Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2545 – 2555
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doi:10.55730/1300-0098.3287

On the 2-class group of some number fields of 2-power degree

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Received: 11.01.2022 • **Accepted/Published Online:** 23.05.2022 • **Final Version:** 05.09.2022

Abstract: Let K be an imaginary cyclic quartic number field whose 2-class group is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, and let K^* denote the genus field of K. In this paper, we compute the rank of the 2-class group of K_n^* the n-th layer of the cyclotomic $\mathbb{Z}_2$-extension of K^*.

Key words: Iwasawa theory, cyclotomic $\mathbb{Z}_2$-extension, cyclic quartic field, 2-class group

1. Introduction

Let k be a number field. A cyclotomic $\mathbb{Z}_2$-extension of k is an extension k_∞/k defined by

$$k=k_0\subset k_1=k(\sqrt{\pi_2})\subset\cdots\subset k_n=k(\sqrt{\pi_{n+1}})\subset\cdots\subset k_\infty=\bigcup_{n\geq 0}k_n,$$

where $\pi_2=2$ and $\pi_{n+1}=2+\sqrt{\pi_n}$ for all $n\geq 2$, this sequence of fields is called the cyclotomic Iwasawa tower of k. Note that the Galois group $\mathrm{Gal}(k_\infty/k)$ is isomorphic to $\mathbb{Z}_2$, the additive group of 2-adic integers, and $\mathrm{Gal}(k_n/k)\simeq\mathbb{Z}/2^n\mathbb{Z}$ (see [16, p. 264]). For each integer n, denote by A_n the 2-part of the class group of k_n, the n-th layer of cyclotomic $\mathbb{Z}_2$-extension of k. Let $\lambda,\mu\geq 0$ and ν be the Iwasawa invariants, then the order of A_n is $2^{\lambda n+\mu 2^n+\nu}$, for n large enough, by Iwasawa's theorem [16, Theorem 13.13, p. 276].

Given a number field K, the computation of the rank of the 2-class group of K is one of classical and difficult problems in number theory, especially for the fields of higher degree. However, if K is a quadratic extension of a number field k whose class number is odd, the ambiguous class number formula can be used to compute this rank (see [7]). In the literature, there are many works who dealt with this problem, we cite for example [1, 2] using an arithmetic method based on class field theory described in [13].

Let q and ℓ be two primes satisfying the following conditions:

$$q\equiv 3 \pmod 8,\ \ell\equiv 5 \pmod 8,\ \text{and}\ \left(\frac{q}{\ell}\right)=-1\cdot \tag{1.1}$$

Let $K=\mathbb{Q}\left(\sqrt{-q\varepsilon\sqrt{\ell}}\right)$ be an imaginary cyclic quartic field, where ε denotes the fundamental unit of $\mathbb{Q}(\sqrt{\ell})$. Then the genus field of K is $K^*=K(\sqrt{q},\sqrt{-1})$, thus its n-th layer of the cyclotomic $\mathbb{Z}_2$-extension is

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2010 *AMS Mathematics Subject Classification:* 11R16, 11R18, 11R20, 11R23, 11R29

$K_n^* = K(\sqrt{q}, \zeta_{2^{n+2}})$. Brown and Parry [4, Theorem 3, p. 66] showed that the 2-class group $\mathrm{Cl}_2(K)$ of K is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by using genus theory. In this work, we first determine the structure of the 2-class group of K_1^*. Next, we compute the rank of the 2-class group of K_n^*, for $n \geq 3$, by using Iwasawa theory. Finally, we deduce all possible cases of the rank, for $n = 2$.

Notations

Let k be a number field and n be an integer ≥ 0. The following notations will be used for the rest of this paper:

- ▷ $\mathbb{Q}_n = \mathbb{Q}(\sqrt{\pi_{n+1}})$: the maximal real subfield of $\mathbb{Q}(\zeta_{2^{n+2}})$;
- ▷ k_n: the n-th layer of the $\mathbb{Z}_2$-extension of k;
- ▷ $k_\infty = \bigcup_{n\geq 0} k_n$;
- ▷ A_n: the 2-part of the class group of k_n;
- ▷ $A_\infty = \varprojlim A_n$;
- ▷ τ: a topological generator of $\mathrm{Gal}(k_\infty/k)$;
- ▷ $\Lambda = \mathbb{Z}_2[\![T]\!]$ for $T = \tau - 1$;
- ▷ $\mu(M), \lambda(M)$: the Iwasawa invariants for a Λ-torsion module M;
- ▷ $\mu(k) = \mu(A_\infty)$;
- ▷ $\lambda(k) = \lambda(A_\infty)$;
- ▷ $\lambda^-(k) = \lambda(A_\infty^-)$ (the definition of A_∞^- will be given later);
- ▷ $h(k)$: the class number of k;
- ▷ $h_2(k)$: the 2-class number of k;
- ▷ $\mathcal{O}_k$: the ring of integers of k;
- ▷ E_k: the unit group of k;
- ▷ W_k: the group of roots of unity contained in k;
- ▷ w_k: the order of W_k;
- ▷ k^+: the maximal real subfield of a CM-field k;
- ▷ $Q_k = [E_k : W_k E_{k^+}]$: the Hasse's unit index of a CM-field k;
- ▷ $N_{L/k}$: the relative norm for an extension L/k;
- ▷ $\mathrm{Cl}_2(k)$: the 2-part of the class group of k;
- ▷ $\mathfrak{q}_k$: the prime ideal of k above q;

▷ $\left(\frac{x}{\mathfrak{p}}\right)$: the quadratic residue symbol for k;

▷ $\left(\frac{x,y}{\mathfrak{p}}\right)$: the Hilbert symbol for k;

▷ $\left(\frac{a}{p}\right)$: the quadratic residue (Legendre) symbol.

2. Some preliminary results in Iwasawa theory

In this section, we collect some results in Iwasawa theory that will be used in what follows.

Proposition 2.1 ([6], p. 3) *Let $n \geq 2$ be a positive integer. Then we have*

(1) *If p is a prime such that $p \equiv 3 \pmod 8$, then, p decomposes into the product of 2 prime ideals of $\mathbb{Q}(\zeta_{2^{n+2}})$ while it is inert in $\mathbb{Q}_n$.*

(2) *If p is a prime such that $p \equiv 7 \pmod{16}$, then, p decomposes into the product of 4 prime ideals of $\mathbb{Q}(\zeta_{2^{n+2}})$ while it decomposes into the product of 2 prime ideals of $\mathbb{Q}_n$.*

Recall that A_n^+ is the group of strongly ambiguous classes with respect to the extension k_n/k_n^+, where k_n^+ is the totally real subfield of k_n, and $A_n^- = A_n/A_n^+$. Let A_∞^- denote the projective limit of A_n^-. We have:

Theorem 2.2 ([15], Theorem 2.5, p. 374) *Let k be a CM-field containing the fourth roots of unity. Then there is no finite Λ-submodule in A_∞^-.*

Lemma 2.3 ([8], Lemma 3.4) *If the extension k_n/k_n^+ is unramified and $h(k_n^+)$ is odd, for all $n \geq 0$. Then $A_\infty^- = A_\infty$.*

Theorem 2.4 ([11], Theorem 3, p. 341) *Let L/F a finite 2-extension of abelian CM-fields. Then we have*

$$\lambda^-(L) - \delta(L) = [L_\infty : F_\infty] \cdot (\lambda^-(F) - \delta(F)) + \sum_{\beta \nmid 2}(e_\beta - 1) - \sum_{\beta^+ \nmid 2}(e_{\beta^+} - 1), \tag{2.1}$$

where $\delta(k)$ takes the values 1 or 0 according to whether k_∞ contains the fourth roots of unity or not, and e_β (resp. e_{β^+}) is the ramification index in L_∞/F_∞ (resp. L_∞^+/F_∞^+) of a finite prime β of L_∞ (resp. β^+ of L_∞^+).

Theorem 2.5 ([5], Theorem 3.3, p. 8) *Let k_∞ be a $\mathbb{Z}_2$-extension of a number field k and assume that any prime of k lying above 2 is totally ramified in k_∞/k. If $\mu(k) = 0$ and A_∞ is an elementary Λ-module, then $rank_2(A_n) = \lambda(k)$ for all $n \geq \lambda(k)$.*

Proposition 2.6 ([16], Proposition 13.22, p. 284) *Let k_∞ be a $\mathbb{Z}_2$-extension of a number field k and assume that there exists only one prime of k lying above 2 and that this prime is totally ramified in k_∞/k. Then*

$$2 \nmid h(k) \Longleftrightarrow 2 \nmid h(k_n), \text{ for all } n \geq 0.$$

3. The 2-class group of K_1^*

In all this section, we assume that q and ℓ are two primes satisfying the conditions (1.1). This section is one of the steps of the proof of Theorem 4.2, in which we shall determine the structure of the 2-class group of $K_1^* = K(\sqrt{q}, \zeta_8)$ of degree 32 over $\mathbb{Q}$.

Proposition 3.1 ([2], Proposition 5.1, p. 270) *Consider $M = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{-1}\right)$. Then the class number $h(M^+)$ of M^+ is odd. Moreover, Q_M the Hasse's unit index of M equals 2 and $h(M)$ is odd too.*

Proposition 3.2 *The class number of $F = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{q}\right)$ is odd.*

Proof Put $L = \mathbb{Q}(\sqrt{\varepsilon\sqrt{l}})$. So from Proposition 3.1, the class number of L is odd. Then, by [7, p. 25], the 2-rank of class group $\mathrm{Cl}(F)$ of F is given by the following formula:

$$\mathrm{rank}_2(\mathrm{Cl}(F)) = t - 1 - e,$$

where t is the number of primes of L which ramify in F and the integer e is defined by $2^e = [E_L : E_L \cap N_{F/L}(F^\times)]$. There exists only one prime of L lying above q which ramifies in F (i.e. $t = 1$), because $\left(\frac{\varepsilon\sqrt{\ell}}{\mathfrak{q}}\right) = \left(\frac{\ell}{q}\right) = -1$ where $\mathfrak{q}$ is a prime ideal of $\mathbb{Q}(\sqrt{\ell})$ dividing q. Therefore $\mathrm{rank}_2(\mathrm{Cl}(F)) = -e$ must be equal to 0. So $h(F)$ is odd. □

For the rest of this section, we need the following proposition:

Proposition 3.3 ([12], p. 355) *Let K/k be a V_4-extension of CM-fields; let K', K'' and K^+ be its three quadratic subfields. Then*

$$h(K) = \frac{Q_K}{Q_{K'}Q_{K''}} \cdot \frac{w_K}{w_{K'}w_{K''}} \cdot \frac{h(K')h(K'')h(K^+)}{h(k)^2}.$$

Lemma 3.4 *The 2-class number of $\mathbb{K} = \mathbb{Q}(\sqrt{\ell}, \sqrt{-2q})$ is equal to 4.*

Proof Consider Figure 1 below:

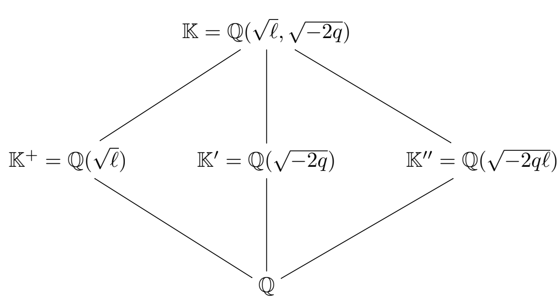

Figure 1. Subextensions of $\mathbb{K}/\mathbb{Q}$.

So $\mathbb{K}/\mathbb{Q}$ is a V_4-extension of CM-fields, of quadratic subextensions $\mathbb{K}'$, $\mathbb{K}''$ and $\mathbb{K}^+$. Then by Proposition 3.3, we have

$$\begin{aligned} h_2(\mathbb{K}) &= \frac{Q_{\mathbb{K}}}{Q_{\mathbb{K}'} \cdot Q_{\mathbb{K}''}} \cdot \frac{w_{\mathbb{K}}}{w_{\mathbb{K}'} \cdot w_{\mathbb{K}''}} \cdot \frac{h_2(\mathbb{K}') \cdot h_2(\mathbb{K}'') \cdot h_2(\mathbb{K}^+)}{h_2(\mathbb{Q})^2} \\ &= \frac{1}{1 \cdot 1} \cdot \frac{2}{2 \cdot 2} \cdot \frac{2 \cdot 4 \cdot 1}{1} \\ &= 4, \end{aligned}$$

where $h_2(\mathbb{K}') = 2$ (cf. Kaplan [9]), $h_2(\mathbb{K}'') = 4$ and $h_2(\mathbb{K}^+) = 1$ (cf. Kaplan [10]); and where $Q_{\mathbb{K}} = Q_{\mathbb{K}'} = Q_{\mathbb{K}''} = 1$ (using [12, Theorem 1]), because $\mathbb{K}/\mathbb{K}^+$, $\mathbb{K}'/\mathbb{Q}$ and $\mathbb{K}''/\mathbb{Q}$ are essentially ramified. □

Lemma 3.5 *The* 2*-class number of* $\mathbb{L} = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{-2q}\right)$ *is equal to* 8*.*

Proof Here, consider Figure 2 below:

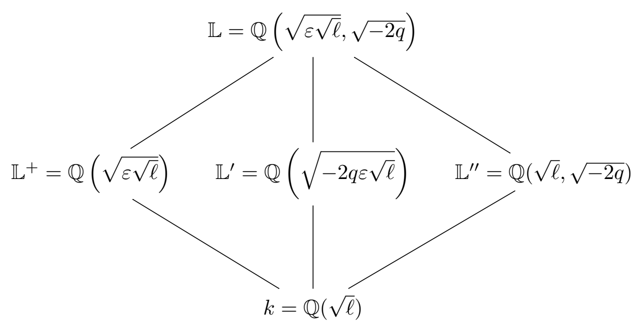

Figure 2. Subextensions of $\mathbb{L}/k$.

So $\mathbb{L}/k$ is a V_4-extension of CM-fields, of quadratic subextensions $\mathbb{L}'$, $\mathbb{L}''$ and $\mathbb{L}^+$. Then by Proposition 3.3, we have

$$\begin{aligned} h_2(\mathbb{L}) &= \frac{Q_{\mathbb{L}}}{Q_{\mathbb{L}'} \cdot Q_{\mathbb{L}''}} \cdot \frac{w_{\mathbb{L}}}{w_{\mathbb{L}'} \cdot w_{\mathbb{L}''}} \cdot \frac{h_2(\mathbb{L}') \cdot h_2(\mathbb{L}'') \cdot h_2(\mathbb{L}^+)}{h_2(k)^2} \\ &= \frac{1}{1 \cdot 1} \cdot \frac{2}{2 \cdot 2} \cdot \frac{4 \cdot 4 \cdot 1}{1} \\ &= 8, \end{aligned}$$

where $h_2(\mathbb{L}') = 4$ (cf. Brown and Parry [4]), $h_2(\mathbb{L}'') = 4$ (by Lemma 3.4) and $h_2(\mathbb{L}^+) = 1$ (by Proposition 3.1); and where $Q_{\mathbb{L}} = Q_{\mathbb{L}'} = Q_{\mathbb{L}''} = 1$ (using [12, Theorem 1]), because $\mathbb{L}/\mathbb{L}^+$, $\mathbb{L}'/k$ and $\mathbb{L}''/k$ are essentially ramified. □

Lemma 3.6 *The* 2*-class number of* $\mathbb{F} = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{-q}, \sqrt{2}\right)$ *is equal to* 8*.*

Proof Now, consider Figure 3 below:

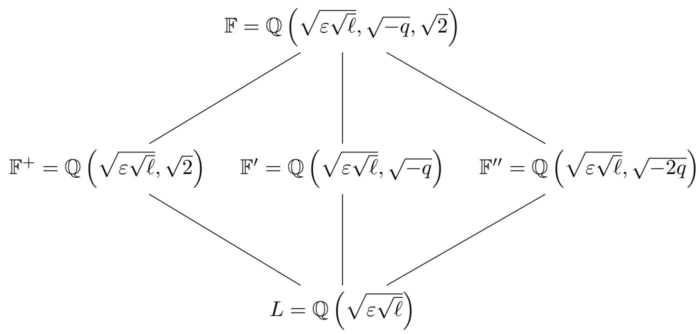

Figure 3. Subextensions of $\mathbb{F}/L$.

So $\mathbb{F}/L$ is a V_4-extension of CM-fields, of quadratic subextensions $\mathbb{F}'$, $\mathbb{F}''$ and $\mathbb{F}^+$. Then by Proposition 3.3, we have

$$\begin{aligned} h_2(\mathbb{F}) &= \frac{Q_{\mathbb{F}}}{Q_{\mathbb{F}'} \cdot Q_{\mathbb{F}''}} \cdot \frac{w_{\mathbb{F}}}{w_{\mathbb{F}'} \cdot w_{\mathbb{F}''}} \cdot \frac{h_2(\mathbb{F}') \cdot h_2(\mathbb{F}'') \cdot h_2(\mathbb{F}^+)}{h_2(L)^2} \\ &= \frac{1}{1 \cdot 1} \cdot \frac{w_{\mathbb{F}}}{w_{\mathbb{F}'} \cdot 2} \cdot \frac{2 \cdot 8 \cdot 1}{1} \\ &= 8, \end{aligned}$$

where $h_2(\mathbb{F}') = 2$ (cf. [3, Theorem 5.19]), $h_2(\mathbb{F}'') = 8$ (by Lemma 3.5) and $h_2(\mathbb{F}^+) = 1$ (cf. [8, Proposition 4.2]); and where $Q_{\mathbb{F}} = Q_{\mathbb{F}'} = Q_{\mathbb{F}''} = 1$ (using [12, Theorem 1]), because $\mathbb{F}/\mathbb{F}^+$, $\mathbb{F}'/L$ and $\mathbb{F}''/L$ are essentially ramified. Moreover, $w_{\mathbb{F}} = w_{\mathbb{F}'}$ and $w_{\mathbb{F}''} = 2$. □

Proposition 3.7 *Let $\mathfrak{q}$ be the prime ideal of $\mathbb{F}$ above q, then $\mathfrak{q}$ is not principal in $\mathbb{F}$.*

Proof Keep the notations of the previous proof. Assume that $\mathfrak{q}$ is principal in $\mathbb{F}$, then there exists $\delta \in \mathbb{F}$ such that $\mathfrak{q} = \delta\mathcal{O}_{\mathbb{F}}$, this implies that $\mathfrak{q}^2 = \delta^2\mathcal{O}_{\mathbb{F}} = \pi\mathcal{O}_{\mathbb{F}}$ where $\pi \in \mathbb{F}^+$ (because $\mathfrak{q}_{\mathbb{F}^+} = \pi\mathcal{O}_{\mathbb{F}^+}$ and $\mathfrak{q}_{\mathbb{F}^+}\mathcal{O}_{\mathbb{F}} = \mathfrak{q}^2$), therefore there exists ε' a unit of $\mathbb{F}$ such that $\delta^2 = \varepsilon'\pi$. Since $Q_{\mathbb{F}} = 1$, we have two cases to discuss:
<u>First case:</u> If $\varepsilon' \in E_{\mathbb{F}^+}$, then $\varepsilon'\pi = (a + b\sqrt{-q})^2 = a^2 - qb^2 + 2ab\sqrt{-q}$ with a and b in $\mathbb{F}^+$, thus $a = 0$ or $b = 0$ because $\varepsilon'\pi \in \mathbb{F}^+$.

(1) If $a = 0$, then $\varepsilon'\pi = -qb^2$, applying the norm in $\mathbb{F}^+/L$, we get $N_{\mathbb{F}^+/L}(\varepsilon')q = q^2 N_{\mathbb{F}^+/L}(b)^2$, this yields that $\beta = qN_{\mathbb{F}^+/L}(b)^2$ where β denotes $N_{\mathbb{F}^+/L}(\varepsilon')$. We know that $\beta \in E_L$, then we have:

   (a) If $\beta \in \{-1, 1\}$, then $\sqrt{\pm q} \in L$ which is absurd.

   (b) If $\beta = \varepsilon$, then εq is a square in L, this means that there exist $x, y \in \mathbb{Q}(\sqrt{\ell})$ such that $\varepsilon q = (x + y\sqrt{\varepsilon\sqrt{\ell}})^2 = x^2 + y^2\varepsilon\sqrt{\ell} + 2xy\sqrt{\varepsilon\sqrt{\ell}}$, therefore $x = 0$ or $y = 0$ because $\varepsilon q \in \mathbb{Q}(\sqrt{\ell})$. If $x = 0$, then $q = y^2\sqrt{\ell}$, applying the norm in $\mathbb{Q}(\sqrt{\ell})/\mathbb{Q}$, we get $\sqrt{-\ell} \in \mathbb{Q}$ which is absurd. Similarly, if $y = 0$, we find that $\sqrt{-1} \in \mathbb{Q}$ which is absurd.

   (c) If $\beta \notin \{-1, 1, \varepsilon\}$, by applying the norm in $L/\mathbb{Q}(\sqrt{\ell})$, we get $\sqrt{-1} \in \mathbb{Q}(\sqrt{\ell})$ which is absurd.

(2) If $b=0$, with the same argument as for $a=0$, we find a contradiction.

Second case: If $\varepsilon' \in W_{\mathbb{F}}$ and $q=3$, (i.e. $\varepsilon' = j$ where j is a root of the polynomial X^2+X+1), we have $j\pi = \delta^2$, applying the norm in $\mathbb{F}/\mathbb{F}'$, we find that $j^2 q = N_{\mathbb{F}/\mathbb{F}'}(\delta)^2$, this implies that $\sqrt{q} \in \mathbb{F}'$ which is absurd. □

Proposition 3.8 *The class number of* $\mathbb{H}^\alpha = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{\alpha}, \sqrt{2}\right)$ *is odd, where* $\alpha \in \{-1, q\}$.

Proof Put $H^\alpha = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{\alpha}\right)$. We first need to count the number of primes of H^α above 2 ramifying in $\mathbb{H}^\alpha$. For this, let $\mathfrak{2}_L$ be a unique prime ideal of $L = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}\right)$ lying above 2, then $\mathfrak{2}_L$ remains inert in H^α. In fact, we can write $H^\alpha = L\left(\sqrt{\alpha\varepsilon\sqrt{\ell}}\right)$ and we find

$$\begin{aligned}
\left(\frac{\alpha\varepsilon\sqrt{\ell}}{\mathfrak{2}_L}\right) &= \left(\frac{\alpha\varepsilon\sqrt{\ell}}{\mathfrak{2}_{\mathbb{Q}(\sqrt{\ell})}}\right) \\
&= \left(\frac{\alpha\varepsilon\sqrt{\ell}, 2}{\mathfrak{2}_{\mathbb{Q}(\sqrt{\ell})}}\right) \\
&= \left(\frac{N_{\mathbb{Q}(\sqrt{\ell})/\mathbb{Q}}(\alpha\varepsilon\sqrt{\ell}), 2}{2}\right) \\
&= \left(\frac{\alpha^2\ell, 2}{2}\right) \\
&= \left(\frac{\ell, 2}{2}\right) \\
&= \left(\frac{2}{\ell}\right) \qquad \text{(cf. [14, Lemma 2.27, p. 63])} \\
&= -1.
\end{aligned}$$

Moreover, the prime ideal $\mathfrak{2}_{H^\alpha}$ of H^α dividing $\mathfrak{2}_L$ ramifies in $\mathbb{H}^\alpha$ which is the first layer of the cyclotomic $\mathbb{Z}_2$-extension of H^α, because $\mathfrak{2}_L$ ramifies in $L(\sqrt{2})$. So we conclude that there exists only one prime of H^α lying above 2 which is totally ramified in H^α_∞. On the other hand, Propositions 3.1 and 3.2 show that $h(H^\alpha)$ is odd, thus $h(\mathbb{H}^\alpha)$ is odd using Proposition 2.6. □

Corollary 3.9 *The class number of* H_n^α *is odd, for all* n.

Proposition 3.10 *The* 2*-class number of* $K_1^* = K(\sqrt{q}, \zeta_8)$ *is equal to* 4.

Proof Keep the previous notations. We can write $K_1^* = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{q}, \sqrt{-1}, \sqrt{2}\right)$, so we can regard $K_1^*/\mathbb{F}^+$ as a V_4-extension of CM-fields, of quadratic subextensions $\mathbb{F}$, $\mathbb{H}^{-1}$ and $\mathbb{H}^q$. Then by Proposition 3.3, we have

$$\begin{aligned}
h_2(K_1^*) &= \frac{Q_{K_1^*}}{Q_{\mathbb{F}} \cdot Q_{\mathbb{H}^{-1}}} \cdot \frac{w_{K_1^*}}{w_{\mathbb{F}} \cdot w_{\mathbb{H}^{-1}}} \cdot \frac{h_2(\mathbb{H}^{-1}) \cdot h_2(\mathbb{H}^q) \cdot h_2(\mathbb{F})}{h_2(\mathbb{F}^+)^2} \\
&= \frac{2}{1 \cdot 2} \cdot \frac{w_{K_1^*}}{2 \cdot w_{K_1^*}} \cdot \frac{1 \cdot 1 \cdot 8}{1} \\
&= 4.
\end{aligned}$$

In fact; we have $Q_{\mathbb{F}} = 1$ because $\mathbb{F}/\mathbb{F}^+$ is essentially ramified, and $Q_{K_1^*} = Q_{\mathbb{H}^{-1}} = 2$ because $K_1^*/\mathbb{H}^q$ and $\mathbb{H}^{-1}/\mathbb{F}^+$ are not essentially ramified and $h_2(\mathbb{H}^q) = h_2(\mathbb{F}^+) = 1$ (using [12, Theorem 1]). Moreover, $w_{\mathbb{F}} \cdot w_{\mathbb{H}^{-1}} = 2 \cdot w_{K_1^*}$, and we have $h_2(\mathbb{F}) = 8$ (by Lemma 3.6) and $h_2(\mathbb{H}^{-1}) = h_2(\mathbb{H}^q) = 1$ (by Proposition 3.8). □

Proposition 3.11 *The 2-class group of K_1^* is cyclic of order 4.*

Proof Let $\mathcal{Q}$ be the prime ideal of $\mathbb{H}^{-1}$ above q, then $v_{\mathcal{Q}}(q) = 1$. Since $K_1^* = \mathbb{H}^{-1}(\sqrt{-q})$, so we have

$$\begin{aligned}\left(\frac{-q, q}{\mathcal{Q}}\right) &= \left(\frac{-1, q}{\mathcal{Q}}\right) \cdot \left(\frac{q, q}{\mathcal{Q}}\right) \\ &= \left(\frac{-1}{\mathcal{Q}}\right)^{v_{\mathcal{Q}}(q)} \cdot \left(\frac{-1}{\mathcal{Q}}\right) \\ &= \left(\frac{-1}{\mathcal{Q}}\right)^{2} \\ &= 1.\end{aligned}$$

By genus theory, the ideal class of $\mathfrak{q}_{K_1^*}$ is a square in K_1^*, hence there exists an ideal I of K_1^* such that $\mathfrak{q}_{K_1^*} \sim I^2$, then the ideal class $[I]$ is of order 4. In fact, if $I^2 \sim 1$ then $\mathfrak{q}_{K_1^*} \sim 1$, applying the norm in $K_1^*/\mathbb{F}$, we get $\mathfrak{q}_{\mathbb{F}} \sim 1$, and this contradicts Proposition 3.7. Finally, we conclude that the 2-class group of K_1^* is cyclic of order 4 generated by the ideal class $[I]$. □

4. Main results

Let q and ℓ be two primes satisfying the conditions (1.1). Let A_n denote the 2-class group of the n-th layer of the cyclotomic $\mathbb{Z}_2$-extension of the genus field $K^* = K(\sqrt{q}, \sqrt{-1})$. The main results of this paper are the following:

Theorem 4.1 *The Iwasawa module A_∞ is isomorphic to $\mathbb{Z}_2^3$.*

Proof Let H and F denote $\mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{-q}\right)$ and $\mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{-1}\right)$, respectively; and let L denote K^* which coincides with $\mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{q}, \sqrt{-1}\right)$. Consider Figure 4 below:

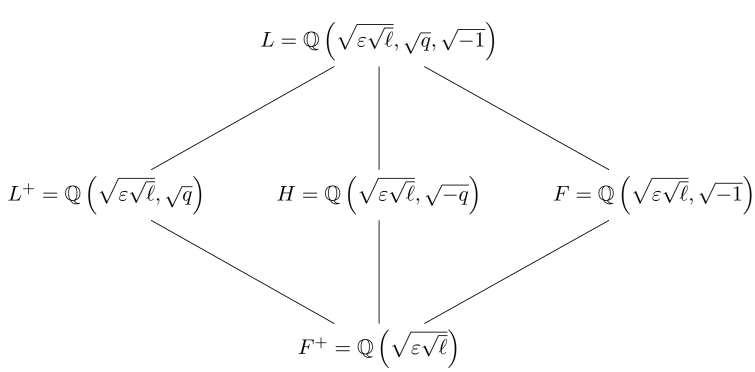

Figure 4. Subextensions of L/F^+.

By Corollary 3.9, the class number of F_n is odd, this implies that $\lambda(F)=0$, therefore $\lambda^-(F)=0$. On the other hand, we have q splits into 2 prime ideals of F. In fact, let $\mathfrak{q}$ be the unique prime ideal of F^+ lying above q, so we have:

$$\left(\frac{-1}{\mathfrak{q}}\right) = \left(\frac{N_{F^+/\mathbb{Q}(\sqrt{\ell})}(-1)}{\mathfrak{q}_{\mathbb{Q}(\sqrt{\ell})}}\right) = 1.$$

Since $q \equiv 3 \pmod 8$, by Proposition 2.1, we have q splits into 2 primes of $\mathbb{Q}(\zeta_{16})$ and it is inert in $\mathbb{Q}_2 = \mathbb{Q}(\sqrt{\pi_3})$, then q splits into the product of 8 primes in $F_2 = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \zeta_{16}\right)$ because it splits into 4 primes in $F_2^+ = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{\pi_3}\right)$. Thus q splits into 8 primes in $F_n = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \zeta_{2^{n+2}}\right)$ while it decomposes into 4 primes in $F_n^+ = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{\ell}}, \sqrt{\pi_{n+1}}\right)$, for all $n \geq 2$. Note that $[L_\infty : F_\infty] = [L_\infty^+ : F_\infty^+] = 2$ and $e_\beta = e_{\beta^+} = 2$, then by Theorem 2.4, we have

$$\lambda^-(L) - 1 = 2\cdot(0-1) + 8 - 4,$$

thus

$$\lambda^-(L) = 3.$$

By Corollary 3.9, we have the class number of L_n^+ is odd, this means that $\lambda(L^+) = 0$, therefore $\lambda^+(L) = \lambda(L^+) = 0$. Then,

$$\lambda(L) = \lambda^+(L) + \lambda^-(L) = 3.$$

Since the extension L_n/L_n^+ is unramified and $h(L_n^+)$ is odd, for all $n \geq 0$, then by Lemma 2.3, $A_\infty^- = A_\infty$. By Theorem 2.2 there is no finite Λ-submodule in A_∞^-, hence A_∞ is a Λ-module without finite part. So,

$$A_\infty \simeq \mathbb{Z}_2^3.$$

□

Theorem 4.2 *The structure of A_n is given by:*

$$A_n \simeq \begin{cases} 0 & \textit{for } n=0, \\ \mathbb{Z}/4\mathbb{Z} & \textit{for } n=1, \\ \mathbb{Z}/2^a\mathbb{Z}\times\mathbb{Z}/2^b\mathbb{Z} \quad \textit{or} \quad \mathbb{Z}/2^a\mathbb{Z}\times\mathbb{Z}/2^b\mathbb{Z}\times\mathbb{Z}/2^c\mathbb{Z} & \textit{for } n=2, \\ \mathbb{Z}/2^{a_n}\mathbb{Z}\times\mathbb{Z}/2^{b_n}\mathbb{Z}\times\mathbb{Z}/2^{c_n}\mathbb{Z} & \textit{for all } n\geq 3, \end{cases}$$

where $\{a, b, c, a_n, b_n, c_n\} \subset \mathbb{N}^*$.

Proof Keep the notations we have introduced in the previous proof. By Theorem 4.1, we have

$$A_\infty \simeq \mathbb{Z}_2^{\lambda(L)} \simeq \bigoplus_j \Lambda/(g_j(T)),$$

where each g_j is distinguished and $\sum_j \deg g_j = \lambda(L)$, this shows that A_∞ is an elementary Λ-module. Moreover, we have $L/\mathbb{Q}$ is an abelian extension, i.e. $\mu(L)=0$. Then, by Theorem 2.5,

$$\operatorname{rank}_2(A_n) = 3, \quad \text{for all } n \geq 3,$$

this implies that

$$A_n \simeq \mathbb{Z}/2^{a_n}\mathbb{Z} \times \mathbb{Z}/2^{b_n}\mathbb{Z} \times \mathbb{Z}/2^{c_n}\mathbb{Z}, \quad \text{for all } n \geq 3.$$

Since the 2-class group of K_1^* is cyclic of order 4 by Proposition 3.11, then

$$A_1 \simeq \mathbb{Z}/4\mathbb{Z}.$$

So we conclude all possible structures of A_2 which are:

$$A_2 \simeq \mathbb{Z}/2^a\mathbb{Z} \times \mathbb{Z}/2^b\mathbb{Z} \quad \text{or} \quad A_2 \simeq \mathbb{Z}/2^a\mathbb{Z} \times \mathbb{Z}/2^b\mathbb{Z} \times \mathbb{Z}/2^c\mathbb{Z}.$$

Finally, we know that the class number of K^* is odd by [3, Theorem 5.19], then

$$A_0 \simeq 0.$$

□

The following example was computed using PARI/GP:

Example 4.3 *Let* $K = \mathbb{Q}\left(\sqrt{-3\varepsilon\sqrt{5}}\right)$ *where* $\varepsilon = \frac{1+\sqrt{5}}{2}$. *Since* $5 \equiv 5 \pmod 8$, $3 \equiv 3 \pmod 8$ *and* $\left(\frac{3}{5}\right) = -1$, *we have*

$$A_n \simeq \begin{cases} 0 & \text{for } n = 0, \\ \mathbb{Z}/4\mathbb{Z} & \text{for } n = 1, \\ \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{for } n = 2. \end{cases}$$

Moreover, $A_\infty \simeq \mathbb{Z}_2^3$ *where* A_∞ *is attached to* K^*.

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