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A characterization of abelian group codes in terms of their parameters

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Abstract: In 1979, Miller proved that for a group G of odd order, two minimal group codes in $\mathbb{F}_2G$ are G-equivalent if and only if they have identical weight distribution. In 2014, Ferraz-Guerreiro-Polcino Milies disproved Miller's result by giving an example of two non-G-equivalent minimal codes with identical weight distribution. In this paper, we give a characterization of finite abelian groups so that over a specific set of group codes, equality of important parameters of two codes implies the G-equivalence of these two codes. As a corollary, we prove that two minimal codes with the same weight distribution are G-equivalent if and only if for each prime divisor p of $|G|$, the Sylow p-subgroup of G is homocyclic.

Key words: Abelian group codes, weight distribution, G-equivalence, homocyclic group

1. Introduction

An abelian code over a field is an ideal in a finite group algebra of an abelian group. This definition is given by Berman [2] and MacWilliams [9]. More generally, **a group code** is an ideal in a finite group algebra. One can easily show that cyclic codes are ideals in finite group algebras of cyclic groups. Reed-Muller codes are group codes for elementary abelian p-groups (see [3]). There are many important linear codes which can be viewed as group codes [8]. It is proved in [4] that group codes are asymptotically good over any field. Besides this, group codes have more algebraic structures than linear codes. Because of all these, they are of interests for many researchers.

The weight, weight distribution and dimension of codes carry important information about the error correcting capacity of them. Let G be an abelian group and $\mathbb{F}$ a finite field of characteristic coprime to the order of G. For an element $\alpha \in \mathbb{F}G$, the **support of** α is the set $\mathrm{supp}(\alpha) = \{g \in G| \ \alpha_g \neq 0\}$ and the **weight of** α is $w(\alpha) = |\,\mathrm{supp}(\alpha)\,|$. If I is an ideal in $\mathbb{F}G$, then the **weight** of I is $w(I) = \min\{w(\alpha)| \ \alpha \in I, \alpha \neq 0\}$. Under some equivalence relations, these three parameters are preserved. One of these equivalences is G-equivalence.

As defined in [11], two abelian codes I and J in $\mathbb{F}G$ are **G-equivalent** if there is a group automorphism $\varphi : G \to G$ whose linear extension to the group algebra maps I onto J. If two codes are G-equivalent, then they have the same weight, the identical weight distribution and the same dimension.

In [10], MacWilliams shows that for a cyclic group G of odd order, if two binary minimal codes have identical weight distribution, then these codes are G-equivalent. In [11] (see Theorem 3.9), it is proved that for

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an abelian group of odd order, two minimal abelian codes in $\mathbb{F}_2G$ are G-equivalent if and only if they have the identical weight distribution. However, in [6] (see Proposition IV.2), the authors show that for $\mathbb{F}_2(C_9 \times C_3)$, there are non-$G$-equivalent minimal codes having the identical weight distribution. Now it is natural to ask for which abelian groups and fields, equality of weight distribution implies the G-equivalence of codes.

In this paper, by this motivation, for a semisimple finite abelian group algebra $\mathbb{F}_qG$, we are concerned with the following conditions and characterize finite abelian groups satisfying these conditions.

Let $\mathcal{I}$ be an arbitrary set of codes in $\mathbb{F}_qG$.

Condition A: Let I_1, I_2 be in $\mathcal{I}$. I_1 is G-equivalent to I_2 if and only if I_1 and I_2 have the same weight.

Condition B: Let I_1 and I_2 be in $\mathcal{I}$. I_1 is G-equivalent to I_2 if and only if I_1 and I_2 have the identical weight distribution.

Besides the weight and the weight distribution of codes, we also consider dimension of codes.

Condition C: Let I_1 and I_2 be in $\mathcal{I}$. I_1 is G-equivalent to I_2 if and only if I_1 and I_2 have the same dimension.

Note that the forward direction of the conditions above always holds, because G-equivalence preserves the important parameters such as dimension, weight and weight distribution of the codes.

In this paper, for a specific family of ideals in $\mathbb{F}_qG$, we characterize finite abelian groups for which Conditions A, B, C holds. The structure of the paper is as follows. In Section 2, we give some needed material for the subject. In Section 3, we concentrate on the problem for p-groups. In the last part, we consider the problem for the composite groups.

2. Preliminaries

Let G be a finite abelian group and $\mathbb{F}_q$ a field such that $(q, |G|) = 1$. For an arbitrary subgroup $H \leq G$, the element

$$\widehat{H} = \frac{1}{|H|} \sum_{h \in H} h \in \mathbb{F}_qG$$

is an idempotent element.

We say that $H \leq G$ is **cocyclic** if $H \neq G$ and G/H is a cyclic group. When G is an abelian p-group and H is a cocyclic subgroup of G, there is a unique subgroup $H^* \leq G$ containing H with $H^*/H \cong C_p$.

Suppose that G is an abelian p-group. For a cocyclic subgroup $H < G$, there is a corresponding idempotent $e_H = \widehat{H} - \widehat{H^*}$. When $H = G$, set $e_G = \widehat{G}$. When G is an abelian composite group, set $G = G_{p_1} \times \ldots \times G_{p_k}$ where G_{p_i} is a Sylow p_i-subgroup of G for any i. Any cocyclic subgroup H of G can be written as

$$H = H_{p_1} \times \ldots \times H_{p_k},$$

where for any i, either H_{p_i} is a cocyclic subgroup of G_{p_i} or $H_{p_i} = G_{p_i}$. Then for such H,

$$e_H = e_{H_{p_1}} \ldots e_{H_{p_k}}$$

is the corresponding idempotent. For $H = G$, set $e_G = \widehat{G}$. These idempotents are defined in [6]. Let $\mathcal{S}_{cc}(G)$ denote the set of all cocyclic subgroups of G. By Proposition II.6 and Lemma II.7 in [6], the set $\{e_H |\ H \in \mathcal{S}_{cc}(G)\} \cup \{\widehat{G}\}$ is a set of orthogonal idempotents in $\mathbb{F}_qG$ and we have $\sum_{H \in \mathcal{S}_{cc}(G)} e_H + \widehat{G} = 1$.

For any finite abelian group G and for a cocyclic subgroup $H < G$, set $I_H = (\mathbb{F}_q G)e_H$, $I_G = (\mathbb{F}_q G)\widehat{G}$ and

$$\mathcal{I}_{\mathbb{F}_q G} = \{I_H \mid H < G \text{ is cocyclic or } H = G\}.$$

Let p be a prime integer, G a finite abelian group of exponent p^n and $\mathbb{F}_q$ a finite field so that $\mathbb{F}_q G$ is semisimple. If q is a generator of the unit group $U(\mathbb{Z}_{p^n})$, then $\mathcal{I}_{\mathbb{F}_q G}$ is the set of minimal ideals in $\mathbb{F}_q G$ (see Theorem 4.1 in [7]).

For any finite abelian group G, it is easy to see that $\{g\widehat{H} \mid gH \in G/H\}$ is a basis for $(\mathbb{F}_q G)\widehat{H}$ over $\mathbb{F}_q$. Recall that from Proposition 3.6.7 of [12] we have a ring isomorphism

$$\varphi_H : (\mathbb{F}_q G)\widehat{H} \to \mathbb{F}_q(G/H),$$

where φ_H is defined to be the linear extension of the map $g\widehat{H} \mapsto gH$.

We begin with some simple observations.

Lemma 2.1 *Let G be a finite group and let T be a normal subgroup of G. If $\alpha \in (\mathbb{F}_q G)\widehat{T}$ then $w(\alpha) = |T|\ w(\varphi_T(\alpha))$. Moreover, if $I \subseteq (\mathbb{F}_q G)\widehat{T}$ then $w(I) = |T|\ w(\varphi_T(I))$.*

Proof Since $\mathcal{B} = \{g\widehat{T} \mid gT \in G/T\}$ is a basis for $(\mathbb{F}_q G)\widehat{T}$, α can be written as follows:

$$\alpha = \sum_{g\widehat{T} \in \mathcal{B}} \alpha_{gT}\ g\widehat{T}.$$

Also, different elements in $\mathcal{B}$ have disjoint supports, so we have that

$$w(\alpha) = k|T|,$$

where k is the number of nonzero α_{gT} in the above presentation of α. On the other hand,

$$\varphi_T(\alpha) = \sum_{g\widehat{T} \in \mathcal{B}} \alpha_{gT}\ gT \in \mathbb{F}_q(G/T)$$

has weight equal to k. Hence, the claim follows. □

Recall that two subgroups H and K are called G**-isomorphic** if there is an automorphism θ of G such that $\theta(H) = K$.

Lemma 2.2 *Let G be a finite abelian group. Then two codes I_H and I_K in $\mathcal{I}_{\mathbb{F}_q G}$ are G-equivalent if and only if H and K are G-isomorphic.*

Proof If H and K are G-isomorphic, then by definition there exists $\theta \in \mathrm{Aut}(G)$ such that $\theta(H) = K$. We will also use θ to denote its extension to $\mathbb{F}_q G$. By Lemma III.1 in [6], we have that for the extension of θ on the group algebras,

$$\theta(I_H) = I_{\theta(H)} = I_K$$

holds. That is I_H is G-equivalent to I_K.

Conversely, assume that $I_H = (\mathbb{F}_q G)e_H$ and $I_K = (\mathbb{F}_q G)e_K$ are G-equivalent. This implies that there exists an automorphism θ of G such that $\theta(I_H) = I_K$. In this case, either $I_H = (\mathbb{F}_q G)\widehat{G} = I_K$ or both I_H and I_K are different from $(\mathbb{F}_q G)\widehat{G}$. By Lemma III.1 in [6] we have that

$$\theta(I_H) = (\mathbb{F}_q G)e_{\theta(H)} = I_K = (\mathbb{F}_q G)e_K.$$

Then we have $e_K = e_K e_{\theta(H)} = e_{\theta(H)}$ as $e_{\theta(H)}$ and e_K are identity elements of $(\mathbb{F}_q G)e_K$. Since we have $e_K = e_{\theta(H)}$, it follows that $\theta(H) = K$. □

The following result will be used frequently in the paper.

Proposition 2.3 (Proposition 1.1 of [1]) *Let G be a finite group and let H, K be cocyclic subgroups of G. Then H and K are G-isomorphic if and only if they are isomorphic.*

Hence, the number of non-G-equivalent codes in $\mathcal{I}_{\mathbb{F}_q G}$ is equal to the number of nonisomorphic subgroups in $\mathcal{S}_{cc}(G) \cup \{G\}$. Letting $\tau(G)$ denote the number of divisors of the exponent of G, it is easy to see that the number of nonisomorphic subgroups in $\mathcal{S}_{cc}(G) \cup \{G\}$ is at least $\tau(G)$. Hence, Theorem 1.7 in [1] can be written as follows.

Theorem 2.4 *[Theorem 1.7 [1]] Let G be a finite abelian group. The number of nonisomorphic subgroups in $\mathcal{S}_{cc}(G) \cup \{G\}$ is equal to the number of divisors of the exponent of G if and only if for each prime p dividing the order of G, the Sylow p-subgroups of G are homocyclic.*

Recall that a **homocyclic group** is a direct product of pairwise isomorphic cyclic groups. Theorem 1.7 in [1] shows that homocyclic p-groups play an important role in determining abelian groups having the least possible number of non-G-equivalent minimal abelian codes. Theorem 2.4 implies the following result.

Corollary 2.5 ***(i)*** *Let G be a finite abelian group. If one of Sylow p-subgroups of G is not homocyclic, then there exist two cocyclic subgroups H and K such that H is not isomorphic to K and $|H| = |K|$.*

(ii) *Suppose that Sylow p-subgroups of G are homocyclic and H, K are cocyclic subgroups of G, then H is isomorphic to K if and only if $|H| = |K|$.*

3. The case for p-groups

Theorem 3.1 *Let p be an odd prime and let q be a prime power where $(q, p) = 1$. Assume that G is an abelian p-group. Then the following are equivalent.*

(i) *Condition A holds for $\mathcal{I}_{\mathbb{F}_q G}$;*

(ii) *Condition B holds for $\mathcal{I}_{\mathbb{F}_q G}$;*

(iii) *G is homocyclic.*

Proof **(i) ⇒ (ii)** Assume Condition A holds for $\mathcal{I}_{\mathbb{F}_q G}$. We need to prove that if weight distributions of $I_{H_1}, I_{H_2} \in \mathcal{I}_{\mathbb{F}_q G}$ are identical, then I_{H_1} and I_{H_2} are G-equivalent. Assume that I_{H_1} is not G-equivalent to I_{H_2}. As Condition A holds we have $w(I_{H_1}) \neq w(I_{H_2})$. This means their weight distributions are different which gives a contradiction.

(ii) ⇒ (iii) Suppose that Condition B holds for $\mathcal{I}_{\mathbb{F}_qG}$. Suppose to the contrary that G is not homocyclic. Then by Corollary 2.5 (i) there exists two nonisomorphic cocyclic subgroups H_1 and H_2 of G such that $|H_1| = |H_2|$. So we have that $G/H_1 \cong G/H_2 \cong C_{p^r}$ for some $r \geq 1$. By Proposition 2.3, H_1 and H_2 are not G-isomorphic. Then by Lemma 2.2, I_{H_1} and I_{H_2} are not G-equivalent. But since Condition B holds for $\mathcal{I}_{\mathbb{F}_qG}$, the weight distribution of these codes are different. Moreover since $e_{H_i}\widehat{H_i} = e_{H_i}$, we have that $I_{H_i} \subseteq (\mathbb{F}_qG)\widehat{H_i}$ for $i = 1, 2$. By considering the isomorphism $\varphi_{H_i} : (\mathbb{F}_qG)\widehat{H_i} \to \mathbb{F}_q(G/H_i)$ we get that

$$\varphi_{H_i}(e_{H_i}) = \widehat{H_i/H_i} - \widehat{H_i^*/H_i} = \widehat{1} - \widehat{C_p} \in \mathbb{F}_qC_{p^r}$$

for $i = 1, 2$. Hence, $\varphi_{H_i}(I_{H_i}) \cong (\mathbb{F}_qC_{p^r})(\widehat{1} - \widehat{C_p})$ for $i = 1, 2$. By Lemma 2.1, the weight distributions of I_{H_1} and I_{H_2} are identical. This is a contradiction. Therefore, G is homocyclic.

(iii) ⇒ (i) Assume G is homocyclic and assume that for codes $I_1, I_2 \in \mathcal{I}_{\mathbb{F}_qG}$ we have $w(I_1) = w(I_2)$. Note that $w((\mathbb{F}_qG)\widehat{G}) = |G|$ and by Proposition 2.1 in [5], $w(\mathbb{F}_qGe_H) = 2|H|$ whenever H is a cocyclic subgroup of G. Thus, since $|G|$ is odd, either both of I_1 and I_2 are $(\mathbb{F}_qG)\widehat{G}$ or there are cocylic subgroups H_1, H_2 of G, so that $I_1 = (\mathbb{F}_qG)e_{H_1}$ and $I_2 = (\mathbb{F}_qG)e_{H_2}$. If the second case holds, Proposition 2.1 in [5] implies that $w(I_1) = 2|H_1| = 2|H_2| = w(I_2)$. As H_1 and H_2 are cocyclic subgroups of a homocyclic group and $|H_1| = |H_2|$, we get that H_1 is isomorphic to H_2 by Corollary 2.5 (ii). Then Proposition 2.3 implies that H_1 and H_2 are G-isomorphic. Hence, I_1 is G-equivalent to I_2 by Lemma 2.2.

□

In the proof of the part **(iii) ⇒ (i)** the condition on the prime p is necessary as the following example shows.

Example 3.1 *Let $G = \langle a \rangle \cong C_2$, and consider $\mathbb{F}_3G$. Then consider the ideals I_1 and I_2 generated by the idempotents corresponding to subgroups G and 1, which are equal to $2 + 2a$ and $2 + a$, respectively. Then $I_1 = \{0, 1 + a, 2 + 2a\}$ and $I_2 = \{0, 2 + a, 1 + 2a\}$. Now, it is easy to see that the weights of I_1 and I_2 are equal but they are not G-equivalent by Lemma 2.2. Note that, I_1 and I_2 have identical weight distribution and equal dimension.*

Corollary 3.2 *Let p be an odd prime, G an abelian p-group with exponent p^n and q a prime power where $\langle q \rangle = U(\mathbb{Z}_{p^n})$. Then the following are equivalent.*

(i) *Condition A holds for the set of minimal ideals of $\mathbb{F}_qG$;*

(ii) *Condition B holds for the set of minimal ideals of $\mathbb{F}_qG$;*

(iii) *G is homocyclic.*

Proof Follows from Theorem 4.1 of [7] and Theorem 3.1. □

Theorem 3.3 *Let p be an odd prime, G an abelian p-group and $\mathbb{F}_q$ a finite field of q elements such that $(p, q) = 1$. Condition C holds for $\mathcal{I}_{\mathbb{F}_qG}$ if and only if G is homocyclic.*

Proof Assume Condition C holds and suppose to the contrary that G is not homocyclic. By Corollary 2.5 (i), there are nonisomorphic cocyclic subgroups H_1 and H_2 of G which have equal order. Let $I_1 = \mathbb{F}_qGe_{H_1}$

and $I_2 = (\mathbb{F}_q G)e_{H_2}$ be the corresponding ideals in $\mathcal{I}_{\mathbb{F}_q G}$. Then by Lemma 2.2, I_1 is not G-equivalent to I_2. By using Proposition 2.1 in [5] we have $\dim(I_1) = \dim(I_2)$ which means Condition C does not hold for $\mathcal{I}_{\mathbb{F}_q G}$.

Assume G is homocyclic. Let I_1 and I_2 be two ideals of the same dimension in $\mathcal{I}_{\mathbb{F}_q G}$. The dimension of $(\mathbb{F}_q G)\widehat{G}$ is equal to one and for a cocyclic subgroup H, the ideal $(\mathbb{F}_q G)e_H$ is even dimensional as we have $\dim((\mathbb{F}_q G)e_H) = |G/H| - |G/H^*|$ by Proposition 2.1 in [5]. Since G has odd order, either both I_1 and I_2 are $(\mathbb{F}_q G)\widehat{G}$ or both of them are different from $(\mathbb{F}_q G)\widehat{G}$. If the second case holds, there are cocyclic subgroups H_1 and H_2 of G so that $I_1 = (\mathbb{F}_q G)e_{H_1}$ and $I_2 = (\mathbb{F}_q G)e_{H_2}$. Then by using Proposition 2.1 in [5] we have $\dim(I_1) = |G/H_1| - |G/{H_1}^*| = |G/H_2| - |G/{H_2}^*| = \dim(I_2)$. As $|H_i^*| = p|H_i|$ we can easily conclude that $|H_1| = |H_2|$. In this case as H_1, H_2 are cocyclic subgroups in a homocyclic group, Corollary 2.5 (ii) implies that they are isomorphic. Hence they are G-isomorphic by Proposition 2.3. As a result, I_1 and I_2 are G-equivalent. □

Corollary 3.4 *Let p be an odd prime, G an abelian p-group with exponent p^n and q a prime power where $\langle q \rangle = U(\mathbb{Z}_{p^n})$. Then Condition C holds on the set of all minimal codes of $\mathbb{F}_q G$ if and only if G is homocyclic.*

Proof Follows from Theorem 4.1 in [7] and Theorem 3.3. □

We should emphasize here that if we take the field as a splitting field for an abelian p-group which is homocyclic, Corollary 3.2 and Corollary 3.4 are not true anymore.

Example 3.2 *Let $G = C_3 =< g >$ be the cyclic group of order 3. Consider $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ where α is a primitive third root of unity. $\mathbb{F}_4$ is a splitting field for G. Then the character table of G is*

G	1	g	g^2
χ_0	1	1	1
χ_1	1	α	α^2
χ_2	1	α^2	α

The primitive idempotents will be in the form $e_{\chi_i} = \frac{1}{3}\sum_{g \in G} \chi_i(g)g$ for $i = 0, 1, 2$. Then $e_{\chi_0} = 1 + g + g^2$ and $e_{\chi_1} = 1 + \alpha g + \alpha^2 g^2$. The corresponding minimal ideals are

$$I_0 = \{0, 1 + g + g^2, \alpha + \alpha g + \alpha g^2, \alpha^2 + \alpha^2 g + \alpha^2 g^2\}$$

$$I_1 = \{0, 1 + \alpha g + \alpha^2 g^2, \alpha + \alpha^2 g + g^2, \alpha^2 + g + \alpha g^2\}.$$

Now $w(I_0) = 3 = w(I_1)$, $\dim(I_1) = \dim(I_2)$ and weight distributions of I_0 and I_1 are identical. As any automorphism of G fixes any element of I_0, I_1 and I_0 are not G-equivalent. So none of the Conditions A, B, C holds for this example.

We have a natural question:

Question 3.5 *Let G be an abelian p-group and $\mathbb{F}$ a finite splitting field for G of characteristic coprime to the order of G. What are the conditions on G so that minimal ideals in $\mathbb{F}G$ satisfy Condition A, Condition B and Condition C ?*

Let $\mathcal{I}_{min}(\mathbb{F}G)$ be the set of minimal ideals in $\mathbb{F}G$ generated by the primitive idempotents corresponding to the nontrivial irreducible characters. That is we disclude the minimal ideal generated by e_{χ_0} where χ_0 is the trivial character. We give an answer for the Question 3.5 as follows.

Theorem 3.6 *Let G be an abelian p-group and let $\mathbb{F}$ be a finite spliting field for G of characteristic coprime to the order of G. Then the following are equivalent.*

(i) Condition A holds on $\mathcal{I}_{min}(\mathbb{F}G)$;

(ii) Condition B holds on $\mathcal{I}_{min}(\mathbb{F}G)$;

(iii) G is an elementary abelian p-group.

Proof Assume Condition A holds. Let I_1, I_2 be two ideals in $\mathcal{I}_{min}(\mathbb{F}G)$ having identical weight distributions. Suppose to the contrary that I_1 is not G-equivalent to I_2. As Condition A holds, we have $w(I_1) \neq w(I_2)$. This is a contradiction as they have same weight distribution. So I_1 is G-equivalent to I_2, hence Condition B holds.

Assume Condition B holds. Assume G is not elementary abelian. Then G has a direct factor of order p^n where $n > 1$. Then there are elements $g, h \in G$ such that $h \notin \langle g \rangle$ and g has order p^n and h has order p. Recall that for an irreducible character χ of G, the corresponding idempotent is $e_\chi = \frac{1}{|G|}\sum_{g\in G}\chi(g)g$. Let χ_g be the irreducible character with $\chi_g(g) = \alpha$ and $\chi_g(h) = 1$ and let χ_h be the irreducible character with $\chi_h(g) = 1$ and $\chi_h(h) = \beta$ where α is primitive p^n-th root of unity and β is primitive p-th root of unity. Then it is easy to see that the idempotents e_{χ_g} and e_{χ_h} generate non-G-equivalent minimal codes. However these minimal codes have identical weight distribution by the classfication of irreducible representations of abelian groups over their splitting field. This means Condition B does not hold which is a contradiction.

Assume G is an elementary abelian p-group. By considering its nontrivial characters and corresponding primitive idempotents, we can conclude that each minimal ideal in $\mathcal{I}_{min}(\mathbb{F}G)$ has weight equal to $|G|$ and since every nonidentity element can be sent to another nonidentity element of G by an automorphism of G, all these minimal ideals are G-equivalent to each other. Hence Condition A holds. □

Theorem 3.7 *Let G be an abelian p-group and let $\mathbb{F}$ be a finite spliting field for G of characteristic coprime to the order of G. Then Condition C holds on $\mathcal{I}_{min}(\mathbb{F}G)$ if and only if G is elementary abelian.*

Proof Assume G is an elemantary abelian p-group. By classification of irreducible representations of abelian groups over their splitting field, we can conclude that all minimal ideals in $\mathcal{I}_{min}(\mathbb{F}G)$ are one dimensional. Moreover, each minimal ideal in $\mathcal{I}_{min}(\mathbb{F}G)$ is G-equivalent to another minimal ideal in $\mathcal{I}_{min}(\mathbb{F}G)$. This is because any nonidentity element can be sent to another nonidentity element of G by an automorphism of G. That is Condition C holds.

Conversely assume that Condition C holds and assume G is not elementary abelian p-group. Then G has a direct factor isomorphic to C_{p^n} where $n > 1$. As in the proof of the previous result, we can conclude that all minimal ideals are one dimensional but there are non-G-equivalent ones. □

4. The case for composite groups

The following lemma can be seen as a generalization of the Proposition 2.3 (ii) of [5]. Note that in some results below we use the sign $\prod$ for both direct product of groups and product of elements in the form $\widehat{H}$ for some subgroup H of the given group.

Lemma 4.1 *Let $G = G_{p_1} \times \ldots \times G_{p_k}$ be an abelian group where each G_{p_i} is a Sylow p_i-subgroup of G. Let H be a cocyclic subgroup of G and write H as*

$$H = (\prod_{i \in S} G_{p_i}) \times (\prod_{i \in \{1,\ldots,k\} \setminus S} H_{p_i}),$$

where $S \subseteq \{1, \ldots, k\}$ and for each $i \in \{1, \ldots, k\} \setminus S$, H_{p_i} is a cocyclic subgroup of G_{p_i}. Consider $I_H = (\mathbb{F}_q G)e_H$ where $(q, p_i) = 1$ for any $i \in \{1, \ldots, k\}$. Then

$$w(I_H) = 2^{k-|S|}|H|.$$

Proof Let $T = (\prod_{i \in S} G_{p_i}) \leq H$, then we have that $\widehat{T} = (\prod_{i \in S} \widehat{G_{p_i}})$ clearly. So

$$e_H \widehat{T} = \left(\prod_{i \in S} \widehat{G_{p_i}}\right)\left(\prod_{i \in \{1,\ldots,k\} \setminus S} (\widehat{H_{p_i}} - \widehat{H^*_{p_i}})\right)\left(\prod_{i \in S} \widehat{G_{p_i}}\right) = e_H$$

and it follows that $I_H \subseteq (\mathbb{F}_q G)\widehat{T}$. Note that

$$\varphi_T(e_H) = \left(\prod_{i \in \{1,\ldots,k\} \setminus S} (\widehat{H_{p_i}} - \widehat{H^*_{p_i}})\right) = e_{H/T}$$

implies that $\varphi_T(I_H) = I_{H/T} \subseteq \mathbb{F}_q(G/T)$. From Proposition 2.3 (ii) of [5], it follows that

$$w(I_{H/T}) = 2^{k-|S|}|H|/|T|$$

and then Lemma 2.1 implies that $w(I_H) = 2^{k-|S|}|H|$. □

Theorem 4.2 *Let n be an odd integer and let G be an abelian group of order n. Let q be a prime power with $(q, n) = 1$. Then Condition A holds for $\mathcal{I}_{\mathbb{F}_q G}$ if and only if every Sylow p-subgroup of G is homocyclic.*

Proof Let $G = G_{p_1} \times \ldots \times G_{p_k}$ where each G_{p_i} is a Sylow p_i-subgroup of G. Assume that G_{p_i} is homocyclic for each $i = 1, \ldots, k$. Suppose that the weights of I_H and I_K are equal. If one of H or K is equal to G, then the other one should also be equal to G. Indeed, since G has odd order, the equality of weights and Lemma 4.1 imply that if $H = G$ then $K = G$. So, we can assume without loss of generality that both of H and K are different from G. In this case again Lemma 4.1 implies that

$$2^{k-|S|}|H| = w(I_H) = w(I_K) = 2^{k-|S'|}|K|$$

and since G has odd order, it follows that $|H| = |K|$. Because H and K are cocyclic subgroups of G and each Sylow p-subgroup of G is homocyclic, it follows that $H \cong K$ by Corollary 2.5 (ii). Hence, by Proposition 2.3, we have that H is G-isomorphic to K. Therefore, by Lemma 2.2, we get that I_H is G-equivalent to I_K. Thus, Condition A holds for $\mathcal{I}_{\mathbb{F}_q G}$.

Suppose that Condition A holds for $\mathcal{I}_{\mathbb{F}_qG}$ and suppose to the contrary that at least one of Sylow p-subgroups is not homocyclic, say G_{p_j}. Then by Corollary 2.5 (i), there exist two nonisomorphic cocyclic subgroups H_{p_j} and K_{p_j} of G_{p_j} such that $|H_{p_j}| = |K_{p_j}|$. Consider, $H = H_{p_j} \times (\prod_{i\neq j} G_{p_i})$ and $K = K_{p_j} \times (\prod_{i\neq j} G_{p_i})$. Then since $|H| = |K|$ we have that $w(I_H) = w(I_K)$ by Lemma 4.1. However, since H is not isomorphic to K, by Proposition 2.3, we have that H is not G-isomorphic to K. Then Lemma 2.2 implies that I_H is not G-equivalent to I_K, that is Condition A does not hold for $\mathcal{I}_{\mathbb{F}_qG}$, which is a contradiction. So each Sylow p-subgroup of G should be homocyclic.

□

Likewise as in the p-group case, we did not use the fact that G has odd order in the forward direction of the proof of the theorem above. But in the reverse direction, we need the odd order condition as the following example shows.

Example 4.1 *Let $G = \langle a\rangle \cong C_6$ and consider $\mathbb{F}_5G$. Then consider the ideals I_1 and I_2 generated by the idempotents $\widehat{G}$ and $e_{\langle a^2\rangle} = \widehat{\langle a^2\rangle}\,(1-\widehat{\langle a^3\rangle})$, respectively. Then by Lemma 3.1, we have that $w(I_2) = 6$. Notice that $w(I_1) = 6$, also I_1 and I_2 are not G-equivalent since G and $\langle a^2\rangle$ are not G-isomorphic. So even though every Sylow p-subgroup of G is homocyclic, Condition A is not satisfied by $\mathcal{I}_{\mathbb{F}_5G}$.*

Theorem 4.3 *Let n be an odd integer and let G be an abelian group of order n. Let q be a prime power with $(q,n) = 1$. Then Condition B holds for $\mathcal{I}_{\mathbb{F}_qG}$ if and only if every Sylow p-subgroup of G is homocyclic.*

Proof Let $G = G_{p_1} \times \ldots \times G_{p_k}$ be an abelian group where each G_{p_i} is a Sylow p_i-subgroup of G. Assume that G_{p_i} is homocyclic for each $i = 1,\ldots,k$. Suppose that the weight distribution of I_H and I_K are identical, then the weight of I_H and I_K are equal. Then, it follows from Theorem 4.2 that I_H and I_K are G-equivalent. So Condition B holds for $\mathcal{I}_{\mathbb{F}_qG}$.

Assume that Condition B holds for $\mathcal{I}_{\mathbb{F}_qG}$. Suppose to the contrary that G_{p_j} is not homocyclic for some $1 \leq j \leq k$. Then by Corollary 2.5(i), there exist two nonisomorphic cocyclic subgroups H_{p_j} and K_{p_j} of G_{p_j} such that $|H_{p_j}| = |K_{p_j}|$. Consider, $H = H_{p_j} \times (\prod_{i\neq j} G_{p_i})$ and $K = K_{p_j} \times (\prod_{i\neq j} G_{p_i})$. Then H and K are cocyclic subgroups of G such that $H \ncong K$ and $|H| = |K|$. In this case H is not G-isomorphic to K by Proposition 2.3. Then Lemma 2.2 implies that I_H is not G-equivalent to I_K. Since Condition B holds, weight distributions of I_H and I_K are different.

Now, note that since $e_H\widehat{H} = e_H$, we have that $I_H = \mathbb{F}_qGe_H \subseteq (\mathbb{F}_qG)\widehat{H}$ and similarly $I_K = \mathbb{F}_qGe_K \subseteq (\mathbb{F}_qG)\widehat{K}$. Moreover, since

$$e_H = (\widehat{H_{p_j}} - \widehat{H^*_{p_j}})(\prod_{i\neq j}\widehat{G_{p_i}}) \text{ and } e_K = (\widehat{K_{p_j}} - \widehat{K^*_{p_j}})(\prod_{i\neq j}\widehat{G_{p_i}})$$

and $G/H \cong G/K \cong C_{p_j{}^r}$ for some integer $r \geq 1$. We have that

$$\varphi_H(e_H) = 1 - \widehat{C_{p_j}} \text{ and } \varphi_K(e_K) = 1 - \widehat{C_{p_j}}.$$

This implies that the weight distributions of $\varphi_H(I_H)$ and $\varphi_K(I_K)$ are identical. On the other hand, since the weight distributions of I_H and I_K are different and $|H| = |K|$, Lemma 2.1 implies that the weight distributions of $\varphi_H(I_H)$ and $\varphi_K(I_K)$ are different. This is a contradiction. Hence, we conclude that each Sylow p-subgroup of G is homocyclic. □

The following lemma is a kind of generalization of the Proposition 2.3 (i) in [5].

Lemma 4.4 *Let $G = G_{p_1} \times \ldots \times G_{p_k}$ be an abelian group where each G_{p_i} is a Sylow p_i-subgroup of G. Let H be a cocyclic subgroup of G and write H as*

$$H = (\prod_{i \in S} G_{p_i}) \times (\prod_{i \in \{1,\ldots,k\} \setminus S} H_{p_i}),$$

where $S \subseteq \{1, \ldots, k\}$ and for each $i \in \{1, \ldots, k\} \setminus S$, H_{p_i} is a cocyclic subgroup of G_{p_i}. Consider $I_H = (\mathbb{F}_q G)e_H$ where $(q, p_i) = 1$ for any $i \in \{1, \ldots, k\}$. Then

$$\dim(I_H) = \frac{|G|}{|H|} \prod_{i \in \{1,\ldots,k\} \setminus S} (1 - \frac{|H_{p_i}|}{|H^*_{p_i}|}) = \frac{|G|}{|H|} \prod_{i \in \{1,\ldots,k\} \setminus S} (1 - \frac{1}{p_i}).$$

In particular, if H, K are cocyclic subgroups of G such that $|H| = |K|$, then $\dim(I_H) = \dim(I_K)$.

Proof As in the proof of Lemma 4.1, let $T = (\prod_{i \in S} G_{p_i}) \leq H$. It is clear that we have $e_H \widehat{T} = e_H$ and $I_H \subseteq (\mathbb{F}_q G)\widehat{T}$. As $H/T \cong \prod_{i \in \{1,\ldots,k\} \setminus S} H_{p_i}$ and $G/T \cong \prod_{i \in \{1,\ldots,k\} \setminus S} G_{p_i}$, it follows that H/T is a cocyclic subgroup of G/T. So we have $e_{H/T} = \prod_{i \in \{1,\ldots,k\} \setminus S} e_{H_{p_i}} \in \mathbb{F}_q(G/T)$. Then by Proposition 2.3 in [5], we have

$$\dim(\mathbb{F}_q(G/T))e_{H/T} = \frac{|G/T|}{|H/T|} \prod_{i \in \{1,\ldots,k\} \setminus S} (1 - \frac{1}{p_i}) = \frac{|G|}{|H|} \prod_{i \in \{1,\ldots,k\} \setminus S} (1 - \frac{1}{p_i}).$$

Since $\mathbb{F}_q(G/T)e_{H/T} = \varphi_T((\mathbb{F}_q G)e_H) = \varphi_T(I_H)$ and φ_T preserves the dimension, we have

$$\dim(I_H) = \frac{|G|}{|H|} \prod_{i \in \{1,\ldots,k\} \setminus S} (1 - \frac{1}{p_i}).$$

If H, K are cocyclic subgroups of G such that $|H| = |K|$, by the dimension formula above, we get $\dim(I_H) = \dim(I_K)$.

□

The converse of the last statement of the Lemma 4.4 is not true in general.

Example 4.2 *Let $G = < a > \times < b > \times < c > \cong C_9 \times C_9 \times C_{49}$. Consider the cocyclic subgroups $H = < a > \times < c > \cong C_9 \times C_{49}$ and $K = < a > \times < b > \times < c^2 > \cong C_9 \times C_9 \times C_7$ of G. By Lemma 4.4 we have $\dim(I_H) = \frac{|G|}{|H|}(1 - \frac{1}{3}) = 6$ and $\dim(I_K) = \frac{|G|}{|K|}(1 - \frac{1}{7}) = 6$. On the other hand we have $|K| \neq |H|$.*

However there is an infinite family of abelian groups which satisfies the desired property if we add some extra conditions. Note that the set of tuples (p_1, p_2) in the next theorem are infinite since for example there are infinitely many prime tuples of the form $(3, 3k + 2)$.

Proposition 4.5 *Let p_1, p_2 be odd primes such that $p_1 < p_2$ and $p_2 \not\equiv 1 (\mathrm{mod}\ p_1)$. Let $G = G_{p_1} \times G_{p_2}$ be an abelian group where each G_{p_i} is Sylow p_i-subgroup of G and H, K are cocyclic subgroups of G. Then if $\dim(I_H) = \dim(I_K)$, then $|H| = |K|$.*

Proof

There are cases we need to consider.

Case 1 : Let $H = H_{p_1} \times H_{p_2}$ and $K = K_{p_1} \times K_{p_2}$ where $H_{p_i}, K_{p_i} < G_{p_i}$ for $i \in \{1, 2\}$ are cocyclic subgroups. Let $h_i, k_i \geq 1$ be the integers such that $|G_{p_i} : H_{p_i}| = {p_i}^{h_i}$ and $|G_{p_i} : K_{p_i}| = {p_i}^{k_i}$. Then by Lemma 4.4, we have

$$\dim(I_H) = {p_1}^{h_1} {p_2}^{h_2} (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) = {p_1}^{k_1} {p_2}^{k_2} (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) = \dim(I_K).$$

It follows that $h_i = k_i$ for $i \in \{1, 2\}$. So $|H| = |K|$.

Case 2 : Let $H = H_{p_1} \times G_{p_2}$ and $K = K_{p_1} \times G_{p_2}$ where for $H_{p_1}, K_{p_1} < G_{p_1}$ are cocyclic subgroups. If we have $|G_{p_1} : H_{p_1}| = {p_1}^{h_1}$ and $|G_{p_1} : K_{p_1}| = {p_1}^{k_1}$, then we get

$$\dim(I_H) = {p_1}^{h_1} (1 - \frac{1}{p_1}) = {p_1}^{k_1} (1 - \frac{1}{p_1}) = \dim(I_K).$$

So we get $h_1 = k_1$ and $|H| = |K|$.

Similarly if $H = G_{p_1} \times H_{p_2}$ and $K = G_{p_1} \times K_{p_2}$ where for $H_{p_2}, K_{p_2} < G_{p_2}$ are cocyclic subgroups such that $\dim(I_H) = \dim(I_K)$, then $|H| = |K|$.

Case 3 : If $H = H_{p_1} \times H_{p_2}$ and $K = G_{p_1} \times K_{p_2}$ where $H_{p_1} < G_{p_1}$ and $H_{p_2}, K_{p_2} < G_{p_2}$ are cocyclic subgroups, by Lemma 4.4, $\dim(I_H) \neq \dim(I_K)$. It follows similarly if $H = H_{p_1} \times H_{p_2}$ and $K = K_{p_1} \times G_{p_2}$.

Case 4 : Let $H = H_{p_1} \times G_{p_2}$ and $K = G_{p_1} \times K_{p_2}$ where for $H_{p_1} < G_{p_1}$, $H_{p_2} < G_{p_2}$ are cocyclic subgroups and $|G_{p_1} : H_{p_1}| = {p_1}^{h_1}$ and $|G_{p_2} : K_{p_2}| = {p_2}^{k_2}$. Then we have $\dim(I_H) = {p_1}^{h_1}(1 - \frac{1}{p_1})$, $\dim(I_K) = {p_2}^{k_2}(1 - \frac{1}{p_2})$ by Lemma 4.4. Assume $\dim(I_H) = \dim(I_K)$. We have ${p_1}^{h_1 - 1}(p_1 - 1) = {p_2}^{k_2 - 1}(p_2 - 1)$. If $h_1 > 1$, then p_1 should divide $p_2^{k_2}$ which is a contradiction. If $h_1 = 1$, then $p_1 - 1 = {p_2}^{k_2 - 1}(p_2 - 1)$ which is not possible as $p_1 < p_2$. So $\dim(I_H) \neq \dim(I_K)$.

□

Proposition 4.6 *Let p_1, p_2 be odd primes such that $p_1 < p_2$ and $p_2 \not\equiv 1 (\mathrm{mod}\ p_1)$. Let $G = G_{p_1} \times G_{p_2}$ be an abelian group where each G_{p_i} is Sylow p_i-subgroup of G. If G_{p_1} and G_{p_2} are homocyclic, then Condition C holds for $\mathcal{I}_{\mathbb{F}_q G}$.*

Proof Assume G_{p_1} and G_{p_2} are homocyclic. Recall that $\mathcal{I}_{\mathbb{F}_q G} = \{I_H \mid H < G \text{ is cocyclic or } H = G\}$. Let I_H and I_K be ideals in $\mathcal{I}_{\mathbb{F}_q G}$. Assume that $\dim(I_H) = \dim(I_K)$. As $\dim(\mathbb{F}_q \widehat{G}) = 1$ and order of G is odd, by dimension formula in Lemma 4.4, either both I_H and I_K are equal to $\mathbb{F}_q \widehat{G}$ or both of I_H and I_K are different from $\mathbb{F}_q \widehat{G}$. By Proposition 4.5, we have $|H| = |K|$. As the Sylow p-subgroups of G are homocyclic, by Corollary 2.5(ii) we have that $H \cong K$. Then by Proposition 2.3, H is G-isomorphic to K. So by Lemma 2.2, I_H is G-equivalent to I_K. Hence Condition C holds for $\mathcal{I}_{\mathbb{F}_q G}$. □

We have also prove the following theorem.

Theorem 4.7 *Let n be an odd integer and let G be an abelian group of order n. Let q be a prime power with $(q,n)=1$. If Condition C holds for $\mathcal{I}_{\mathbb{F}_qG}$, then every Sylow p-subgroup of G is homocyclic.*

Proof

Assume that Condition C holds for $\mathcal{I}_{\mathbb{F}_qG}$. Suppose to the contrary that one of the Sylow p-subgroup is not homocyclic. Then by Corollary 2.5 (i), there exist two nonisomorphic cocyclic subgroups H and K of G such that $H \ncong K$ and $|H| = |K|$. In this case H is not G-isomorphic to K by Proposition 2.3. Then Lemma 2.2 implies that I_H is not G-equivalent to I_K. Since Condition C holds, their dimensions are not equal. On the other hand by Lemma 4.4, we have $\dim(I_H) = \dim(I_K)$ which gives a contradiction. □

We end the paper with the following question.

Question 4.8 *Let G be an abelian group of odd order and $\mathbb{F}_q$ a field such that $\mathbb{F}_qG$ is semisimple. Assume $\mathbb{F}_q$ is not a splitting field for G. Is there any other set $\mathcal{I}$ of ideals in $\mathbb{F}_qG$ such that Conditions A, B holds on $\mathcal{I}$ if and only if for each prime divisor p of $|G|$ the Sylow p-subgroup of G is homocyclic?*

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