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Birotational hypersurface and the second Laplace–Beltrami operator in the four dimensional Euclidean space $\mathbb{E}^4$

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Abstract: We consider the birotational hypersurface $\mathbf{x}(u,v,w)$ with the second Laplace–Beltrami operator in the four dimensional Euclidean space $\mathbb{E}^4$. We give the i-th curvatures of $\mathbf{x}$. In addition, we compute the second Laplace–Beltrami operator of the birotational hypersurface satisfying $\Delta^{II}\mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix $\mathcal{A}$.

Key words: Euclidean spaces, four space, birotational hypersurface, Gauss map, i-th curvature, second Laplace–Beltrami operator

1. Introduction

With Chen [13–16], the researches of the submanifolds of the finite type whose immersion into the $\mathbb{E}^m$ (or $\mathbb{E}^m_\nu$) by using a finite number of eigenfunctions of their Laplacian have been examined for almost 50 years.

Takahashi [46] introduced that a connected Euclidean submanifold is of 1-type, iff it is either minimal in $\mathbb{E}^m$ or minimal in some hypersphere of $\mathbb{E}^m$. Submanifolds of the finite type closest in simplicity to the minimal ones are the 2-type spherical submanifolds (where spherical means into a sphere). Some results of the 2-type spherical closed submanifolds were studied by [9, 10, 14]. Garay [28] worked an extension of the Takahashi's theorem in $\mathbb{E}^m$. Cheng and Yau gave the hypersurfaces with constant scalar curvature; Chen and Piccinni [17] considered the submanifolds with the finite type Gauss map in $\mathbb{E}^m$. Dursun [23] focused on the hypersurfaces with pointwise 1-type Gauss map in $\mathbb{E}^{n+1}$.

In $\mathbb{E}^3$; Takahashi [46] gave that the minimal surfaces and spheres are the only surfaces satisfying the condition $\Delta r = \lambda r,\ \lambda \in \mathbb{R}$; Ferrandez et al. [25] classified that the surfaces satisfying $\Delta H = AH$, $A \in Mat(3,3)$ are either minimal, or an open piece of sphere or of a right circular cylinder; Choi and Kim [20] found the minimal helicoid in terms of the pointwise 1-type Gauss map of the first kind; Garay [27] worked on the certain class of the finite type surfaces of revolution; Dillen et al. [21] focused that the only surfaces satisfying $\Delta r = Ar + B$, $A \in Mat(3,3)$, $B \in Mat(3,1)$ are the minimal surfaces, the spheres and the circular cylinders; Stamatakis and Zoubi [45] obtained the surfaces of revolution satisfying $\Delta^{III}x = Ax$; Senoussi and Bekkar [44] introduced the helicoidal surfaces M^2 which are of the finite type with respect to the fundamental forms I, II and III, i.e.

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their position vector field $r(u,v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A \in Mat(3,3)$; Kim et al. [37] gave the Cheng–Yau's operator and the Gauss map of the surfaces of revolution.

In $\mathbb{E}^4$; Moore [41, 42] gave the general rotational surfaces; Hasanis and Vlachos [34] studied the hypersurfaces with the harmonic mean curvature vector field; Cheng and Wan [18] considered the complete hypersurfaces with CMC; Kim and Turgay [38] studied the surfaces with the L_1-pointwise 1-type Gauss map; Arslan et al. [3] introduced Vranceanu surface with pointwise 1-type Gauss map; Arslan et al. [4] introduced generalized rotational surfaces; Arslan et al. [5] obtained the tensor product surfaces with pointwise 1-type Gauss map; Kahraman Aksoyak and Yaylı [35] considered the rotational surfaces with the pointwise 1-type Gauss map; Güler et al. [32] worked the helicoidal hypersurfaces; Güler et al. [31] introduced the Gauss map and the third Laplace–Beltrami operator of the rotational hypersurface; Güler and Turgay [33] obtained the Cheng–Yau's operator and the Gauss map of the rotational hypersurfaces; Güler [30] worked the rotational hypersurfaces satisfying $\Delta^I R = AR$, where $A \in Mat(4,4)$. He [29] also examined the fundamental form IV and the curvature formulas of the hypersphere; Arslan et al. [7] introduced the rotational λ-hypersurfaces in the Euclidean spaces.

In Minkowski 4-space $\mathbb{E}^4_1$; Ganchev and Milousheva [26] indicated analogue of surfaces of [41, 42]; Arvanitoyeorgos et al. [8] studied that if the mean curvature vector field of M^3_1 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M^3_1 has CMC; Arslan and Milousheva [6] considered the meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map; Turgay [47] introduced some classifications of the Lorentzian surfaces with the finite type Gauss map; Dursun and Turgay [24] gave the space-like surfaces in with the pointwise 1-type Gauss map. Kahraman Aksoyak and Yaylı [36] worked the general rotational surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4_2$; Bektaş et al. [11] considered surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map in $\mathbb{E}^5_2$. They [12] also gave the pseudo-spherical submanifolds with 1-type pseudo-spherical Gauss map.

We consider the birotational hypersurface with the second Laplace-Beltrami operator in the four dimensional Euclidean space $\mathbb{E}^4$. In Section 2, we indicate the fundamental notions of the four dimensional Euclidean geometry. We obtain the curvature formulas of a hypersurface in $\mathbb{E}^4$ in Section 3. In Section 4, we give the birotational hypersurface. Additionally, we examine the birotational hypersurface satisfying $\Delta^{II}\mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix $\mathcal{A}$ in $\mathbb{E}^4$ in Section 5. Finally, we give some results in Section 6.

2. Preliminaries

In this section, giving some of basic facts and definitions, we describe notations used whole paper. Let $\mathbb{E}^m$ denote the Euclidean m-space with the canonical Euclidean metric tensor given by $\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$. Consider an m-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote the Levi–Civita connections of $\mathbb{E}^m$ and M by $\widetilde{\nabla}$ and ∇, respectively. We shall use the letters X, Y, Z, W (resp., ξ, η) to denote the vectors fields tangent (resp., normal) to M. The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \tag{2.2}$$

where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

For each $\xi \in T_p^{\perp}M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM at $p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X,Y,)Z,W\rangle = \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{2.3}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.4}$$

where R, R^D are the curvature tensors associated with connections ∇ and D, respectively, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Hypersurfaces of Euclidean space

Now, let M be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $\mathbf{S}$ its shape operator (i.e. Weingarten map) and x its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ of consisting of principal directions of M corresponding from the principal curvature k_i for $i = 1, 2, \dots n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \dots, \theta_n\}$. Then the first structural equation of Cartan is given by

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i,j = 1,2,\dots,n, \tag{2.5}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. We denote the Levi–Civita connection of M and $\mathbb{E}^{n+1}$ by ∇ and $\widetilde{\nabla}$, respectively. Then, from the Codazzi equation (2.3), we have

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \tag{2.6}$$

$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \tag{2.7}$$

for distinct $i, j, l = 1, 2, \dots, n$.

We put $s_j = \sigma_j(k_1, k_2, \dots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} a_{i_1} a_{i_2} \dots a_{i_j}.$$

We use the following notation

$$r_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, k_{i+2}, \dots, k_n).$$

By the definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. We call the function s_k as the k-th mean curvature of M. We would like to note that functions $H = \frac{1}{n}s_1$ and $K = s_n$ are called the mean curvature and the Gauss–Kronecker curvature of M, respectively. In particular, M is said to be the j-minimal if $s_j \equiv 0$ on M.

In $\mathbb{E}^{n+1}$, to find the i-th curvature formulas $\mathfrak{C}_i$ (Curvature formulas sometimes are represented as the mean curvature H_i, and sometimes as the Gaussian curvature K_i by different writers, such as [1] and [39]. We will call it just the i-th curvature $\mathfrak{C}_i$ in this paper.), where $i = 0, ..., n$, firstly, we use the characteristic polynomial of $\mathbf{S}$:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda \mathcal{I}_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \tag{2.8}$$

where $i = 0, ..., n$, $\mathcal{I}_n$ denotes the identity matrix of order n. Then, we get curvature formulas $\binom{n}{i}\mathfrak{C}_i = s_i$. That is, $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1}\mathfrak{C}_1 = s_1, \ldots, \binom{n}{n}\mathfrak{C}_n = s_n = K$.

For a Euclidean submanifold $x\colon\ M \longrightarrow \mathbb{E}^m$, the immersion (M, x) is called finite type, if x can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of (M, x), i.e. $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \ldots, x_k$ nonconstant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. If λ_i are different, M is called *k-type*. See [14] for details.

Let $\mathbf{x} = \mathbf{x}(u, v, w)$ be an isometric immersion from $M^3 \subset \mathbb{E}^3$ to $\mathbb{E}^4$. The triple vector product of $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ of $\mathbb{E}^4$ is defined by:

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

For a hypersurface $\mathbf{x}$ in 4-space, we see $(g_{ij})_{3\times 3}$, $(h_{ij})_{3\times 3}$, where (g_{ij}) and (h_{ij}) are the first, and the second fundamental form matrices, respectively, and $g_{11} = \mathbf{x}_u \cdot \mathbf{x}_u$, $g_{12} = \mathbf{x}_u \cdot \mathbf{x}_v$, $g_{22} = \mathbf{x}_v \cdot \mathbf{x}_v$, $g_{13} = \mathbf{x}_u \cdot \mathbf{x}_v$, $g_{23} = \mathbf{x}_v \cdot \mathbf{x}_w$, $g_{33} = \mathbf{x}_w \cdot \mathbf{x}_w$, $h_{11} = \mathbf{x}_{uu} \cdot e$, $h_{12} = \mathbf{x}_{uv} \cdot e$, $h_{22} = \mathbf{x}_{vv} \cdot e$, $h_{13} = \mathbf{x}_{uw} \cdot e$, $h_{23} = \mathbf{x}_{vw} \cdot e$, $h_{33} = \mathbf{x}_{ww} \cdot e$. Here,

$$e = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\|} \tag{2.9}$$

is the unit normal (i.e. the Gauss map) of the hypersurface $\mathbf{x}$.

The product matrices $(g_{ij})^{-1} \cdot (h_{ij})$ gives the matrix of the shape operator $\mathbf{S}$ of the hypersurface $\mathbf{x}$ in 4-space. See [31–33] for details.

3. i-th curvatures

In $\mathbb{E}^4$, to compute the i-th mean curvature formula $\mathfrak{C}_i$, where $i = 0, 1, 2, 3,$ we use the characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$:

$$P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda I_3) = 0.$$

Then, obtain $\mathfrak{C}_0 = 1$ (by definition), $\binom{3}{1}\mathfrak{C}_1 = \binom{3}{1}H = -\frac{b}{a}$, $\binom{3}{2}\mathfrak{C}_2 = \frac{c}{a}$, $\binom{3}{3}\mathfrak{C}_3 = K = -\frac{d}{a}$.

Therefore, we find i-th curvature folmulas depends on the coefficients of the first and second fundamental forms in 4-space.

Theorem 3.1 *Any hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^4$ *has the following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{\left\{ \begin{array}{c} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})g_{33} + (g_{11}g_{22} - g_{12}^2)h_{33} - g_{23}^2 h_{11} - g_{13}^2 h_{22} \\ -2(g_{13}h_{13}g_{22} - g_{23}h_{13}g_{12} - g_{13}h_{23}g_{12} + g_{11}g_{23}h_{23} - g_{13}g_{23}h_{12}) \end{array} \right\}}{3\left[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2\right]}, \tag{3.1}$$

$$\mathfrak{C}_2 = \frac{\left\{ \begin{array}{c} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})\,h_{33} + \left(h_{11}h_{22} - g_{12}^2\right)g_{33} - g_{11}h_{23}^2 - g_{22}h_{13}^2 \\ -2\left(g_{13}h_{13}h_{22} - g_{23}h_{13}h_{12} - g_{13}h_{23}h_{12} + g_{23}h_{23}h_{11} - h_{13}h_{23}g_{12}\right) \end{array} \right\}}{3\left[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2\right]}, \tag{3.2}$$

$$\mathfrak{C}_3 = \frac{\left(h_{11}h_{22} - h_{12}^2\right)h_{33} - h_{11}h_{23}^2 + 2h_{12}h_{13}h_{23} - h_{22}h_{13}^2}{(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2}. \tag{3.3}$$

See [29] for details.

4. Birotational hypersurface

In this section, we define the rotational hypersurface, then find its differential geometric properties in Euclidean 4-space $\mathbb{E}^4$. We would like to note that the definition of the rotational hypersurfaces in Riemannian space forms were defined in [22]. A rotational hypersurface $M \subset \mathbb{E}^{n+1}$ generated by a curve γ around an axis γ that does not meet γ is obtained by taking the orbit of γ under those orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed (See [22, Remark 2.3]).

We use curve γ as $(\mathbf{f}(u), 0, \mathbf{g}(u), 0)$ with the following rotation matrix

$$\begin{pmatrix} \cos v & -\sin v & 0 & 0 \\ \sin v & \cos v & 0 & 0 \\ 0 & 0 & \cos w & -\sin w \\ 0 & 0 & \sin w & \cos w \end{pmatrix},$$

and give the following definition:

Definition 4.1 *A birotational hypersurface in* $\mathbb{E}^4$ *is defined by*

$$\mathbf{x}(u, v, w) = (\mathbf{f}(u)\cos v, \mathbf{f}(u)\sin v, \mathbf{g}(u)\cos w, \mathbf{g}(u)\sin w), \tag{4.1}$$

where $\mathbf{f}, \mathbf{g}$ *are differentiable functions, and* $0 \leq v, w \leq 2\pi$.

Remark 4.2 *While* $\mathbf{f}(u) = \mathbf{g}(u) = 1$ *in* (4.1)*, we obtain the Clifford torus in* $\mathbb{E}^4$*. See [2, 48] for details. Moreover, when* $v = w$ *in* (4.1)*, we get the tensor product surface in* $\mathbb{E}^4$*. See [5, 43] for details.*

Considering the following first order derivative of (4.1) with respect to u, v, w, respectively,

$$\mathbf{x}_u = \begin{pmatrix} \mathbf{f}' \cos v \\ \mathbf{f}' \sin v \\ \mathbf{g}' \cos w \\ \mathbf{g}' \sin w \end{pmatrix}, \quad \mathbf{x}_v = \begin{pmatrix} -\mathbf{f} \sin v \\ \mathbf{f} \cos v \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_w = \begin{pmatrix} 0 \\ 0 \\ -\mathbf{g} \sin w \\ \mathbf{g} \cos w \end{pmatrix},$$

we find the following first quantities of (4.1):

$$(g_{ij}) = diag\left(\mathbf{f}'^2 + \mathbf{g}'^2, \mathbf{f}^2, \mathbf{g}^2\right), \tag{4.2}$$

where $\mathbf{f}'$ and $\mathbf{g}'$ denote the first order derivative of $\mathbf{f}$ and $\mathbf{g}$ respect to u, respectively. Here,

$$g = \det(g_{ij}) = \mathbf{f}^2\mathbf{g}^2\left(\mathbf{f}'^2 + \mathbf{g}'^2\right).$$

Using (2.9), we get the following Gauss map of the birotational hypersurface (4.1):

$$e = \frac{1}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}\left(-\mathbf{g}'\cos v, -\mathbf{g}'\sin v, \mathbf{f}'\cos w, \mathbf{f}'\sin w\right). \tag{4.3}$$

With the help of the second differentials of (4.1) with respect to u, v, w, and the Gauss map (4.3) of the birotational hypersurface (4.1), we have the following second quantities:

$$(h_{ij}) = diag\left(\frac{\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}, \frac{\mathbf{f}\mathbf{g}'}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}, -\frac{\mathbf{g}\mathbf{f}'}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}\right), \tag{4.4}$$

where $\mathbf{f}''$ and $\mathbf{g}''$ denote the second order derivative of $\mathbf{f}$ and $\mathbf{g}$ respect to u, respectively. So, we get

$$h = \det(h_{ij}) = -\frac{\mathbf{f}\mathbf{g}\mathbf{f}'\mathbf{g}'\left(\mathbf{f}'\mathbf{g}'' - \mathbf{f}''\mathbf{g}'\right)}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{3/2}}.$$

We calculate the shape operator matrix of the birotational hypersurface (4.1), by using (4.2) and (4.4), then, obtain the following

$$\mathbf{S} = diag\left(\frac{\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{3/2}}, \frac{\mathbf{g}'}{\mathbf{f}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}, -\frac{\mathbf{f}'}{\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}\right).$$

Finally, by using (3.1), (3.2), and (3.3), with (4.2), (4.4), respectively, we find the following curvatures of the birotational hypersurface (4.1):

Corollary 4.3 *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. The* $\mathbf{x}$ *has the following (mean) 1-curvature*

$$\mathfrak{C}_1 = \frac{\left(\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''\right)\mathbf{f}\mathbf{g} - \left(\mathbf{f}'^2 + \mathbf{g}'^2\right)\left(\mathbf{f}\mathbf{f}' - \mathbf{g}\mathbf{g}'\right)}{3\mathbf{f}\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{3/2}}.$$

Corollary 4.4 *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. The* $\mathbf{x}$ *has the following 2-curvature*

$$\mathfrak{C}_2 = \frac{\left(\mathbf{f}\mathbf{f}' - \mathbf{g}\mathbf{g}'\right)\left(\mathbf{g}'\mathbf{f}'' - \mathbf{f}'\mathbf{g}''\right) - \left(\mathbf{f}'^2 + \mathbf{g}'^2\right)\mathbf{f}'\mathbf{g}'}{3\mathbf{f}\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^2}.$$

Corollary 4.5 *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. The* $\mathbf{x}$ *has the following (Gaussian) 3-curvature*

$$\mathfrak{C}_3 = -\frac{\mathbf{f}'\mathbf{g}'\left(\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''\right)}{\mathbf{f}\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{5/2}}.$$

Example 4.6 *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. When the curve* γ *of* $\mathbf{x}$ *is parametrized by the arc length, and* $\mathbf{f}(u) = \cos u$*,* $\mathbf{g}(u) = \sin u$*, the birotational hypersurface has the following curvatures*

$$\begin{aligned} \mathfrak{C}_1 &= 1, \text{ i.e. } \mathbf{x} \text{ has positive CMC,} \\ \mathfrak{C}_2 &= 1, \text{ i.e. } \mathbf{x} \text{ has positive contant 2-curvature,} \\ \mathfrak{C}_3 &= 1, \text{ i.e. } \mathbf{x} \text{ has positive CGC.} \end{aligned}$$

Example 4.7 *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. When the curve* γ *of* $\mathbf{x}$ *is parametrized by* $\mathbf{f}(u) = \mathbf{g}(u) = \frac{u}{\sqrt{2}}$*, the birotational hypersurface has the following curvatures*

$$\begin{aligned} \mathfrak{C}_1 &= 0, \text{ i.e. } \mathbf{x} \text{ is 1-minimal,} \\ \mathfrak{C}_2 &= -\frac{1}{3u^2}, \text{ i.e. } \mathbf{x} \text{ has negative 2-curvature,} \\ \mathfrak{C}_3 &= 0, \text{ i.e. } \mathbf{x} \text{ is 3-minimal.} \end{aligned}$$

5. Birotational hypersurface satisfying $\Delta^{II}\mathbf{x} = \mathcal{A}\mathbf{x}$

In this section, we give the second Laplace–Beltrami operator of a smooth function. Therefore, we calculate the second Laplace–Beltrami operator of the birotational hypersurface.

The inverse of the matrix II, i.e.

$$(h_{ij}) = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$$

is given by

$$\frac{1}{h}\begin{pmatrix} h_{22}h_{33} - h_{23}h_{32} & -(h_{12}h_{33} - h_{13}h_{32}) & h_{12}h_{23} - h_{13}h_{22} \\ -(h_{21}h_{33} - h_{31}h_{23}) & h_{11}h_{33} - h_{13}h_{31} & -(h_{11}h_{23} - h_{21}h_{13}) \\ h_{21}h_{32} - h_{22}h_{31} & -(h_{11}h_{32} - h_{12}h_{31}) & h_{11}h_{22} - h_{12}h_{21} \end{pmatrix},$$

where

$$\begin{aligned} h &= \det(h_{ij}) \\ &= h_{11}h_{22}h_{33} - h_{11}h_{23}h_{32} + h_{12}h_{31}h_{23} - h_{12}h_{21}h_{33} + h_{21}h_{13}h_{32} - h_{13}h_{22}h_{31}. \end{aligned}$$

Definition 5.1 *The second Laplace−Beltrami operator of a smooth function* $\varphi = \varphi(x^1, x^2, x^3)\mid_{\mathbf{D}}$ $(\mathbf{D} \subset \mathbb{R}^3)$ *of class* C^3 *with respect to the second fundamental form of a hypersurface* $\mathbf{x}$ *is the operator* Δ^{II} *defined by*

$$\Delta^{II}\varphi = \frac{1}{h^{1/2}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i}\left(h^{1/2}h^{ij}\frac{\partial\varphi}{\partial x^j}\right), \tag{5.1}$$

where $\left(h^{ij}\right) = (h_{kl})^{-1}$ *and* $h = \det(h_{ij})$*.*

We can write (5.1), clearly, as follows:

$$\Delta^{II}\varphi = \frac{1}{|h|^{1/2}}\left\{ \begin{array}{c} \frac{\partial}{\partial x^1}\left(|h|^{1/2} h^{11} \frac{\partial\varphi}{\partial x^1}\right) - \frac{\partial}{\partial x^1}\left(|h|^{1/2} h^{12} \frac{\partial\varphi}{\partial x^2}\right) + \frac{\partial}{\partial x^1}\left(|h|^{1/2} h^{13} \frac{\partial\varphi}{\partial x^3}\right) \\ -\frac{\partial}{\partial x^2}\left(|h|^{1/2} h^{21} \frac{\partial\varphi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(|h|^{1/2} h^{22} \frac{\partial\varphi}{\partial x^2}\right) - \frac{\partial}{\partial x^2}\left(|h|^{1/2} h^{23} \frac{\partial\varphi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^3}\left(|h|^{1/2} h^{31} \frac{\partial\varphi}{\partial x^1}\right) - \frac{\partial}{\partial x^3}\left(|h|^{1/2} h^{32} \frac{\partial\varphi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(|h|^{1/2} h^{33} \frac{\partial\varphi}{\partial x^3}\right) \end{array} \right\}.$$

For any rotational hypersurface $h_{ij}=0$, when $i\neq j$. Hence, we can rewrite the second Laplace–Beltrami operator:

$$\Delta^{II}\varphi=\frac{1}{|h|^{1/2}}\left\{\frac{\partial}{\partial x^1}\left(|h|^{1/2}h^{11}\frac{\partial\varphi}{\partial x^1}\right)+\frac{\partial}{\partial x^2}\left(|h|^{1/2}h^{22}\frac{\partial\varphi}{\partial x^2}\right)+\frac{\partial}{\partial x^3}\left(|h|^{1/2}h^{33}\frac{\partial\varphi}{\partial x^3}\right)\right\}.$$

Therefore, more clear form of the second Laplace–Beltrami operator of any rotational hypersurface $\mathbf{x}(u,v,w)$ is given by

$$\Delta^{II}\mathbf{x}=\frac{1}{|h|^{1/2}}\left\{\frac{\partial}{\partial u}\left(\frac{h_{22}h_{33}}{|h|^{1/2}}\mathbf{x}_u\right)+\frac{\partial}{\partial v}\left(\frac{h_{11}h_{33}}{|h|^{1/2}}\mathbf{x}_v\right)+\frac{\partial}{\partial w}\left(\frac{h_{11}h_{22}}{|h|^{1/2}}\mathbf{x}_w\right)\right\}. \tag{5.2}$$

Differentiating $\frac{h_{22}h_{33}}{|h|^{1/2}}\mathbf{x}_u$, $\frac{h_{11}h_{33}}{|h|^{1/2}}\mathbf{x}_v$, $\frac{h_{11}h_{22}}{|h|^{1/2}}\mathbf{x}_w$, with respect to u,v,w, respectively, and substituting them into (5.2), we get the following.

Theorem 5.2 *The second Laplace–Beltrami operator of the birotational hypersurface* (4.1) *is given by*

$$\Delta^{II}\mathbf{x}=\begin{pmatrix}\Delta^{II}\mathbf{x}_1\\ \Delta^{II}\mathbf{x}_2\\ \Delta^{II}\mathbf{x}_3\\ \Delta^{II}\mathbf{x}_4\end{pmatrix}=\begin{pmatrix}\mathfrak{f}(u)\cos v\\ \mathfrak{f}(u)\sin v\\ \mathfrak{g}(u)\cos w\\ \mathfrak{g}(u)\sin w\end{pmatrix},$$

where

$$\mathfrak{f}(u)=\frac{\left\{\begin{array}{c}-\mathbf{fgf'g'^2}\left(\mathbf{f'^2}+\mathbf{g'^2}\right)\mathbf{f'''}+\mathbf{fgf'^2g'}\left(\mathbf{f'^2}+\mathbf{g'^2}\right)\mathbf{g'''}\\ +\left(\begin{array}{c}-\mathbf{fgg'}\left(4\mathbf{f'^2}+5\mathbf{g'^2}\right)\mathbf{f''}\\ +\left(\begin{array}{c}\mathbf{fgf'}\left(\mathbf{f'^2}+2\mathbf{g'^2}\right)\mathbf{g''}\\ -\mathbf{f'g'}\left(\mathbf{f'^2}+\mathbf{g'^2}\right)\left(\mathbf{fg'}+\mathbf{gf'}\right)\end{array}\right)\end{array}\right)\left(\mathbf{f'g''}-\mathbf{f''g'}\right)\end{array}\right\}}{\mathbf{fgg'}\left(\mathbf{f'^2}+\mathbf{g'^2}\right)^{1/2}\left(\mathbf{f'g''}-\mathbf{f''g'}\right)^2},$$

$$\mathfrak{g}(u)=\frac{\left\{\begin{array}{c}-\mathbf{fgf'g'^2}\left(\mathbf{f'^2}+\mathbf{g'^2}\right)\mathbf{f'''}+\mathbf{fgf'^2g'}\left(\mathbf{f'^2}+\mathbf{g'^2}\right)\mathbf{g'''}\\ +\left(\begin{array}{c}+\mathbf{fgg'}\left(2\mathbf{f'^2}+\mathbf{g'^2}\right)\mathbf{f''}\\ -\left(\begin{array}{c}\mathbf{fgf'}\left(5\mathbf{f'^2}+4\mathbf{g'^2}\right)\mathbf{g''}\\ +\mathbf{f'g'}\left(\mathbf{f'^2}+\mathbf{g'^2}\right)\left(\mathbf{fg'}+\mathbf{gf'}\right)\end{array}\right)\end{array}\right)\left(\mathbf{f'g''}-\mathbf{f''g'}\right)\end{array}\right\}}{\mathbf{fgf'}\left(\mathbf{f'^2}+\mathbf{g'^2}\right)^{1/2}\left(\mathbf{f'g''}-\mathbf{f''g'}\right)^2},$$

and $\mathbf{f'''}$ *and* $\mathbf{g'''}$ *denote the third order derivative of* $\mathbf{f}$ *and* $\mathbf{g}$ *respect to* u, *respectively.*

6. Conclusion

Considering the findings in the previous section, we obtain the following results:

Corollary 6.1 *Let* $\mathbf{x}:M^3\longrightarrow\mathbb{E}^4$ *be an immersion given by* (4.1). *The birotational hypersurface* $\mathbf{x}$ *satisfies* $\Delta^{II}\mathbf{x}=\mathcal{A}\mathbf{x}$, *where*

$$\mathcal{A}=diag\left(\frac{\mathfrak{f}}{\mathbf{f}}\mathcal{I}_2,\frac{\mathfrak{g}}{\mathbf{g}}\mathcal{I}_2\right),$$

and $\mathcal{A}\in Mat\,(4,4)$, $\mathcal{I}_2$ *is the identity matrix.*

Corollary 6.2 *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. When the curve* γ *of birotational hypersurface* $\mathbf{x}$ *is parametrized by the arc length, the* $\mathbf{x}$ *holds* $\Delta^{II}\mathbf{x} = \mathcal{B}\mathbf{x}$*, where*

$$\mathcal{B} = diag\left(\mathfrak{p}\mathcal{I}_2, \mathfrak{q}\mathcal{I}_2\right),$$

and

$$\begin{aligned}
\mathfrak{p}(u) &= \mathbf{f}'\left(\mathbf{f}'\mathbf{g}''' - \mathbf{g}'\mathbf{f}'''\right) - \left(4\mathbf{f}'^2 + 5\mathbf{g}'^2\right)\mathbf{f}'' + \frac{\mathbf{f}'\left(\mathbf{f}'^2 + 2\mathbf{g}'^2\right)}{\mathbf{g}'}\mathbf{g}'' - \frac{\mathbf{f}'\left(\mathbf{f}\mathbf{g}' + \mathbf{g}\mathbf{f}'\right)}{\mathbf{f}\mathbf{g}}, \\
\mathfrak{q}(u) &= \mathbf{g}'\left(\mathbf{f}'\mathbf{g}''' - \mathbf{g}'\mathbf{f}'''\right) + \frac{\mathbf{g}'\left(2\mathbf{f}'^2 + \mathbf{g}'^2\right)}{\mathbf{f}'}\mathbf{f}'' - \left(5\mathbf{f}'^2 + 4\mathbf{g}'^2\right)\mathbf{g}'' - \frac{\mathbf{g}'\left(\mathbf{f}\mathbf{g}' + \mathbf{g}\mathbf{f}'\right)}{\mathbf{f}\mathbf{g}},
\end{aligned}$$

and $\mathcal{B} \in Mat(4,4)$, $\mathcal{I}_2$ *is the identity matrix.*

Corollary 6.3 *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. When the curve* γ *of* $\mathbf{x}$ *is parametrized* $\mathbf{f}(u) = \cos u$, $\mathbf{g}(u) = \sin u$, *the birotational hypersurface* $\mathbf{x}$ *supplies* $\Delta^{II}\mathbf{x} = \mathcal{C}\mathbf{x}$*, where*

$$\mathcal{C} = 3\ diag\left(\cos u\ \mathcal{I}_2, \sin u\ \mathcal{I}_2\right),$$

and $\mathcal{C} \in Mat(4,4)$, $\mathcal{I}_2$ *is the identity matrix.*

Example 6.4 *Considering the hypersphere* $S^3(r) = \left\{\xi \in \mathbb{E}^4 \mid \langle \xi, \xi \rangle = r^2\right\}$ *for radius* $r > 0$*:*

$$\xi(u,v,w) = \left(r\cos u\cos v, r\cos u\sin v, r\sin u\cos w, r\sin u\sin w\right), \tag{6.1}$$

we have the shape operator $\mathbf{S} = \frac{1}{r}\mathcal{I}_3$, *and find the following curvatures of it:*

$$\mathfrak{C}_0 = 1,\ \mathfrak{C}_1 = H = \frac{1}{r},\ \mathfrak{C}_2 = \frac{1}{r^2},\ \mathfrak{C}_3 = K = \frac{1}{r^3}.$$

Here, $H\mathfrak{C}_2 = K$, $H^2 = \mathfrak{C}_2$, *and* $H^3 = K$, *i.e. the hypersurface* (6.1) *is the birotational umbilical hypersphere.*

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