Main results

We begin this section by introducing a new subclass of $\mathcal{F}$, and so generalize the family $\mathcal{F}^*$. Let (Λ, ρ) be a quasi metric space, $\emptyset \neq U, V \subseteq \Lambda$, $\Upsilon : U \to P(V)$ be a mapping and $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ be a function. Define

$$ {}_{\Upsilon}\mathcal{F}_{\rho}^{\alpha} = \{F \in \mathcal{F} : F_{\eta,\sigma}^{\varkappa,\rho} \neq \emptyset \text{ for all } \varkappa \in U_\alpha,\ \eta \in V \text{ with } \rho(\eta, \Upsilon\varkappa) > 0 \text{ and } \sigma > 0\}, $$

where

$$ F_{\eta,\sigma}^{\varkappa,\rho} = \{\xi \in \Upsilon\varkappa : F(\rho(\eta, \xi)) \leq F(\rho(\eta, \Upsilon\varkappa)) + \sigma\} $$

and

$$ U_\alpha = \{\varkappa \in U : \alpha(\varkappa, \zeta) \geq 1 \text{ for some } \zeta \in \Lambda\}. $$

We claim that $\mathcal{F}^* \subseteq {}_{\Upsilon}\mathcal{F}_{\rho}^{\alpha}$, but the converse may not be true. To see this, we first show that $\mathcal{F}^* \subseteq {}_{\Upsilon}\mathcal{F}_{\rho}^{\alpha}$. Let $F \in \mathcal{F}^*$ be an arbitrary mapping. Then, if $U_\alpha = \emptyset$, we get $F \in {}_{\Upsilon}\mathcal{F}_{\rho}^{\alpha}$. Now, assume that $U_\alpha \neq \emptyset$. Let $\varkappa \in U_\alpha$, $\eta \in V$ with $\rho(\eta, \Upsilon\varkappa) > 0$. Then since $\rho(\eta, \Upsilon\varkappa) > 0$ and F satisfies the condition (F_4), we have

$$ \begin{aligned} F_{\eta,\sigma}^{\varkappa,\rho} &= \{\xi \in \Upsilon\varkappa : F(\rho(\eta, \xi)) \leq F(\rho(\eta, \Upsilon\varkappa)) + \sigma\} \\ &= \{\xi \in \Upsilon\varkappa : F(\rho(\eta, \xi)) \leq F\left(\inf\{\rho(\eta, \xi') : \xi' \in \Upsilon\varkappa\}\right) + \sigma\} \\ &= \{\xi \in \Upsilon\varkappa : F(\rho(\eta, \xi)) \leq \inf\{F(\rho(\eta, \xi')) : \xi' \in \Upsilon\varkappa\} + \sigma\} \end{aligned} $$

which implies that $F_{\eta,\sigma}^{\varkappa,\rho} \neq \emptyset$ for all $\sigma > 0$. Hence, we get $F \in {}_{\Upsilon}\mathcal{F}_{\rho}^{\alpha}$. Now, for the second part of the proof let us show $\mathcal{F}^* \neq {}_{\Upsilon}\mathcal{F}_{\rho}^{\alpha}$. Let $\Lambda = \{1, 2\} \cup [3, +\infty)$ be endowed with a quasi metric metric $\rho : \Lambda \times \Lambda \to [0, +\infty)$ defined by

$$ \rho(\varkappa, \eta) = \begin{cases} \eta - \varkappa & , \quad \varkappa \leq \eta \\ \varkappa & , \quad \varkappa > \eta \end{cases}. $$

Consider the subsets $U=\{1\}$ and $V=\{2\}\cup[3,+\infty)$. Define $\Upsilon:U\to P(V)$ and $F:(0,+\infty)\to\mathbb{R}$ by $\Upsilon 1=[3,+\infty)$ and

$$F(\beta)=\begin{cases}\ln\beta & , \quad \beta\leq 1\\ \beta & , \quad \beta>1\end{cases},$$

respectively. Hence, we deduced that $F\in\mathcal{F}\backslash\mathcal{F}^*$. Now, for an arbitrary function $\alpha:\Lambda\times\Lambda\to[0,+\infty)$, we have either $U_\alpha=\emptyset$ or $U_\alpha=\{1\}$. If $U_\alpha=\emptyset$, we get $F\in{}_\Upsilon\mathcal{F}_\rho^\alpha$. Now, assume that $U_\alpha=\{1\}$. Choose $\varkappa=1$ and $\eta\in V$ satisfying $\rho(\eta,\Upsilon\varkappa)>0$. This implies $\eta=2$. Then, we have

$$\begin{aligned}F_{2,\sigma}^{1,\rho} &= \{\xi\in\Upsilon 1: F(\rho(2,\xi))\leq F(\rho(2,\Upsilon 1))+\sigma\}\\ &= \{\xi\in[3,+\infty): F(\xi-2)\leq F(1)+\sigma\}.\end{aligned}$$

Since $3\in F_{2,\sigma}^{1,\rho}$ for all $\sigma>0$, we obtain $F\in{}_\Upsilon\mathcal{F}_\rho^\alpha$.

Now, in this part of this section, because of the nonsymmetry condition of a quasi metric, we update some concepts connected with the best proximity point theory. Consider the following ones:

$$U_0^{\mathcal{L}}=\{\varkappa\in U:\rho(\varkappa,\eta)=\rho(U,V)\text{ for some }\eta\in V\},$$

$$U_0^{\Re}=\{\varkappa\in U:\rho(\eta,\varkappa)=\rho(V,U)\text{ for some }\eta\in V\},$$

and

$$V_0^{\mathcal{L}}=\{\eta\in V:\rho(\eta,\varkappa)=\rho(V,U)\text{ for some }\varkappa\in U\},$$

$$V_0^{\Re}=\{\eta\in V:\rho(\varkappa,\eta)=\rho(U,V)\text{ for some }\varkappa\in U\}.$$

Definition 2.1 *Let $\emptyset\neq U,V$ be subsets of a quasi metric space (Λ,ρ) and $\Upsilon:U\to P(V)$ be a mapping. Then, an element $\varkappa\in U$ is said to be a right (left) best proximity point of Υ if $\rho(\Upsilon\varkappa,\varkappa)=\rho(V,U)$ $(\rho(\varkappa,\Upsilon\varkappa)=\rho(U,V))$.*

It has been shown very recently by Almeida et al. [4] that the existence of a best proximity point under weak P-property can be obtained by the corresponding fixed point result. However, we would also like to point out that the results given by Almeida et al. [4] cannot be applicable to our results due to the discontinuity of a quasi metric so that our results are the real generalizations of fixed point results in the literature. In addition, we modify the definition of weak P-property as follows:

Definition 2.2 *Let $\emptyset\neq U,V$ be subsets of a quasi metric (Λ,ρ). Then, the pair (U,V) is said to have generalized weak P_ρ-property if*

$$\left.\begin{array}{l}\rho(\varkappa_1,\eta_1)=\rho(U,V)\\ \rho(\varkappa_2,\eta_2)=\rho(U,V)\end{array}\right\}\textit{imply }\rho(\varkappa_1,\varkappa_2)\leq\rho(\eta_1,\eta_2)$$

for all $\varkappa_1,\varkappa_2\in U$ with $\varkappa_1\neq\varkappa_2$ and $\eta_1,\eta_2\in V$.

Now, we give the concept of α_ρ-proximal admissible mapping.

Definition 2.3 *Let $\emptyset \neq U, V$ be subsets of a quasi metric (Λ, ρ), $\Upsilon : U \to P(V)$ be a mapping and $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ be a function. Then, Υ is called α_ρ-proximal admissible mapping if it is satisfied*

$$\left.\begin{array}{c}\alpha(\varkappa_1, \varkappa_2) \geq 1 \\ \rho(u_1, \eta_1) = \rho(U, V) \\ \rho(u_2, \eta_2) = \rho(U, V)\end{array}\right\} \text{imply } \alpha(u_1, u_2) \geq 1 \tag{2.1}$$

for all $\varkappa_1, \varkappa_2, u_1, u_2 \in U$ and $\eta_1 \in \Upsilon\varkappa_1, \eta_2 \in \Upsilon\varkappa_2$.

We weaken the completeness condition on quasi metric spaces with the following definition.

Definition 2.4 *Let (Λ, ρ) be a quasi metric and $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ be a function. Then,*

(i) if every left (right) K-Cauchy sequence $\{\varkappa_n\}$ in Λ with $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ for all $n \geq 0$ is convergent w.r.t. τ_ρ, then (Λ, ρ) is said to be left (right) K_α-complete.

(ii) if every left (right) K-Cauchy sequence $\{\varkappa_n\}$ in Λ with $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ for all $n \geq 0$ is convergent w.r.t. $\tau_{\rho^{-1}}$, then (Λ, ρ) is said to be left (right) M_α-complete.

It can be easily seen that every left (right) K-complete (M-complete) quasi metric space is left (right) K_α-complete (M_α-complete), but the converse may not be true. Indeed, let $\Lambda = (0, +\infty)$ and $\rho : \Lambda \times \Lambda \to [0, +\infty)$ be a function defined as

$$\rho(\varkappa, \eta) = \left\{\begin{array}{ccc} \eta - \varkappa & , & \varkappa \leq \eta \\ \\ \varkappa & , & \varkappa > \eta \end{array}\right. .$$

Then, (Λ, ρ) is a quasi metric space. However, it is not left K-complete. Indeed, if we take a left K-Cauchy sequence $\{\varkappa_n\} = \left\{\frac{1}{n}\right\}$ in Λ, then although $\frac{1}{n} \to 0$ w.r.t. τ_ρ, $0 \notin \Lambda$. Now, consider the subset $U = [1, +\infty)$ of Λ. If we define a function $\alpha : \Lambda \times \Lambda \to \mathbb{R}$ by

$$\alpha(\varkappa, \eta) = \left\{\begin{array}{ccc} 1 & , & \varkappa, \eta \in U \\ \\ 0 & , & \text{otherwise} \end{array}\right. ,$$

then since U is a closed subset w.r.t. usual metric, for each left K-Cauchy sequence $\{\varkappa_n\}$ in Λ with for all $n \geq 0$, $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ there exists a point $\varkappa \in U$ such that $\varkappa_n \to \varkappa$ w.r.t. τ_ρ. The relation between other completeness can be seen in a similar way.

Now, we present our main results.

Theorem 2.5 *Let $\emptyset \neq U, V$ be subsets of a quasi metric (Λ, ρ) where U is closed w.r.t. τ_ρ. Suppose that $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ is a function, $\Upsilon : U \to CB_\rho(V)$ is a α_ρ-proximal admissible mapping and $F \in {}_\Upsilon\mathcal{F}_\rho^\alpha$. Assume that (Λ, ρ) is left K_α-complete, $\Upsilon\varkappa \subseteq V_0^{\Re}$ for all $\varkappa \in U_0^{\mathcal{L}}$ and the pair (U, V) has the generalized weak P_ρ-property. If the following conditions are satisfied:*

(i) there are $\varkappa_0, \varkappa_1 \in U_0^{\mathcal{L}}$ and $\eta_0 \in \Upsilon\varkappa_0$ satisfying

$$\alpha(\varkappa_0, \varkappa_1) \geq 1 \text{ and } \rho(\varkappa_1, \eta_0) = \rho(U, V),$$

(ii) there exists $\tau > 0$ *satisfying*

$$\tau + F(H_\rho(\Upsilon\varkappa, \Upsilon\eta)) \leq F(\rho(\varkappa, \eta)) \tag{2.2}$$

for all $(\varkappa, \eta) \in \Upsilon_\alpha^\rho$ *where*

$$\Upsilon_\alpha^\rho = \left\{(\varkappa, \eta) \in U \times U : \alpha(\varkappa, \eta) \geq 1 \text{ and } H_\rho(\Upsilon\varkappa, \Upsilon\eta) > 0\right\},$$

then Υ *has a left best proximity point in* U *providing that the function* $\varkappa \to \rho(\varkappa, \Upsilon\varkappa)$ *is lower semicontinuous (l.s.c.) on* U *w.r.t.* τ_ρ*.*

Proof From the hypothesis (i), there exist $\varkappa_0, \varkappa_1 \in U_0^{\mathcal{L}}$ and $\eta_0 \in \Upsilon\varkappa_0$ satisfying

$$\alpha(\varkappa_0, \varkappa_1) \geq 1 \text{ and } \rho(\varkappa_1, \eta_0) = \rho(U, V). \tag{2.3}$$

If $\eta_0 \in \Upsilon\varkappa_1$, then from (2.3) we have $\rho(\varkappa_1, \Upsilon\varkappa_1) = \rho(U, V)$. Hence, the proof is complete. Now, assume that $\eta_0 \notin \Upsilon\varkappa_1 = \overline{\Upsilon\varkappa_1}^\rho$. Therefore, we have $H_\rho(\Upsilon\varkappa_0, \Upsilon\varkappa_1) \geq \rho(\eta_0, \Upsilon\varkappa_1) > 0$. Since $\varkappa_1 \in U_\alpha$ and $F \in {}_\Upsilon\mathcal{F}_\rho^\alpha$, we have $F_{\eta_0, \frac{\tau}{2}}^{\varkappa_1, \rho} \neq \emptyset$, and so there is $\eta_1 \in \Upsilon\varkappa_1$ satisfying

$$F(\rho(\eta_0, \eta_1)) \leq F(\rho(\eta_0, \Upsilon\varkappa_1)) + \frac{\tau}{2}. \tag{2.4}$$

Also, since $\alpha(\varkappa_0, \varkappa_1) \geq 1$ and $H_\rho(\Upsilon\varkappa_0, \Upsilon\varkappa_1) > 0$, we have $(\varkappa_0, \varkappa_1) \in \Upsilon_\alpha^\rho$. On the other hand, since $\eta_1 \in \Upsilon\varkappa_1 \subseteq V_0^{\Re}$, there exists $\varkappa_2 \in U_0^{\mathcal{L}}$ such that

$$\rho(\varkappa_2, \eta_1) = \rho(U, V). \tag{2.5}$$

If $\varkappa_1 = \varkappa_2$, then from (2.5) the proof is complete. So, we get $\varkappa_1 \neq \varkappa_2$. Because of the fact that Υ is α_ρ-proximal admissible mapping and (U, V) has the generalized weak P_ρ-property, from (2.3) and (2.5) we have

$$\alpha(\varkappa_1, \varkappa_2) \geq 1 \text{ and } \rho(\varkappa_1, \varkappa_2) \leq \rho(\eta_0, \eta_1). \tag{2.6}$$

Therefore, using the condition (F_1) and the inequalities (2.2), (2.4) we get

$$\begin{aligned} F(\rho(\varkappa_1, \varkappa_2)) &\leq F(\rho(\eta_0, \eta_1)) \\ &\leq F(\rho(\eta_0, \Upsilon\varkappa_1)) + \frac{\tau}{2} \\ &\leq F(H_\rho(\Upsilon\varkappa_0, \Upsilon\varkappa_1)) + \frac{\tau}{2} \\ &\leq F(\rho(\varkappa_0, \varkappa_1)) - \tau + \frac{\tau}{2} \\ &= F(\rho(\varkappa_0, \varkappa_1)) - \frac{\tau}{2}. \end{aligned}$$

Hence, we get

$$F(\rho(\varkappa_1, \varkappa_2)) \leq F(\rho(\varkappa_0, \varkappa_1)) - \frac{\tau}{2}.$$

Repeating this process, we can show that there are two sequences $\{\varkappa_n\} \subseteq U_0^{\mathcal{L}}$ whose consecutive terms are different and $\{\eta_n\} \subseteq V_0^{\Re}$ with $\eta_n \in \Upsilon\varkappa_n$ for all $n \geq 0$ such that

$$\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1 \text{ and } \rho(\varkappa_{n+1}, \eta_n) = \rho(U, V), \tag{2.7}$$

$$\rho(\varkappa_{n+1}, \varkappa_{n+2}) \leq \rho(\eta_n, \eta_{n+1}) \tag{2.8}$$

and

$$F(\rho(\varkappa_{n+1}, \varkappa_{n+2})) \leq F(\rho(\varkappa_n, \varkappa_{n+1})) - \frac{\tau}{2} \tag{2.9}$$

for all $n \geq 0$. Therefore, we have

$$\begin{aligned} F(\rho(\varkappa_n, \varkappa_{n+1})) &\leq F(\rho(\varkappa_{n-1}, \varkappa_n)) - \frac{\tau}{2} \\ &\leq F(\rho(\varkappa_{n-2}, \varkappa_{n-1})) - \frac{2\tau}{2} \\ &\vdots \\ &\leq F(\rho(\varkappa_0, \varkappa_1)) - \frac{n\tau}{2} \end{aligned} \tag{2.10}$$

for all $n \geq 0$. Taking limit as $n \to +\infty$, we have $\lim_{n\to+\infty} F(\rho(\varkappa_n, \varkappa_{n+1})) = -\infty$, and so from the condition (F_2) we get

$$\lim_{n\to+\infty} \rho(\varkappa_n, \varkappa_{n+1}) = 0. \tag{2.11}$$

Using (F_3), we say that there exists k in $(0,1)$ such that

$$\lim_{n\to+\infty} \rho(\varkappa_n, \varkappa_{n+1})^k F(\rho(\varkappa_n, \varkappa_{n+1})) = 0. \tag{2.12}$$

From (2.10) we have

$$\rho(\varkappa_n, \varkappa_{n+1})^k F(\rho(\varkappa_n, \varkappa_{n+1})) - \rho(\varkappa_n, \varkappa_{n+1})^k F(\rho(\varkappa_0, \varkappa_1)) \leq -\frac{\rho(\varkappa_n, \varkappa_{n+1})^k n\tau}{2} \leq 0$$

for all $n \geq 0$. So, taking limit as $n \to +\infty$ from (2.11) and (2.12) we get

$$\lim_{n\to+\infty} \rho(\varkappa_n, \varkappa_{n+1})^k n = 0.$$

Hence, there exists $n_0 \geq 0$ such that $\rho(\varkappa_n, \varkappa_{n+1})^k n \leq 1$ for all $n \geq n_0$, and so we have

$$\rho(\varkappa_n, \varkappa_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}$$

for all $n \geq n_0$. Let n, m be arbitrary natural numbers with $m > n \geq n_0$. Then, we get

$$\begin{aligned} \rho(\varkappa_n, \varkappa_m) &\leq \rho(\varkappa_n, \varkappa_{n+1}) + \rho(\varkappa_{n+1}, \varkappa_{n+2}) + \cdots + \rho(\varkappa_{m-1}, \varkappa_m) \\ &\leq \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + \cdots + \frac{1}{(m-1)^{\frac{1}{k}}} \\ &\leq \sum_{i=n}^{+\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since the series $\sum_{i=1}^{+\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, the sequence $\{\varkappa_n\} \subseteq U$ is a left K-Cauchy satisfying $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ for all $n \geq 1$. Since $U \subseteq \Lambda$ is closed w.r.t. τ_ρ and (Λ, ρ) is a left K_α-complete quasi metric space, there is $\varkappa^*$ in U with $\lim_{n\to+\infty} \rho(\varkappa^*, \varkappa_n) = 0$. Moreover, from (2.7) we get

$$\begin{aligned} \rho(U,V) &\leq \rho(\varkappa_n, \Upsilon\varkappa_n) \\ &\leq \rho(\varkappa_n, \eta_n) \\ &\leq \rho(\varkappa_n, \varkappa_{n+1}) + \rho(\varkappa_{n+1}, \eta_n) \\ &= \rho(\varkappa_n, \varkappa_{n+1}) + \rho(U,V) \end{aligned}$$

for all $n \geq 0$. Letting limit as $n \to +\infty$, from (2.11) it is obtained

$$\lim_{k\to+\infty} \rho(\varkappa_n, \Upsilon\varkappa_n) = \rho(U,V).$$

Since the function $\varkappa \to \rho(\varkappa, \Upsilon\varkappa)$ is l.s.c. on U w.r.t. τ_ρ, we have

$$\begin{aligned} \rho(U,V) &\leq \rho(\varkappa^*, \Upsilon\varkappa^*) \\ &= \lim_{n\to+\infty} \inf \rho(\varkappa_n, \Upsilon\varkappa_n) \\ &= \rho(U,V). \end{aligned}$$

Therefore, we get $\rho(\varkappa^*, \Upsilon\varkappa^*) = \rho(U,V)$, and so the proof is complete. □

Since the proof of the following result is similar to Theorem 2.5, we will give it without proof.

Theorem 2.6 *Let $\emptyset \neq U, V$ be subsets of a quasi metric (Λ, ρ) where U is closed with respect to $\tau_{\rho^{-1}}$. Assume that $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ is a function, $\Upsilon : U \to CB_{\rho^{-1}}(V)$ is a $\alpha_{\rho^{-1}}$-proximal admissible mapping and $F \in {}_\Upsilon\mathcal{F}^\alpha_{\rho^{-1}}$. Suppose that (Λ, ρ) is right M_α-complete, $\Upsilon\varkappa \subseteq V_0^{\mathcal{L}}$ for all $\varkappa \in U_0^{\Re}$ and the pair (U,V) has the generalized weak $P_{\rho^{-1}}$-property. If the following conditions are satisfied:*

(i) there are $\varkappa_0, \varkappa_1 \in U_0^{\Re}$ and $\eta_0 \in \Upsilon\varkappa_0$ satisfying

$$\alpha(\varkappa_0, \varkappa_1) \geq 1 \text{ and } \rho(\eta_0, \varkappa_1) = \rho(V,U),$$

(ii) there exists $\tau > 0$ satisfying

$$\tau + F(H_{\rho^{-1}}(\Upsilon\varkappa, \Upsilon\eta)) \leq F(\rho(\eta, \varkappa))$$

for all $(\varkappa, \eta) \in \Upsilon_\alpha^{\rho^{-1}}$,

then Υ has a right best proximity point in U providing that the function $\varkappa \to \rho(\Upsilon\varkappa, \varkappa)$ is l.s.c. on U w.r.t. $\tau_{\rho^{-1}}$.

Now, taking into account Theorem 2.5, we present a best proximity point result on quasi metric spaces including the well known result of Nadler [22].

Corollary 2.7 *Let $\emptyset \neq U, V$ be subsets of a quasi metric (Λ, ρ) where U is closed w.r.t. τ_ρ. Suppose that $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ is a function and $\Upsilon : U \to CB_\rho(V)$ is a α_ρ-proximal admissible mapping. Assume that (Λ, ρ) is left K_α-complete, $\Upsilon\varkappa \subseteq V_0^{\Re}$ for all $\varkappa \in U_0^{\mathcal{L}}$ and (U, V) has the generalized weak P_ρ-property. If the following ones hold:*

(i) there are $\varkappa_0, \varkappa_1 \in U_0^{\mathcal{L}}$ and $\eta_0 \in \Upsilon\varkappa_0$ satisfying

$$\alpha(\varkappa_0, \varkappa_1) \geq 1 \text{ and } \rho(\varkappa_1, \eta_0) = \rho(U, V),$$

(ii) there is $q \in [0, 1)$ such that

$$H_\rho(\Upsilon\varkappa, \Upsilon\eta) \leq q\rho(\varkappa, \eta) \tag{2.13}$$

for all $\varkappa, \eta \in U$ with $\alpha(\varkappa, \eta) \geq 1$,

then Υ has a left best proximity point in U providing that the function $\varkappa \to \rho(\varkappa, \Upsilon\varkappa)$ is l.s.c. on U w.r.t. τ_ρ.

Proof Assume that $\varkappa$ and η are arbitrary elements in Υ_α^ρ. So, using the hypothesis (ii) there is $q \in [0, 1)$ satisfying

$$H_\rho(\Upsilon\varkappa, \Upsilon\eta) \leq q\rho(\varkappa, \eta).$$

If we define a mapping $F : (0, +\infty) \to \mathbb{R}$ by $F(t) = \ln t$ for all $t \in (0, +\infty)$, then $F \in {}_\Upsilon\mathcal{F}_\rho^\alpha$. Also, if we take $\tau = -\ln q$, then we have

$$\tau + F(H_\rho(\Upsilon\varkappa, \Upsilon\eta)) \leq F(\rho(\varkappa, \eta)).$$

Hence, all hypotheses of Theorem 2.5 are satisfied. Therefore, Υ has a left best proximity point in U. □

The following example is important to show that Theorem 2.5 is a real generalization of Corollary 2.7.

Example 2.8 *Let $\Lambda = \left(0, \frac{1}{2}\right] \cup \{1, 2, 3, \cdots\}$. Define a function $\rho : \Lambda \times \Lambda \to \mathbb{R}$ by*

$$\rho(\varkappa, \eta) = \begin{cases} 0 & , \quad \varkappa = \eta \\ \varkappa & , \quad \varkappa \neq \eta \end{cases}.$$

Then, (Λ, ρ) is a quasi metric space. Think of the following subsets of Λ:

$$U = \{2k : k \geq 1\}$$

and

$$V = \{2k + 1 : k \geq 0\}.$$

Since τ_ρ is a discrete topology, $U \subseteq \Lambda$ is closed w.r.t. τ_ρ. Further, $\rho(U, V) = 2$, $U_0^{\mathcal{L}} = \{2\}$ and $V_0^{\Re} = V$. Then, (U, V) has the generalized weak P_ρ-property. Define a mapping $\Upsilon : U \to CB_\rho(V)$ and a function $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ by

$$\Upsilon\varkappa = \begin{cases} \{1\} & , \quad \varkappa = 2 \\ \{2k - 3, 2k - 1\} & , \quad \varkappa = 2k, \ k \geq 2 \end{cases}$$

and

$$\alpha(\varkappa,\eta)=\begin{cases}1 & , \quad \varkappa\geq\eta \text{ and } \varkappa,\eta\in U\\ \\ 0 & , \quad \text{otherwise}\end{cases},$$

respectively. Then, (Λ,ρ) *is a left* K_α*-complete and* Υ *is* α_ρ*-proximal admissible mapping satisfying* $\Upsilon\varkappa\subseteq V_0^{\Re}$ *for all* $\varkappa\in U_0^{\mathcal{L}}$*. Also, there are* $\varkappa_0=\varkappa_1=2\in U_0^{\mathcal{L}}$ *and* $\eta_0=1\in\Upsilon\varkappa_0$ *satisfying* $\rho(\varkappa_1,\eta_0)=\rho(U,V)$ *and* $\alpha(\varkappa_0,\varkappa_1)\geq 1$*. Now, if we define* $F:(0,+\infty)\to\mathbb{R}$ *by*

$$F(\beta)=\begin{cases}\ln\beta & , \quad \beta\leq 1\\ 2\beta & , \quad \beta>1\end{cases},$$

then it is clear that $F\in{}_{\Upsilon}\mathcal{F}_\rho^{\alpha}\backslash\mathcal{F}^*$*. Now, we shall demonstrate that the hypothesis (ii) of Theorem 2.5 holds. Notice that*

$$\begin{aligned}\Upsilon_\alpha^\rho &= \{(\varkappa,\eta)\in U\times U:\alpha(\varkappa,\eta)\geq 1 \text{ and } H_\rho(\Upsilon\varkappa,\Upsilon\eta)>0\}\\ &= \{(\varkappa,\eta)\in U\times U:\varkappa\geq\eta \text{ and } H_\rho(\Upsilon\varkappa,\Upsilon\eta)>0\}\\ &= \{(\varkappa,\eta)\in U\times U:\varkappa>\eta\}.\end{aligned}$$

Therefore, it is enough to check the following conditions:

Case 1. Let $\varkappa=2k$*,* $k\geq 2$ *and* $\eta=2$*. Then, we have*

$$\begin{aligned}\tau+F(H_\rho(\Upsilon\varkappa,\Upsilon\eta)) &= 1+F(2k-1)\\ &= 4k-1\\ &\leq 4k\\ &= F(2k)\\ &= F(\rho(\varkappa,\eta)).\end{aligned}$$

Case 2. Let $\varkappa=2k$ *and* $\eta=2l$*,* $k,l\geq 2$*,* $k>l$*. So, we have*

$$\begin{aligned}\tau+F(H_\rho(\Upsilon\varkappa,\Upsilon\eta)) &= 1+F(2k-1)\\ &= 4k-1\\ &\leq F(2k)\\ &= F(\rho(\varkappa,\eta)).\end{aligned}$$

Finally, since τ_ρ *is discrete, the function* $\varkappa\to\rho(\varkappa,\Upsilon\varkappa)$ *is l.s.c. on* U *w.r.t.* τ_ρ*. Hence, whole hypotheses of Theorem 2.5 hold, thus there is an element in* U *such that* $\rho(\varkappa,\Upsilon\varkappa)=\rho(U,V)$*. However, note that the condition (ii) of Corollary 2.7 is not satisfied. Therefore, Corollary 2.7 cannot be applied to this example. Suppose the contrary. So, there exists* q *in* $[0,1)$ *satisfying (2.13). If we take* $\varkappa=2k$*,* $k\geq 2$ *and* $\eta=2$*, then* $\alpha(2k,2)\geq 1$*. Hence, we have*

$$\frac{H_\rho(\Upsilon\varkappa,\Upsilon\eta)}{\rho(\varkappa,\eta)}=\frac{2k-1}{2k}\leq q.$$

So, taking limit as $k\to+\infty$ *we get*

$$1=\lim_{k\to+\infty}\frac{2k-1}{2k}\leq q<1$$

which is a contradiction.

3. Application

It has been realized that homotopy theory, which emerged as a subject in algebraic topology, is closely related to many branches of mathematics such as algebraic mathematics and category theory recently. Therefore, many authors have obtained an application of their fixed point results to homotopy theory [1, 21, 23, 31]. So, in this section, we present an application to homotopy theory via our main results. In this sense, we investigate the existence of best proximity points for homotopic mappings. Now, recall the definition of homotopy:

Definition 3.1 *Let (Λ_1, τ_1) and (Λ_2, τ_2) be topological spaces, $\varphi, \psi : \Lambda_1 \to \Lambda_2$ be continuous mappings. If there exists continuous function $\Upsilon : \Lambda_1 \times [0,1] \to \Lambda_2$ such that $\Upsilon(\varkappa, 0) = \varphi\varkappa$ and $\Upsilon(\varkappa, 1) = \psi\varkappa$ for all $\varkappa \in \Lambda_1$, then it is said to be that φ and ψ are homotopic mappings. Also, the mapping Υ is called homotopy.*

Now, we present the following result:

Theorem 3.2 *Let $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ be a function and $\Upsilon : U \times [0,1] \to CB_\rho(V)$ be a multivalued mapping where (Λ, ρ) is a quasi metric space, $\emptyset \neq U, V \subseteq \Lambda$ and U is closed w.r.t. τ_ρ. Suppose that (Λ, ρ) is a left K_α-complete, (U, V) has the generalized weak P_ρ-property, $F \in {}_{\Upsilon}\mathcal{F}_\rho^\alpha$ and $\emptyset \neq M \subseteq U$. Presume that the following conditions hold:*

(i) $\rho(\varkappa, \Upsilon(\varkappa, \lambda)) > \rho(U, V)$ for all $\varkappa \in U \backslash M$, $\lambda \in [0,1]$ and $\alpha(\varkappa, \eta) \geq 1$ for all $\varkappa, \eta \in M$,

(ii) there exist $\varkappa_0, \varkappa_1 \in U_0^{\mathcal{L}}$ and $\eta_0 \in \Upsilon(\varkappa_0, \lambda)$ for all $\lambda \in [0,1]$ satisfying $\rho(\varkappa_1, \eta_0) = \rho(U, V)$ and $\alpha(\varkappa_0, \varkappa_1) \geq 1$,

(iii) for each $\varkappa, \eta \in U$ satisfying $\alpha(\varkappa, \eta) \geq 1$ and $H_\rho(\Upsilon(\varkappa, \lambda), \Upsilon(\eta, \lambda')) > 0$ for every $\lambda, \lambda' \in [0,1]$ there is $\tau > 0$ such that

$$\tau + F(H_\rho(\Upsilon(\varkappa, \lambda), \Upsilon(\eta, \lambda'))) \leq F(\rho(\varkappa, \eta)), \tag{3.1}$$

(iv) for all $\lambda \in [0,1]$, the mapping $\Upsilon(\cdot, \lambda) : U \to CB_\rho(V)$ is α_ρ-proximal admissible,

(v) for each $\lambda \in [0,1]$ such that $\rho(\varkappa, \Upsilon(\varkappa, \lambda)) = \rho(U, V)$ for some $\varkappa \in M$, there exists $\varepsilon_\lambda > 0$ satisfying $\Upsilon(\varkappa, \lambda^*) \subseteq V_0^{\Re}$ for all $\varkappa \in U_0^{\mathcal{L}}$ and $\lambda^* \in (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)$,

(vi) if $\rho(\varkappa, \Upsilon(\varkappa, \lambda)) = \rho(U, V)$ for some $\varkappa \in U$ and $\lambda \in [0,1]$, then $\Upsilon(\varkappa, \lambda)$ is singleton.

Then, $\Upsilon(\cdot, 1)$ has a left best proximity point if $\Upsilon(\cdot, 0)$ has a left best proximity point providing that for all $\lambda \in [0,1]$, $\varkappa \to \rho(\varkappa, \Upsilon(\varkappa, \lambda))$ is l.s.c. on U w.r.t. τ_ρ.

Proof Consider

$$K = \{\lambda \in [0,1] : \rho(\varkappa, \Upsilon(\varkappa, \lambda)) = \rho(U, V) \text{ for some } \varkappa \in M\}.$$

Since the hypothesis (i) is satisfied and there exists $\varkappa$ in U such that $\rho(\varkappa, \Upsilon(\varkappa, 0)) = \rho(U, V)$, we obtain $0 \in K$, and so $K \neq \emptyset$. If we demonstrate that K is both open and closed, then due to connectedness of $[0,1]$, we get $K = [0,1]$. Now, assume that $\{\lambda_k\} \subseteq K$ is a sequence such that $\lim_{k \to +\infty} \lambda_k = \lambda^*$. Hence, there exists $\varkappa_k$ in M such that

$$\rho(\varkappa_k, \Upsilon(\varkappa_k, \lambda_k)) = \rho(U, V) \tag{3.2}$$

for all $k \geq 0$. We want to demonstrate that $\{\varkappa_k\}$ is left K-Cauchy. Presume the contrary. So, there exist two sequences $\{l_r\}$, $\{k_r\}$ with $\ell_r > k_r \geq r$ and $\varepsilon > 0$ such that

$$\rho(\varkappa_{k_r}, \varkappa_{l_r}) \geq \varepsilon \tag{3.3}$$

for all $r \geq 0$ where ℓ_r is the least integer satisfying the inequality (3.3). Also, since (U, V) has the generalized weak P_ρ-property and $\Upsilon(\varkappa_k, \lambda_k)$ is singleton, from (3.2) we have

$$\varepsilon \leq \rho(\varkappa_{k_r}, \varkappa_{l_r}) \leq \rho(\Upsilon(\varkappa_{k_r}, \lambda_{k_r}), \Upsilon(\varkappa_{l_r}, \lambda_{l_r})) = H_\rho(\Upsilon(\varkappa_{k_r}, \lambda_{k_r}), \Upsilon(\varkappa_{l_r}, \lambda_{l_r})$$

for all $r \geq 0$. Hence, considering $\alpha(\varkappa_{k_r}, \varkappa_{l_r}) \geq 1$ for all $r \geq 0$ and the condition (iii) we say that there exists $\tau > 0$ such that

$$\tau + F(H_\rho(\Upsilon(\varkappa_{k_r}, \lambda_{k_r}), \Upsilon(\varkappa_{l_r}, \lambda_{l_r}))) \leq F(\rho(\varkappa_{k_r}, \varkappa_{l_r})).$$

Hence, we have

$$\begin{array}{rcl} F(\rho(\varkappa_{k_r}, \varkappa_{l_r}) & \leq & F(\rho(\Upsilon(\varkappa_{k_r}, \lambda_{k_r}), \Upsilon(\varkappa_{l_r}, \lambda_{l_r})))) \\ & = & F(H_\rho(\Upsilon(\varkappa_{k_r}, \lambda_{k_r}), \Upsilon(\varkappa_{l_r}, \lambda_{l_r}))) \\ & \leq & F(\rho(\varkappa_{k_r}, \varkappa_{l_r})) - \tau \\ & < & F(\rho(\varkappa_{k_r}, \varkappa_{l_r}) \end{array}$$

which is a contradiction. Hence, $\{\varkappa_k\}$ is a left K-Cauchy sequence satisfying $\alpha(\varkappa_k, \varkappa_{k+1}) \geq 1$. Since $U \subseteq \Lambda$ is closed w.r.t. τ_ρ and (Λ, ρ) is a K_α-complete quasi metric space, there exists $\varkappa^* \in U$ such that $\lim_{k\to+\infty} \rho(\varkappa^*, \varkappa_k) = 0$. Because of the fact that the function $\varkappa \to \rho(\varkappa, \Upsilon(\varkappa, \lambda))$ is l.s.c. w.r.t. τ_ρ, from (3.2) we have

$$\begin{array}{rcl} \rho(U, V) & \leq & \rho(\varkappa^*, \Upsilon(\varkappa^*, \lambda^*)) \\ & = & \lim_{k\to+\infty} \inf \rho(\varkappa_k, \Upsilon(\varkappa_k, \lambda_k)) \\ & = & \rho(U, V). \end{array}$$

Therefore, we get $\lambda^* \in K$, thus $K \subseteq [0, 1]$ is closed.

Now, assume that $\lambda_0 \in K$. So, there exists $\varkappa_0 \in M$ satisfying $\rho(\varkappa_0, \Upsilon(\varkappa_0, \lambda_0)) = \rho(U, V)$. From the condition (v), for $\lambda_0 \in [0, 1]$, there exists $\varepsilon_{\lambda_0} > 0$ such that $\Upsilon(\varkappa, \lambda^*) \subseteq V_0^{\Re}$ for all $\varkappa \in U_0^{\mathcal{L}}$ and $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$. Now, from the condition (iii) the mapping $\Upsilon(\cdot, \lambda^*) : U \to CB_\rho(V)$ for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$ satisfies (2.2). Hence, all assumptions of Theorem 2.5 hold. Therefore, for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$, $\Upsilon(\cdot, \lambda^*)$ has a left best proximity point $\varkappa^*_{\lambda^*} \in U$. Using the condition (i), we get $\varkappa^*_{\lambda^*} \in M$ for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$ and so $\lambda^* \in K$. Hence, we get $\lambda_0 \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}) \subseteq K$, that is, K is open in $[0, 1]$. □

If we take that (Λ, ρ) is a right M_α-complete, then we obtain the following result. Since the proof of this result is similar to the proof of Theorem 3.2, we give it without proof.

Theorem 3.3 *Let $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ be a function and $\Upsilon : U \times [0, 1] \to CB_{\rho^{-1}}(V)$ be a multivalued mapping where (Λ, ρ) is a quasi metric space, $\emptyset \neq U, V \subseteq \Lambda$ and U is closed w.r.t. $\tau_{\rho^{-1}}$. Suppose that (Λ, ρ) is a right M_α-complete, the pair (U, V) has the generalized weak $P_{\rho^{-1}}$-property, $F \in {}_\Upsilon\mathcal{F}^\alpha_{\rho^{-1}}$ and $\emptyset \neq M \subseteq U$. Assume that the following conditions hold:*

(i) $\rho(\Upsilon(\varkappa,\lambda),\varkappa) > \rho(V,U)$ *for all* $\varkappa \in U \backslash M$, $\lambda \in [0,1]$ *and* $\alpha(\varkappa,\eta) \geq 1$ *for all* $\varkappa, \eta \in M$,

(ii) *there exist* $\varkappa_0, \varkappa_1 \in U_0^{\Re}$ *and* $\eta_0 \in \Upsilon(\varkappa_0,\lambda)$ *for all* $\lambda \in [0,1]$ *satisfying* $\rho(\eta_0,\varkappa_1) = \rho(V,U)$ *and* $\alpha(\varkappa_0,\varkappa_1) \geq 1$,

(iii) *for each* $\varkappa, \eta \in U$ *satisfying* $\alpha(\varkappa,\eta) \geq 1$ *and* $H_{\rho^{-1}}(\Upsilon(\varkappa,\lambda),\Upsilon(\eta,\lambda')) > 0$ *for all* $\lambda, \lambda' \in [0,1]$ *there is* $\tau > 0$ *such that*

$$\tau + F(H_{\rho^{-1}}(\Upsilon(\varkappa,\lambda),\Upsilon(\eta,\lambda'))) \leq F(\rho(\eta,\varkappa)),$$

(iv) *for all* $\lambda \in [0,1]$, *the mapping* $\Upsilon(\cdot,\lambda) : U \to CB_{\rho}(V)$ *is* $\alpha_{\rho^{-1}}$*-proximal admissible,*

(v) *for each* $\lambda \in [0,1]$ *with* $\rho(\Upsilon(\varkappa,\lambda),\varkappa) = \rho(V,U)$ *for some* $\varkappa \in M$, *there exists* $\varepsilon_\lambda > 0$ *such that* $\Upsilon(\varkappa,\lambda^*) \subseteq V_0^{\mathcal{L}}$ *for all* $\varkappa \in U_0^{\Re}$ *and* $\lambda^* \in (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)$,

(vi) *if* $\rho(\Upsilon(\varkappa,\lambda),\varkappa) = \rho(V,U)$ *for some* $\varkappa \in U$ *and* $\lambda \in [0,1]$, *then* $\Upsilon(\varkappa,\lambda)$ *is singleton.*

Then, $\Upsilon(\cdot,1)$ *has a right best proximity point if* $\Upsilon(\cdot,0)$ *has a right best proximity point providing that for all* $\lambda \in [0,1]$, *the function* $\varkappa \to \rho(\Upsilon(\varkappa,\lambda),\varkappa)$ *is l.s.c. on* U *w.r.t.* $\tau_{\rho^{-1}}$.

4. Conclusion

In the present paper, we first introduce a new subclass of the family $\mathcal{F}$. Then, we show that $\mathcal{F}^*$, which has often been used to give an affirmative answer to the problem arising from Altun et al. [7] for multivalued F-contractions (closed and bounded valued), is a proper subset of this new class. Taking into account the note given by Almeida et al. [4], we also generalize the definition of weak P-property. Moreover, we modify some concepts in best proximity point theory like α-proximal admissible, by considering the lack of symmetry condition on quasi metric spaces. Hence, we obtain some best proximity point results for multivalued F-contractions via the new class on quasi metric spaces under generalized weak P-property. Therefore, we propose a new and interesting way to overcome the aforementioned problem. Further, we present a real generalization of fixed point results existing in the literature, since the note given by Almeida et al. [4] cannot be applicable to our results due to both the discontinuity of quasi metric spaces and generalized weak P-property. On the other hand, we show that all mappings homotopic to a mapping satisfying all the hypotheses of our best proximity point result have a best proximity point, too.

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