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Initial value problem for elastic system in transversely isotropic inhomogeneous media

Meltem ALTUNKAYNAK*
Department of Mathematics, Faculty of Science, Dokuz Eylül University, İzmir, Turkey

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Abstract: In this paper, we consider an initial value problem (IVP) for three dimensional elasticity system in a transversely isotropic inhomogeneous media. We will rewrite the problem in the form of Fourier images by means of Fourier transform method. After some arrangements, the problem is reduced to integral equations in the vector form. Using the properties of the vector integral equation and successive approximations method, an explicit formula for the solution of the IVP in transversely isotropic inhomogeneous media is constructed, and existence and uniqueness of the solution is stated. By a computational example, we illustrate the robustness of the method.

Key words: Anisotropic inhomogeneous media, transversely isotropic media, elastodynamic system

1. Introduction

The study of wave propagation of elastic waves in anisotropic, inhomogeneous media has great interest to characterize the properties of many important materials in different branches of applied sciences. However, the number of studies dealing with wave propagation in anisotropic inhomogeneous media is scarce [2, 4–7, 9, 13–17, 20, 22, 23]. Anisotropic materials are materials whose properties are directionally dependent. A general anisotropic solid has 21 independent elastic constants. If a material has a symmetry plane, then this leaves decreasing in the number of independent constants. Orthotropic, transversely isotropic, hexagonal, cubic materials are anisotropic materials with various numbers of symmetry planes. Transverse isotropic materials are one of the subset of anisotropic materials. These materials have the same properties in one plane (e.g., the xy-plane) and different properties in the direction normal to this plane (e.g., the z-axis). Thus, they are described by 5 independent elastic constants.

The velocity stress formulation for propagation of elastic seismic waves through 2D heterogeneous transversely isotropic media of arbitrary orientation is considered by Christopher Juhlin in [10]. In the paper, the equations are recast into a finite difference scheme and solved numerically using operators. Numerical modeling of two dimensional, three-component wave propagation in a transversely isotropic medium with arbitrary orientation using finite element modeling in transversely isotropic media is studied in [25] by Jianlin Zhu and Jim Dorman. In [24], Yang et al. transform the seismic wave equations in 2D inhomogeneous anisotropic media into a system of first-order partial differential equations with respect to time t. The space derivatives are calculated by using an interpolation approximation, while the time derivatives are replaced by a truncated

*Correspondence: meltem.topcuoglu@deu.edu.tr
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Taylor expansion. The method enables wave propagation to be simulated in two dimensions through generally anisotropic and heterogeneous models. A solution of the problem of seismic waves propagation from a point source in an inhomogeneous transverse isotropic medium using a method based on a combination of partial separation of variables and finite-difference method is studied by Martynov and Mikhailenko in [13]. Carcione et al. used a pseudospectral time-integration technique to solve the equation of motion, where the propagation is done by a direct expansion of the evolution operator by a Chebycheff polynomial series in [3]. Synthetic seismograms were computed for P SV wave propagation in transversely isotropic media by Tsingas et al. in [18]. The finite difference method used to model wave propagation in anisotropic inhomogeneous media bounded by irregular interfaces. The analytical solutions for body-wave velocity in a continuously inhomogeneous transversely isotropic material, in which Young's moduli, shear modulus, and material density change according to the generalized power law model with the remaining elastic constants are assumed to be constants are set down in [21] by Wang et al.

The explicit formula for the solution of IVP of elastic system has not been constructed and visualized in the case when the elastic modules, initial data and nonhomogeneous terms are arbitrary smooth functions. In [1], the explicit formula for the solution of initial value problem (IVP) of elastic system in inhomogeneous orthotropic media is constructed. As a continuation of the study in [1], the IVP for three dimensional elasticity system in transversely isotropic vertically inhomoeneous media is considered when the elastic modules, initial data and nonhomogeneous terms are arbitrary smooth functions. An explicit formula for the solution of the considered IVP using similar analytical approach is obtained. A computational example is given to compare the exact solution and the solution obtained by the analytical method explained herein.

2. Problem setup

The mathematical model of anisotropic elastic wave propagation is described by

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k} + f_j, \quad j = 1, 2, 3, \tag{2.1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t > 0$, $u_j(x,t)$ are the components of the unknown displacement vector. The constant $\rho > 0$ is the density of the medium [6, 7, 17].

According to the Hooke's law each stress-tensor component σ_{jk} is related to all the strain-tensor components ε_{lm} can be written as

$$\sigma_{jk} = \sum_{l,m=1}^{3} c_{jklm} \varepsilon_{lm}, \tag{2.2}$$

where the components of the strain-tensor is defined as follows:

$$\varepsilon_{lm} = \frac{1}{2} \left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l} \right). \tag{2.3}$$

$\{c_{jklm}\}_{j,k,l,m=1}^{3}$ are the components of a tensor, known as the elasticity tensor. It is a forth-order positive definite tensor and the components of the tensor satisfy the symmetry conditions $c_{jklm} = c_{kjlm} = c_{jkml}$ [7, 8, 11].

Invoking the definition of the strain tensor, given in expression (2.3), equation (2.2) can be rewritten as

$$\begin{aligned}\sigma_{jk} = c_{jk11}\frac{\partial u_1}{\partial x_1} + c_{jk22}\frac{\partial u_2}{\partial x_2} + c_{jk33}\frac{\partial u_3}{\partial x_3} \\ + c_{jk12}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + c_{jk13}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) + c_{jk23}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right).\end{aligned} \tag{2.4}$$

It is convenient and customary to describe the elastic moduli in terms of a 6×6 matrix according to the following conventions relating a pair (j,k) of indices $j,k = 1,2,3$ to a single index $\alpha,\beta = 1,\cdots,6$: [7, 12, 19]

$$\begin{aligned}(1,1)\leftrightarrow 1, \quad (2,2)\leftrightarrow 2, \quad (3,3)\leftrightarrow 3, \\ (2,3),(3,2)\leftrightarrow 4, \quad (1,3),(3,1)\leftrightarrow 5, \quad (1,2),(2,1)\leftrightarrow 6.\end{aligned} \tag{2.5}$$

This correspondence is possible due to the symmetry properties. The additional symmetry property $c_{jklm} = c_{lmjk}$ implies that the matrix

$$c = (c_{\alpha\beta}), \quad \text{where } \alpha = (jk) \text{ and } \beta = (lm),$$

is symmetric.

In transversely isotropic media the physical properties are symmetric about an axis that is normal to a plane of isotropy (xy-plane in the figure). There are three mutually orthogonal planes of reflection symmetry and axial symmetry with respect to z-axis. Number of independent coefficients is 5. The elastic moduli c for the transversely isotropic media can be written as follows [17]:

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) \end{bmatrix}.$$

In this paper, we assume that the components of the elasticity tensor depend on x_3. Substituting (2.4) into equation (2.1) and using the symmetry properties of elastic moduli, the mathematical model of the elastic wave propagation in transversely isotropic media can be formulated as follows:

$$\begin{aligned}\rho\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial c_{44}}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) + c_{11}\frac{\partial^2 u_1}{\partial x_1^2} + c_{12}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + c_{13}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} \\ + \frac{1}{2}(c_{11}-c_{12})\left(\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2}\right) + c_{44}\left(\frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right) + f_1(x,t),\end{aligned} \tag{2.6}$$

$$\begin{aligned}\rho\frac{\partial^2 u_2}{\partial t^2} = \frac{\partial c_{44}}{\partial x_3}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) + \frac{1}{2}(c_{11}-c_{12})\left(\frac{\partial^2 u_1}{\partial x_2 \partial x_1} + \frac{\partial^2 u_2}{\partial x_1^2}\right) \\ + c_{12}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + c_{11}\frac{\partial^2 u_2}{\partial x_2^2} + c_{13}\frac{\partial^2 u_3}{\partial x_2 \partial x_3} + c_{44}\left(\frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial^2 u_3}{\partial x_2 \partial x_3}\right) + f_2(x,t),\end{aligned} \tag{2.7}$$

$$\rho\frac{\partial^2 u_3}{\partial t^2}=\frac{\partial c_{13}}{\partial x_3}\frac{\partial u_1}{\partial x_1}+\frac{\partial c_{13}}{\partial x_3}\frac{\partial u_2}{\partial x_2}+\frac{\partial c_{33}}{\partial x_3}\frac{\partial u_3}{\partial x_3}+c_{44}\left(\frac{\partial^2 u_1}{\partial x_1\partial x_3}+\frac{\partial^2 u_3}{\partial x_1^2}\right)$$
$$+c_{44}\left(\frac{\partial^2 u_2}{\partial x_2\partial x_3}+\frac{\partial^2 u_3}{\partial x_2^2}\right)+c_{13}\frac{\partial^2 u_1}{\partial x_1\partial x_3}+c_{13}\frac{\partial^2 u_2}{\partial x_2\partial x_3}+c_{33}\frac{\partial^2 u_3}{\partial x_3^2}+f_3(x,t). \tag{2.8}$$

In the paper, an initial value problem (IVP) for elastic system (2.6)–(2.8) in transversely isotropic inhomogeneous media with initial conditions

$$u_i(x,0)=0 \quad \text{and} \quad \frac{\partial u_i(x,t)}{\partial t}\Big|_{t=0}=0, \tag{2.9}$$

is considered.

Through the paper, the following assumptions and notations are used: $\Delta(T)$ is a triangular domain defined as

$$\Delta(T)=\{(x_3,t):\ t\in[0,T],\ |x_3|\le a(T-t)\}, \tag{2.10}$$

where T is a given positive number. The elastic constants are twice continuously differentiable functions over $[-aT,aT]$ and depend on x_3. Assume that there exist constants ρ_0, α, and β such that $\rho(x_3)\ge\rho_0>0$ and $0<\alpha\le c_{33},c_{44}\le\beta$, where $a=\sqrt{\beta/\rho_0}$. Further, assume that the functions $f_i(x,t)$, $i=1,2,3$ have the respective Fourier transform $F_i(\nu,x_3,t)$ with respect to x_1,x_2, and let their Fourier images belong to the space $C(\mathbb{R}^2\times\Delta(T))$, $\nu=(\nu_1,\nu_2)$.

Consider now the following formulation of the Fourier transform

$$\mathcal{F}_{x_1,x_2}[u](\nu,x_3,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(x,t)e^{i(\nu_1x_1+\nu_2x_2)}dx_1dx_2.$$

Applying the Fourier transform with respect to x_1,x_2 to the equations of motion (2.6)–(2.8) yields

$$\rho\frac{\partial^2 U_1}{\partial t^2}=\frac{\partial c_{44}}{\partial x_3}\left(\frac{\partial U_1}{\partial x_3}+\mathrm{i}\nu_1U_3\right)-\nu_1^2c_{11}U_1-\nu_1\nu_2c_{12}U_2+\mathrm{i}\nu_1c_{13}\frac{\partial U_3}{\partial x_3}$$
$$+\frac{1}{2}\left(c_{11}-c_{12}\right)\left(-\nu_1\nu_2U_2-\nu_2^2U_1\right)+c_{44}\left(\frac{\partial U_1}{\partial x_3^2}+\mathrm{i}\nu_1\frac{\partial U_3}{\partial x_3}\right)+F_1\left(\nu,x_3,t\right), \tag{2.11}$$

$$\rho\frac{\partial^2 U_2}{\partial t^2}=\frac{\partial c_{44}}{\partial x_3}\left(\frac{\partial U_2}{\partial x_3}+\mathrm{i}\nu_2U_3\right)+\frac{1}{2}\left(c_{11}-c_{12}\right)\left(-\nu_1\nu_2U_1-\nu_1^2U_2\right)$$
$$-c_{12}\nu_1\nu_2U_1-c_{11}\nu_2^2U_2+\mathrm{i}\nu_2c_{13}\frac{\partial U_3}{\partial x_3}+c_{44}\left(\frac{\partial^2 U_2}{\partial x_3^2}+\mathrm{i}\nu_2\frac{\partial U_3}{\partial x_3}\right)+F_2\left(\nu,x_3,t\right), \tag{2.12}$$

$$\rho\frac{\partial^2 U_3}{\partial t^2}=\mathrm{i}\nu_1\frac{\partial c_{13}}{\partial x_3}U_1+\mathrm{i}\nu_2\frac{\partial c_{13}}{\partial x_3}U_2+\frac{\partial c_{33}}{\partial x_3}\frac{\partial U_3}{\partial x_3}+c_{44}\left(\mathrm{i}\nu_1\frac{\partial U_1}{\partial x_3}-\nu_1^2U_3\right)$$
$$+c_{44}\left(\mathrm{i}\nu_2\frac{\partial U_2}{\partial x_3}-\nu_2^2U_3\right)+\mathrm{i}c_{13}\left(\nu_1\frac{\partial U_1}{\partial x_3}+\nu_2\frac{\partial U_2}{\partial x_3}\right)+c_{33}\frac{\partial^2 U_3}{\partial x_3^2}+F_3\left(\nu,x_3,t\right), \tag{2.13}$$

where $\mathcal{F}_{x_1,x_2}[u]=U(\nu,x_3,t)$ and $\mathcal{F}_{x_1,x_2}[f]=F(\nu,x_3,t)$.

3. Reduction of IVP to a vector integral equation

In this section, we consider the following transform:

$$y_j = \tau_j(x_3); \quad \tau_j(x_3) = \int_0^{x_3} c_j(\xi)d\xi; \quad x_3 = \tau_j^{-1}(y_j)$$
$$c_1^2(x_3) = c_2^2(x_3) = \frac{\rho}{c_{44}},\ c_3^2(x_3) = \frac{\rho}{c_{33}},$$

and denote

$$U_j(\nu, x_3, t)\Big|_{x_3=\tau_j^{-1}(y_j)} = V_j(\nu, y_j, t), \quad j = 1, 2, 3,$$
$$F_j\left(\nu, x_3, t\right)\Big|_{x_3=\tau_j^{-1}(y_j)} = \Psi_j\left(\nu, y_j, t\right), \quad j = 1, 2, 3.$$

Using the transformation and the notations given above, the IVP (2.11)–(2.13) becomes

$$\begin{aligned}\frac{\partial^2 V_1}{\partial t^2} &= \frac{\partial^2 V_1}{\partial y_1^2} - M_1\left(y_1\right)\frac{\partial V_1}{\partial y_1} - \frac{1}{\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\left\{\left(\nu_1^2 c_{11} + \nu_2^2 c_{66}\right)V_1 + \nu_1\nu_2\left(c_{12}+c_{66}\right)V_2\right\}_{x_3=\tau_1^{-1}(y_1)} \\ &+ \frac{\mathrm{i}\nu_1}{\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\left\{\frac{\partial c_{44}}{\partial x_3}U_3 + \left(c_{13}+c_{44}\right)\frac{\partial U_3}{\partial x_3}\right\}_{x_3=\tau_1^{-1}(y_1)} + \frac{1}{\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\Psi_1\left(\nu, y_1, t\right),\end{aligned} \tag{3.1}$$

$$\begin{aligned}\frac{\partial^2 V_2}{\partial t^2} &= \frac{\partial^2 V_2}{\partial y_1^2} - M_2\left(y_1\right)\frac{\partial V_2}{\partial y_1} - \frac{1}{\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\left\{\left(\nu_1^2 c_{66} + \nu_2^2 c_{11}\right)V_2 + \nu_1\nu_2\left(c_{66}+c_{12}\right)V_1\right\}_{x_3=\tau_1^{-1}(y_1)} \\ &+ \frac{\mathrm{i}\nu_2}{\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\left\{\frac{\partial c_{44}}{\partial x_3}U_3 + \left(c_{13}+c_{44}\right)\frac{\partial U_3}{\partial x_3}\right\}_{x_3=\tau_1^{-1}(y_1)} + \frac{1}{\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\Psi_2\left(\nu, y_1, t\right),\end{aligned} \tag{3.2}$$

$$\begin{aligned}\frac{\partial^2 V_3}{\partial t^2} &= \frac{\partial^2 V_3}{\partial y_3^2} - M_3\left(y_3\right)\frac{\partial V_3}{\partial y_3} - \frac{1}{\rho\left(\tau_3^{-1}\left(y_3\right)\right)}\left\{\left(\nu_1^2 + \nu_2^2\right)c_{44}V_3 - \mathrm{i}\frac{\partial c_{13}}{\partial x_3}\left(\nu_1 U_1 + \nu_2 U_2\right)\right\}_{x_3=\tau_3^{-1}(y_3)} \\ &+ \frac{\mathrm{i}}{\rho\left(\tau_3^{-1}\left(y_3\right)\right)}\left\{\left(c_{13}+c_{44}\right)\left(\nu_1\frac{\partial U_1}{\partial x_3} + \frac{\partial U_2}{\partial x_3}\right)\right\}_{x_3=\tau_3^{-1}(y_3)} + \frac{1}{\rho\left(x_3^{-1}\left(y_3\right)\right)}\Psi_3\left(\nu, y_3, t\right),\end{aligned} \tag{3.3}$$

where

$$M_1\left(y_1\right) = M_2\left(y_1\right) = \frac{d}{dy_1}\ln\left\{\left[c_{44}\left(\tau_1^{-1}\left(y_1\right)\right)\rho\left(\tau_1^{-1}\left(y_1\right)\right)\right]^{-\frac{1}{2}}\right\},$$
$$M_3\left(y_3\right) = \frac{d}{dy_3}\ln\left\{\left[c_{33}\left(\tau_3^{-1}\left(y_3\right)\right)\rho\left(\tau_3^{-1}\left(y_3\right)\right)\right]^{-\frac{1}{2}}\right\}.$$

We seek a solution of the problem (3.1)–(3.3) in the following form:

$$V_j(\nu, y_j, t) = S_j(y_j)W_j(\nu, y_j, t), \quad j = 1, 2, 3,$$

where the function $S_j(y_j)$ defined by

$$S_j(y_j) = \exp\left(\frac{1}{2}\int_0^{y_j} M_j(\xi)d\xi\right),$$

then the following equations can be obtained

$$\begin{aligned}\frac{\partial^2 W_1}{\partial t^2}-\frac{\partial^2 W_1}{\partial y_1^2}=W_1\left[\frac{M_1'}{2}-\frac{1}{4}M_1^2\right]-\frac{1}{\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\left\{\left(\nu_1^2c_{11}+\nu_2^2c_{66}\right)W_1+\nu_1\nu_2\left(c_{12}+c_{66}\right)W_2\right\}_{x_3=\tau_1^{-1}(y_1)}\\+\frac{\mathrm{i}\nu_1}{\rho\left(\tau_1^{-1}\left(y_1\right)\right)S_1\left(y_1\right)}\left\{\frac{\partial c_{44}}{\partial x_3}U_3+\left(c_{13}+c_{44}\right)\frac{\partial U_3}{\partial x_3}\right\}_{x_3=\tau_1^{-1}(y_1)}\\+\frac{1}{S_1\left(y_1\right)\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\Psi_1\left(\nu,y_3,t\right),\end{aligned}\tag{3.4}$$

$$\begin{aligned}\frac{\partial^2 W_2}{\partial t^2}-\frac{\partial^2 W_2}{\partial y_1^2}=W_2\left[\frac{M_1'}{2}-\frac{1}{4}M_1^2\right]-\frac{1}{\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\left\{\left(\nu_1^2c_{66}+\nu_2^2c_{11}\right)W_2+\nu_1\nu_2\left(c_{66}+c_{12}\right)W_1\right\}_{x_3=\tau_1^{-1}(y_1)}\\+\frac{\mathrm{i}\nu_2}{S_1\left(y_1\right)\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\left\{\frac{\partial c_{44}}{\partial x_3}U_3+\left(c_{13}+c_{44}\right)\frac{\partial U_3}{\partial x_3}\right\}_{x_3=\tau_1^{-1}(y_1)}\\+\frac{1}{S_1\left(y_1\right)\rho\left(\tau_1^{-1}\left(y_1\right)\right)}\Psi_2\left(\nu,y_3,t\right),\end{aligned}\tag{3.5}$$

$$\begin{aligned}\frac{\partial^2 W_3}{\partial t^2}-\frac{\partial^2 W_3}{\partial y_3^2}=W_3\left[\frac{M_3'}{2}-\frac{1}{4}M_3^2\right]-\frac{1}{\rho\left(\tau_3^{-1}\left(y_3\right)\right)}\left\{\left(\nu_1^2+\nu_2^2\right)c_{44}W_3\right\}_{x_3=\tau_3^{-1}(y_3)}\\+\frac{\mathrm{i}}{\rho\left(\tau_3^{-1}\left(y_3\right)\right)S_3\left(y_3\right)}\left\{\frac{\partial c_{13}}{\partial x_3}\left(\nu_1U_1+\nu_2U_2\right)+\left(c_{13}+c_{44}\right)\left(\nu_1\frac{\partial U_1}{\partial x_3}+\frac{\partial U_2}{\partial x_3}\right)\right\}_{x_3=\tau_3^{-1}(y_3)}\\+\frac{1}{\rho\left(\tau_3^{-1}\left(y_3\right)\right)S_3\left(y_3\right)}\Psi_3\left(\nu,y_3,t\right).\end{aligned}\tag{3.6}$$

To simplify the equations (3.4)–(3.6), let us define new notations as follows:

$$q_i\left(y_i\right)=\frac{1}{2}\frac{\partial M_i}{\partial y_i}-\frac{1}{4}M_i^2,\quad L_i\left(y_i\right)=\frac{1}{\rho\left(\tau_i^{-1}\left(y_i\right)\right)},\quad N_i\left(y_i\right)=\frac{1}{\rho\left(\tau_i^{-1}\left(y_i\right)\right)S_i\left(y_i\right)},$$

$$\Phi_i\left(\nu,y_i,t\right)=N_i\left(y_i\right)\Psi_i(\nu,y_3,t),\quad P_{li}(y_i)=C_{ll}(x_3)\Big|_{x_3=\tau_i^{-1}(y_i)},\quad l=1,4,6,$$

$$Q_{mi}(y_i)=C_{1m}(x_3)\Big|_{x_3=\tau_i^{-1}(y_i)},\quad m=2,3.$$

Using these notations, equations (3.4)–(3.6) can be written as follows:

$$\begin{aligned}\frac{\partial^2 W_1}{\partial t^2}-\frac{\partial^2 W_1}{\partial y_1^2}=q_1W_1-L_1\left[\left(\nu_1^2P_{11}+\nu_2^2P_{61}\right)W_1+\nu_1\nu_2\left(Q_{21}+P_{61}\right)W_2\right]\\+\mathrm{i}\nu_1N_1\tilde{C}_1\left[Q_{31}\frac{\partial U_3}{\partial y_1}+\frac{\partial}{\partial y_1}\left(P_{41}U_3\right)\right]+\Phi_1\left(\nu,y_1,t\right),\end{aligned}\tag{3.7}$$

$$\begin{aligned}\frac{\partial^2 W_2}{\partial t^2}-\frac{\partial^2 W_2}{\partial y_1^2}=q_1W_2-L_1\left[\left(\nu_1^2P_{61}+\nu_2^2P_{11}\right)W_2+\nu_1\nu_2\left(P_{61}+Q_{21}\right)W_1\right]\\+\mathrm{i}\nu_2N_1\tilde{C}_1\left[Q_{31}\frac{\partial U_3}{\partial y_1}+\frac{\partial}{\partial y_1}\left(P_{41}U_3\right)\right]+\Phi_2\left(\nu,y_1,t\right),\end{aligned}\tag{3.8}$$

$$\frac{\partial^2 W_3}{\partial t^2} - \frac{\partial^2 W_3}{\partial y_3^2} = q_3 W_3 - L_3\left[\left(\nu_1^2+\nu_2^2\right)P_{43}W_3\right] + \Phi_3\left(\nu, y_3, t\right)$$
$$+\,\mathrm{i}\tilde{C}_3 N_3\left[\nu_1\frac{\partial}{\partial y_3}\left(Q_{33}U_1\right)+\nu_2\frac{\partial}{\partial y_3}\left(Q_{33}U_2\right)+\nu_1 P_{43}\frac{\partial U_1}{\partial y_3}+\nu_2 P_{43}\frac{\partial U_2}{\partial y_3}\right]. \tag{3.9}$$

Using d'Alembert's formula, equations (3.7)–(3.9) together with the initial conditions, (2.9) can be written in the following form of integral equations:

$$W_1(\nu,y_1,t) = \frac{1}{2}\int_0^t\int_{y_1-(t-\eta)}^{y_1+(t-\eta)} q_1W_1 - L_1\left[\left(\nu_1^2P_{11}+\nu_2^2P_{61}\right)W_1+\nu_1\nu_2\left(Q_{21}+P_{61}\right)W_2\right]$$
$$-\,\mathrm{i}\nu_1\frac{\partial}{\partial\xi}\left[N_1\tilde{C}_1Q_{31}\right]U_3-\mathrm{i}\nu_1\frac{\partial}{\partial\xi}\left[N_1\tilde{C}_1\right]P_{41}U_3+\Phi_1(\nu,\xi,\tau)d\xi d\eta \tag{3.10}$$
$$+\frac{1}{2}\int_0^t \mathrm{i}\nu_1N_1\tilde{C}_1(Q_{31}+P_{41})U_3(\nu,\tau_1^{-1}(\xi),\eta)\Big|_{y_1-(t-\tau)}^{y_1+(t-\tau)}d\eta,$$

$$W_2\left(\nu,y_1,t\right) = \frac{1}{2}\int_0^t\int_{y_1-(t-\eta)}^{y_1+(t-\eta)}\{q_1W_2 - L_1\left[\left(\nu_1^2P_{61}+\nu_2^2P_{11}\right)W_2+\nu_1\nu_2\left(P_{61}+Q_{21}\right)W_1\right]$$
$$-\,\mathrm{i}\nu_2\frac{\partial}{\partial\xi}\left[N_1\tilde{C}_1Q_{31}\right]U_3-\mathrm{i}\nu_2\frac{\partial}{\partial\xi}\left[N_1\tilde{c}_1\right]P_{41}U_3+\Phi_2(\nu,\xi,\tau)\}d\xi d\eta \tag{3.11}$$
$$+\frac{1}{2}\int_0^t \mathrm{i}\nu_2N_1\tilde{C}_1\left(Q_{31}+P_{41}\right)U_3\Big|_{y_1-(t-\tau)}^{y_1+(t-\tau)}d\eta,$$

$$W_3\left(\nu,y_3,t\right) = \frac{1}{2}\int_0^t\int_{y_3-(t-\eta)}^{y_3+(t-\eta)}\{q_3W_3 - L_3\left[\left(\nu_1^2+\nu_2^2\right)P_{43}W_3\right]+\Phi_3(\nu,\xi,\tau)$$
$$-\,\mathrm{i}\nu_1\frac{\partial}{\partial\xi}\left[\tilde{C}_3N_3\right]Q_{33}U_1-\mathrm{i}\nu_2\frac{\partial}{\partial\xi}\left[\tilde{C}_3N_3\right]Q_{33}U_2$$
$$-\,\mathrm{i}\nu_1\frac{\partial}{\partial\xi}\left[\tilde{C}_3N_3P_{43}\right]U_1-\mathrm{i}\nu_2\frac{\partial}{\partial\xi}\left[\tilde{C}_3N_3P_{43}\right]U_2\}d\xi d\eta \tag{3.12}$$
$$+\frac{1}{2}\int_0^t \mathrm{i}\tilde{C}_3N_3\left(Q_{33}+P_{43}\right)\left(\nu_1U_1+\nu_2U_2\right)\Big|_{y_3-(t-\tau)}^{y_3+(t-\tau)}d\eta.$$

Since $V_j\left(\nu,y_j,t\right) = S_j\left(y_j\right)W_j\left(\nu,y_j,t\right)$ and $U\left(\nu,x_3,t\right)\Big|_{x_3=\tau_j^{-1}(y_j)} = V_j\left(\nu,y_j,t\right)$, then equations (3.10)–(3.12) take the form

$$U_1(\nu,x_3,t) = \frac{S_1(\tau_1(x_3))}{2}\int_0^t\int_{\tau_1(x_3)-(t-\eta)}^{\tau_1(x_3)+(t-\eta)}\left[q_1(\xi)-L_1(\xi)\left(\nu_1^2P_{11}(\xi)+\nu_2^2P_{61}(\xi)\right)\right]\frac{U_1(\nu,\tau_1^{-1}(\xi),\eta)}{S_1(\xi)}$$
$$-\,\nu_1\nu_2L_1(\xi)\left(Q_{21}(\xi)+P_{61}(\xi)\right)\frac{U_2(\nu,\tau_1^{-1}(\xi),\eta)}{S_1(\xi)}-\mathrm{i}\nu_1\frac{\partial}{\partial\xi}\left[N_1(\xi)\tilde{C}_1(\xi)Q_{31}(\xi)\right]U_3(\nu,\tau_1^{-1}(\xi),\eta)$$
$$-\,\mathrm{i}\nu_1\frac{\partial}{\partial\xi}\left[N_1(\xi)\tilde{C}_1(\xi)\right]P_{41}(\xi)U_3(\nu,\tau_1^{-1}(\xi),\eta)+\Phi_1(\nu,\xi,\eta)d\xi d\eta$$
$$+\frac{\mathrm{i}\nu_1S_1(\tau_1(x_3))}{2}\int_0^t N_1(\xi)\tilde{C}_1(\xi)(Q_{31}(\xi)+P_{41}(\xi))U_3(\nu,\tau_1^{-1}(\xi),\eta)\Big|_{\tau_1(x_3)-(t-\tau)}^{\tau_1(x_3)+(t-\tau)}d\eta, \tag{3.13}$$

$$
\begin{aligned}
U_2(\nu,x_3,t) = \frac{S_1(\tau_1(x_3))}{2}\int_0^t\int_{\tau_1(x_3)-(t-\eta)}^{\tau_1(x_3)+(t-\eta)} &\left[q_1(\xi)-L_1(\xi)\left(\nu_1^2P_{61}(\xi)+\nu_2^2P_{11}(\xi)\right)\right]\frac{U_2(\nu,\tau_1^{-1}(\xi),\eta)}{S_1(\xi)} \\
&-\nu_1\nu_2L_1(\xi)\left(P_{61}(\xi)+Q_{21}(\xi)\right)\frac{U_1(\nu,\tau_1^{-1}(\xi),\eta)}{S_1(\xi)} - \mathrm{i}\nu_2\frac{\partial}{\partial\xi}\left[N_1(\xi)\tilde{C}_1(\xi)Q_{31}(\xi)\right]U_3(\nu,\tau_1^{-1}(\xi),\eta) \\
&-\mathrm{i}\nu_2\frac{\partial}{\partial\xi}\left[N_1(\xi)\tilde{C}_1(\xi)\right]P_{41}(\xi)U_3(\nu,\tau_1^{-1}(\xi),\eta)+\Phi_2(\nu,\xi,\eta)d\xi d\eta \\
&+\frac{\mathrm{i}\nu_2S_1(\tau_1(x_3))}{2}\int_0^t N_1(\xi)\tilde{C}_1(\xi)\left(Q_{31}(\xi)+P_{41}(\xi)\right)U_3(\nu,\tau_1^{-1}(\xi),\eta)\Big|_{\tau_1(x_3)-(t-\tau)}^{\tau_1(x_3)+(t-\tau)}d\eta,
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
U_3(\nu,x_3,t) = \frac{S_3(\tau_3(x_3))}{2}\int_0^t\int_{\tau_3(x_3)-(t-\eta)}^{\tau_3(x_3)+(t-\eta)} &\left[q_3(\xi)-L_3(\xi)\left(\nu_1^2+\nu_2^2\right)P_{43}(\xi)\right]\frac{U_3(\nu,\tau_3^{-1}(\xi),\eta)}{S_3(\xi)} \\
&-\mathrm{i}\nu_1\frac{\partial}{\partial\xi}\left[\tilde{C}_3(\xi)N_3(\xi)\right]Q_{33}(\xi)U_1(\nu,\tau_3^{-1}(\xi),\eta)-\mathrm{i}\nu_2\frac{\partial}{\partial\xi}\left[\tilde{C}_3(\xi)N_3(\xi)\right]Q_{33}(\xi)U_2(\nu,\tau_3^{-1}(\xi),\eta) \\
&-\mathrm{i}\nu_1\frac{\partial}{\partial\xi}\left[\tilde{C}_3(\xi)N_3(\xi)P_{43}(\xi)\right]U_1(\nu,\tau_3^{-1}(\xi),\eta)-\mathrm{i}\nu_2\frac{\partial}{\partial\xi}\left[\tilde{C}_3(\xi)N_3(\xi)P_{43}(\xi)\right]U_2(\nu,\tau_3^{-1}(\xi),\eta)\} \\
&+\Phi_3(\nu,\xi,\eta)d\xi d\eta \\
&+\frac{\mathrm{i}S_3(\tau_3(x_3))}{2}\int_0^t \tilde{C}_3(\xi)N_3(\xi)\left(Q_{33}(\xi)+P_{43}(\xi)\right)\left(\nu_1U_1(\nu,\tau_3^{-1}(\xi),\eta)+\nu_2U_2(\nu,\tau_3^{-1}(\xi),\eta)\right)\Big|_{\tau_3^{-1}-(t-\tau)}^{\tau_3^{-1}+(t-\tau)}d\eta.
\end{aligned}
\tag{3.15}
$$

Using equations (3.13)–(3.15) and the notations

$$
\begin{aligned}
V_j\left(\nu,y_j,t\right) &= S_j\left(y_j\right)W_j\left(\nu,y_j,t\right), \\
U\left(\nu,x_3,t\right)\Big|_{x_3=\tau_j^{-1}(y_j)} &= V_j\left(\nu,y_j,t\right),
\end{aligned}
$$

the system of integral equations with respect to the unknowns U_1,U_2,U_3 can be written as an operator integral equation

$$
\Theta(\nu,x_3,t)=\mathbf{G}(\nu,x_3,t)+\int_0^t(\mathbf{K}\Theta)(\nu,x_3,t)d\tau, \quad \text{where} \quad \Theta=(\Theta_1,\Theta_2,\Theta_3), \tag{3.16}
$$

has components which are unknown functions $\Theta_j=U_j$, $j=1,2,3$ and $\mathbf{G}=(G_1,G_2,G_3)$ is the known vector function with components

$$
G_i(\nu,x_3,t)=\frac{S_i(\tau_i(x_3))}{2}\int_0^t\int_{y_i-(t-\tau)}^{y_i+(t-\tau)}\Phi_i(\nu,\xi,\tau)d\xi d\tau, \quad i=1,2,3, \tag{3.17}
$$

and $\mathbf{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ is the vector-operator with components defined as

$$
\begin{aligned}
(\mathcal{K}_1\Theta)(\nu, x_3, t, \tau) = {} & \frac{S_1(\tau_1(x_3))}{2} \int_{\tau_1(x_3)-(t-\eta)}^{\tau_1(x_3)+(t-\eta)} \left[q_1(\xi) - L_1(\xi)\left(\nu_1^2 P_{11}(\xi) + \nu_2^2 P_{61}(\xi)\right)\right] \frac{\Theta_1(\nu, \tau_1^{-1}(\xi), \eta)}{S_1(\xi)} \\
& - \nu_1\nu_2 L_1(\xi)\left(Q_{21}(\xi) + P_{61}(\xi)\right) \frac{\Theta_2(\nu, \tau_1^{-1}(\xi), \eta)}{S_1(\xi)} - \mathrm{i}\nu_1 \frac{\partial}{\partial \xi}\left[N_1(\xi)\tilde{C}_1(\xi) Q_{31}(\xi)\right] \Theta_3(\nu, \tau_1^{-1}(\xi), \eta) \\
& - \mathrm{i}\nu_1 \frac{\partial}{\partial \xi}\left[N_1(\xi)\tilde{C}_1(\xi)\right] P_{41}(\xi)\Theta_3(\nu, \tau_1^{-1}(\xi), \eta) d\xi \\
& + \frac{\mathrm{i}\nu_1 S_1(\tau_1(x_3))}{2} N_1(\xi)\tilde{C}_1(\xi)(Q_{31}(\xi) + P_{41}(\xi))\Theta_3(\nu, \tau_1^{-1}(\xi), \eta)\Big|_{\tau_1(x_3)-(t-\tau)}^{\tau_1(x_3)+(t-\tau)},
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
(\mathcal{K}_2\Theta)(\nu, x_3, t, \tau) = {} & \frac{S_1(\tau_1(x_3))}{2} \int_{\tau_1(x_3)-(t-\eta)}^{\tau_1(x_3)+(t-\eta)} \left[q_1(\xi) - L_1(\xi)\left(\nu_1^2 P_{61}(\xi) + \nu_2^2 P_{11}(\xi)\right)\right] \frac{\Theta_2(\nu, \tau_1^{-1}(\xi), \eta)}{S_1(\xi)} \\
& - \nu_1\nu_2 L_1(\xi)\left(P_{61}(\xi) + Q_{21}(\xi)\right) \frac{\Theta_1(\nu, \tau_1^{-1}(\xi), \eta)}{S_1(\xi)} - \mathrm{i}\nu_2 \frac{\partial}{\partial \xi}\left[N_1(\xi)\tilde{C}_1(\xi) Q_{31}(\xi)\right] \Theta_3(\nu, \tau_1^{-1}(\xi), \eta) \\
& - \mathrm{i}\nu_2 \frac{\partial}{\partial \xi}\left[N_1(\xi)\tilde{C}_1(\xi)\right] P_{41}(\xi)\Theta_3(\nu, \tau_1^{-1}(\xi), \eta) d\xi \\
& + \frac{\mathrm{i}\nu_2 S_1(\tau_1(x_3))}{2} N_1(\xi)\tilde{C}_1(\xi)\left(Q_{31}(\xi) + P_{41}(\xi)\right)\Theta_3\Theta_3(\nu, \tau_1^{-1}(\xi), \eta)\Big|_{\tau_1(x_3)-(t-\tau)}^{\tau_1(x_3)+(t-\tau)},
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
(\mathcal{K}_3\Theta)(\nu, x_3, t, \tau) = {} & \frac{S_3(\tau_3(x_3))}{2} \int_{\tau_3(x_3)-(t-\eta)}^{\tau_3(x_3)+(t-\eta)} \left[q_3(\xi) - L_3(\xi)\left(\nu_1^2 + \nu_2^2\right) P_{43}(\xi)\right] \frac{\Theta_3(\nu, \tau_3^{-1}(\xi), \eta)}{S_3(\xi)} \\
& - \mathrm{i}\nu_1 \frac{\partial}{\partial \xi}\left[\tilde{C}_3(\xi) N_3(\xi)\right] Q_{33}(\xi)\Theta_1(\nu, \tau_3^{-1}(\xi), \eta) - \mathrm{i}\nu_2 \frac{\partial}{\partial \xi}\left[\tilde{C}_3(\xi) N_3(\xi)\right] Q_{33}(\xi)\Theta_2(\nu, \tau_3^{-1}(\xi), \eta) \\
& - \mathrm{i}\nu_1 \frac{\partial}{\partial \xi}\left[\tilde{C}_3(\xi) N_3(\xi) P_{43}(\xi)\right] \Theta_1(\nu, \tau_3^{-1}(\xi), \eta) - \mathrm{i}\nu_2 \frac{\partial}{\partial \xi}\left[\tilde{C}_3(\xi) N_3(\xi) P_{43}(\xi)\right] \Theta_2(\nu, \tau_3^{-1}(\xi), \eta)\} d\xi \\
& + \frac{\mathrm{i} S_3(\tau_3(x_3))}{2} \tilde{C}_3(\xi) N_3(\xi)\left(Q_{33}(\xi) + P_{43}(\xi)\right)\left(\nu_1\Theta_1(\nu, \tau_3^{-1}(\xi), \eta) + \nu_2\Theta_2(\nu, \tau_3^{-1}(\xi), \eta)\right)\Big|_{\tau_3^{-1}-(t-\tau)}^{\tau_3^{-1}+(t-\tau)}.
\end{aligned}
\tag{3.20}
$$

4. Existence of the unique solution of the problem

It is shown that under some notations and assumptions given in previous sections, the system of equations (2.11)–(2.13) is equivalent to the operator integral equation (3.16). In this section, some properties for this vector integral equation are given, and a theorem on the existence and uniqueness of a solution is stated.

Proposition 4.1 *Let T be a fixed positive number, and let components of* $\mathbf{G}$ *be defined by equation (3.17). Then under assumptions in Section 2, the components of* $\mathbf{G}$ *that are functions $G_i(\nu, x_3, t)$, $i = 1, 2, 3$ belong to the space $C(\mathbb{R}^2; \Delta(T))$.*

Proposition 4.2 *Let T be a fixed positive number and under assumptions in Section 2, $\Theta(\nu, x_3, t)$ be a vector* *function with continuous components in* $\mathbb{R}^2 \times \Delta(T)$. *Then*

$$\int_0^t (\mathcal{K}_i\Theta)(\nu, x_3, t, \tau)d\tau, \quad i = 1, 2, 3, \tag{4.1}$$

are continuous in $\mathbb{R}^2 \times \Delta(T)$ *and satisfy the following inequalities*

$$\left|\int_0^t (\mathcal{K}_i\Theta)(\nu, x_3, t, \tau)d\tau\right| \leq \mathcal{M}\int_0^t \|\Theta\|(\nu, \tau)d\tau,$$

where

$$\|\Theta\|(\nu, \tau) = \max_{i=1,2,3} \max_{\xi \in [-c(T-\tau), c(T-\tau)]} |\Theta_i(\nu, \xi, \tau)|,$$

$(x_3, t) \in \Delta(T)$, $|\nu| \leq \Omega$ *for any positive number* Ω *and a positive number* $\mathcal{M}$ *that depends on* c, T, Ω.

Proof Using formulas (3.18)–(3.20), we find out that the expressions given in (4.1) are continuous functions for $\nu \in \mathbb{R}^2$, $(x_3, t) \in \Delta(T)$. Let Q be defined by

$$\begin{aligned} Q = \max_{j,l,m}\max_{\xi}\Big\{&|q_j(\xi)|, |L_j(\xi)|, |N_j(\xi)|, |S_j(\xi)|, |\tilde{c}_j(\xi)|, |P_{lj}(\xi)|, |R_{3j}(\xi)|, |Q_{mj}(\xi)|,\\ &\frac{\partial}{\partial\xi}(\tilde{c}_j(\xi)N_j(\xi)), \frac{\partial}{\partial\xi}(\tilde{c}_j(\xi)N_j(\xi)Q_{31}(\xi)), \frac{\partial}{\partial\xi}(\tilde{c}_j(\xi)N_j(\xi)R_{32}(\xi)),\\ &\frac{\partial}{\partial\xi}(\tilde{c}_j(\xi)N_j(\xi)P_{43}(\xi)), \frac{\partial}{\partial\xi}(\tilde{c}_j(\xi)N_j(\xi)P_{53}(\xi))\Big\}\\ &j = 1, 2, 3, \quad l = 1, 4, 5, 6, \quad m = 1, 2, \end{aligned}$$

for any $(x_3, t) \in \Delta(T)$, $\tau \in [0, t]$, $\xi \in [\tau_j(x_3) - (t-\tau), \tau_j(x_3) + (t-\tau)]$ and since $\|\Theta\|(\nu, \tau)$ is defined in Proposition 4.2 by

$$\|\Theta\|(\nu, \tau) = \max_{i=1,2,3} \max_{\xi \in [-c(T-\tau), c(T-\tau)]} \big|\Theta_i(\nu, \xi, \tau)\big|,$$

then we have

$$\big|\Theta_i(\nu, \xi, \tau)\big| \leq \|\Theta\|(\nu, \tau)$$

for all $i = 1, 2, 3$, $\tau \in [0, t]$ and $\xi \in [-c(T-\tau), c(T-\tau)]$. Then, from formulas (3.18)–(3.20), the following relations can be obtained:

$$\begin{aligned} |(\mathcal{K}_\alpha\Theta)(\nu, x_3, t, \tau)| &\leq \mathcal{M}_\alpha(T, \Omega)\|\Theta\|(\nu, \tau), \quad \alpha = 1, 2\\ |(\mathcal{K}_3\Theta)(\nu, x_3, t, \tau)| &\leq \mathcal{M}_3(T, \Omega)\|\Theta\|(\nu, \tau), \end{aligned}$$

where $|\nu| < \Omega$ and

$$\begin{aligned} \mathcal{M}_1(T, \Omega) = \mathcal{M}_2(T, \Omega) &= TQ^2(1 + 4\Omega^2 Q + Q\Omega + \Omega) + \Omega Q^4,\\ \mathcal{M}_3(T, \Omega) &= TQ^2(1 + 2Q\Omega^2 + 2Q\Omega + 2\Omega) + \Omega Q^4 + \Omega Q^2,\\ \mathcal{M} &= \max_{i=1,2,3} \mathcal{M}_i(T, \Omega). \end{aligned}$$

□

Theorem 4.3 *Let T be a fixed positive number; $\mathbf{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ be the vector operator defined by (3.18)–(3.20). Then for any $\mathbf{G} = (G_1, G_2, G_3)$ such that $G_j = G_j(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta(c,T))$, $j = 1,2,3$ there exists a solution $\Theta = (\Theta_1, \Theta_2, \Theta_3)$ of the operator integral equation (3.16) such that $\Theta_j = U_j \in C(\mathbb{R}^2 \times \Delta(T))$, $j = 1,2,3$.*

Applying the following successive approximations

$$\Theta^{(0)}(\nu, x_3, t) = \mathbf{G}(\nu, x_3, t),$$

$$\Theta^{(n)}(\nu, x_3, t) = \int_0^t \big(\mathbf{K}\Theta^{(n-1)}\big)(\nu, x_3, t, \tau)d\tau, \quad n = 1, 2, \cdots,$$

the solution of the vector integral equation (3.16) can be constructed. Here $\Theta^{(n)}(\nu, x_3, t)$, $n = 0, 1, 2, \cdots$ are vector functions with continuous components and for each n, $\|\Theta^{(n)}\|(\nu, x_3, t)$ satisfies the inequality

$$|\Theta_i^{(n)}(\nu, x_3, t)| \leq \mathcal{M} \int_0^t \|\Theta^{(n-1)}\|(\nu, \tau)d\tau, \tag{4.2}$$

where $\|\cdot\|(\nu, \tau)$ and $\mathcal{M}$ are defined in Proposition 4.2. Using the inequality (4.2) it is easy to verify

$$|\Theta_i^{(n)}(\nu, x_3, t)| \leq \frac{(\mathcal{M}T)^n}{n!} \max_{|\nu| \leq \Omega} \|\mathbf{G}\|(\nu, T), \quad i = 1, 2, 3, \ n = 0, 1, 2, \cdots. \tag{4.3}$$

From inequality (4.3) and the first Weierstrass theorem, it follows that the series $\sum_{n=0}^{\infty} \Theta_i^{(n)}(\nu, x_3, t)$ converge uniformly to the continuous functions $\Theta_i(\nu, x_3, t)$. The functions $\Theta_i(\nu, x_3, t)$, $i = 1, 2, 3$ are the components of the solution vector function $\Theta(\nu, x_3, t)$ that is a solution of the integral equation (3.11). Using Proposition 4.2, it can also be shown that the constructed solution of (3.11) is unique.

Since equations (2.11)–(2.13) with zero initial conditions are equivalent to (3.11) and right-hand sides of equations (2.11)–(2.13) are from the class $C(\mathbb{R} \times [0, \infty))$, according to the definition given in [20], this problem is called a generalized Cauchy problem and the generalized solution may be written by the d'Alembert formula. Thus, we conclude that $\Theta(\nu, x_3, t)$ is a generalized solution of (2.11)–(2.13).

Hence, the components $\mathbf{U}(\nu, x_3, t)$ of the generalized solution $\Theta(\nu, x_3, t)$ belong to the space $C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); C_0^\infty(\mathbb{R}^2))$ and the solution $\mathbf{u}(x, t)$ of the IVP (2.6)–(2.9) is the inverse Fourier transform of $\mathbf{U}(\nu, x_3, t)$. By means of the real Paley-Wiener theorem, we conclude that the inverse Fourier transform of $\mathbf{U}(\nu, x_3, t)$ ($\mathbf{u}(x, t) = \mathcal{F}_\nu^{-1}[\mathbf{U}]$) is a unique generalized solution of (2.6)–(2.9) such that the components of the solution belong to the class

$$u_j(x, t) \in C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); PW(\mathbb{R}^2)).$$

The following theorem summarizes all the assumptions and results. For further explanation, the readers are referred to [1].

Theorem 4.4 *Let c, T be given positive numbers and $\Delta(T)$ be the triangular domain defined by*

$$\Delta(T) = \{(x_3, t) : \ t \in [0, T], \ |x_3| \leq c(T - t)\}$$

and let the density of the medium $\rho(x_3)$ and elastic constants $c_{33}(x_3), c_{44}(x_3)$ be twice continuously differentiable functions over $[-cT, cT]$. Let $F_i(\nu, x_3, t)$, $i = 1,2,3$ be the Fourier transform of the functions $f_i(x,t)$ with respect to x_1, x_2 that belong to the space $C(\mathbb{R}^2 \times \Delta(T))$ $(x_3, t) \in \Delta(T)$ for any $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$. Then a unique generalized solution $\mathbf{u}(x,t)$ of IVP (2.6)–(2.9) exists with components $u_i(x,t)$, $i = 1,2,3$, which belong to the space

$$C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); PW(\mathbb{R}^2)).$$

5. Computational example

In this section, we compare the Fourier transform (with respect to x_1, x_2) of the exact and approximate solutions. The approximate solution is found using the integral equation (3.16). The program of solving integral equation using succussive approximations is written in Maple.

Let us consider the problem (2.6)–(2.9) with

$$f_1(x,t) = -(x_3+1)(t-1)\delta(x_1)\delta(x_2), \; f_2(x,t) = f_3(x,t) = 0$$

and

$$c_{11} = c_{12} = c_{13} = c_{33} = c_{44} = \frac{1}{2}(x_3+1)^2.$$

Exact solution of the problem is

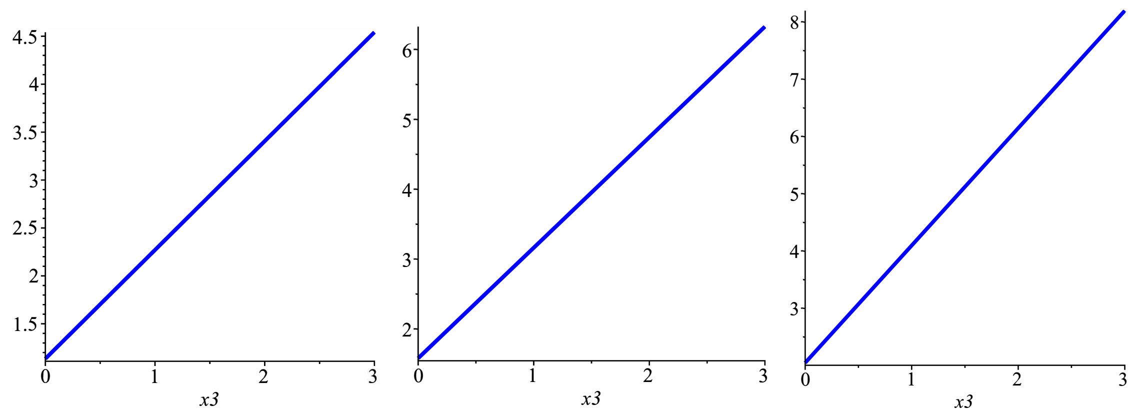
$$u_1^e(x,t) = (x_3+1)(\mathrm{e}^{-t} + t - 1)\delta(x_1)\delta(x_2), \; u_2^e(x,t) = u_3^e(x,t) = 0.$$

The comparison of the Fourier transform (with respect to x_1, x_2) of the exact solution $(u_1^c(x,t))$ and the approximate solution $(u_1(x,t))$ are reported in the table below. Table is written with the number of N=10 iterations of our method of successive approximations in calculating the approximate solution.

Figures 1 and 2 show the difference between the Fourier transform (with respect to x_1, x_2) of the exact and approximate solutions in t and x_3 for the computational example. The numerical results obtained by the proposed method are in good agreement with the exact solution (see, Figures 1 and 2).

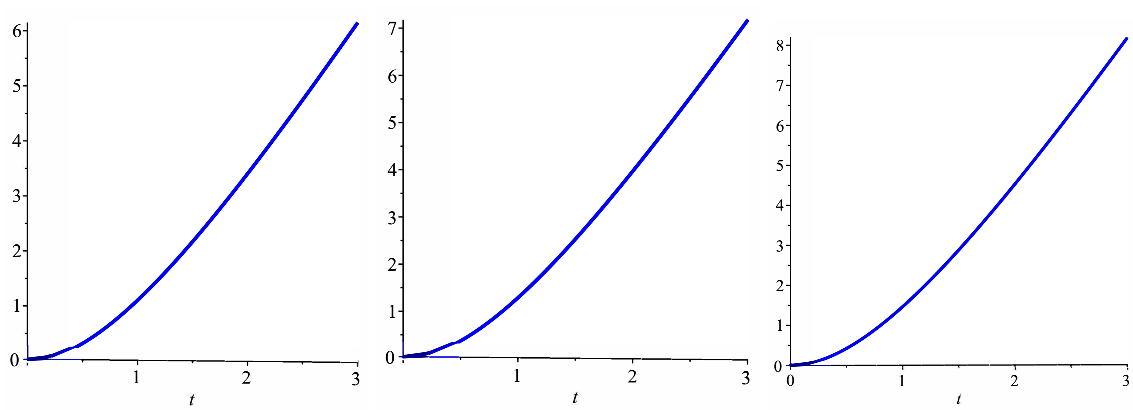
Table . Comparison of the solution $u_1^c(x,t)$ and the approximate solution $u_1(x,t)$.

t	x_3	$u_1^c(x,t)$	$u_1(x,t)$	$\lvert u_1^c(x,t) - u_1(x,t)\rvert$
$\frac{1}{2}$	$\frac{1}{2}$	0.1597959896	0.1597959893	3×10^{-10}
$\frac{1}{2}$	$\frac{3}{2}$	0.2663266490	0.2663266499	9×10^{-10}
1	$\frac{1}{2}$	0.5518191618	0.5518191614	4×10^{-10}
1	$\frac{3}{2}$	0.9196986030	0.9196986005	25×10^{-10}
$\frac{1}{2}$	2	0.3195919790	0.3195919796	6×10^{-10}
1	$\frac{17}{10}$	0.9932744912	0.9932744966	54×10^{-10}
$\frac{3}{2}$	$\frac{17}{10}$	1.9524515170	1.952451440	80×10^{-10}
2	3	4.541341133	4.541341132	10×10^{-10}
$\frac{5}{2}$	3	6.328339994	6.328340012	180×10^{-10}
3	3	8.199148274	8.199148302	280×10^{-10}
3	2	6.149361205	6.149361237	320×10^{-10}
3	$\frac{5}{2}$	7.174254739	7.174254731	110×10^{-10}



(a) $t = 2$ (b) $t = \frac{5}{2}$ (c) $t = 3$

Figure 1. The plot of $u_1(x,t)$ and $u_1^c(x,t)$ for $t = 2, \frac{5}{2}, 3$.



(a) $x_3 = 2$ (b) $x_3 = \frac{5}{2}$ (c) $x_3 = 3$

Figure 2. The plot of $u_1(x,t)$ and $u_1^c(x,t)$ for $x_3 = 2, \frac{5}{2}, 3$.

6. Conclusion

In this paper, the initial value problem for elastic system in transversely isotropic vertically inhomogeneous media is considered. Applying the Fourier transform with respect to variables x_1, x_2, the problem is reduced to integral equations of Volterra type, whose solutions can be obtained by successive approximations. Using the real Paley-Wiener theorem, the inverse images for the solution of the elastic system can be obtained. Using operator integral equations, existence and uniqueness theorem for the IVP is stated and proved. Finally, by a computational example, the robustness of the method is illustrated.

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