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A note on the transfinite diameter of Bernstein sets

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Abstract: A compact set $K \subset \mathbb{C}^n$ is called Bernstein set if, for some constant $M > 0$, the following inequality

$$||D^\alpha P||_K \leq M^{|\alpha|}(\deg P)^{|\alpha|}||P||_K$$

is satisfied for every multiindex $\alpha \in \mathbb{N}^n$ and for every polynomial P. We provide here a lower bound for the transfinite diameter of Bernstein sets by using generalized extremal Leja points.

Key words: Transfinite diameter, Bernstein and Markov sets, Pluripolar sets, Leja points

1. Introduction

Bernstein inequality for the closed unit disc Δ in $\mathbb{C}$ states that

$$||P'||_\Delta \leq \deg P ||P||_\Delta$$

for every polynomial P where $||.||_\Delta$ is the supremum norm on Δ. A compact set $K \subset \mathbb{C}^n$ is called Bernstein set if there exists a constant $M > 0$ such that

$$||D^\alpha P||_K \leq M^{|\alpha|}(\deg P)^{|\alpha|}||P||_K \tag{1.1}$$

for every multiindex $\alpha \in \mathbb{N}^n$ and for every polynomial P. Siciak in [5] showed that Bernstein sets are not pluripolar, that is, they are not contained in an infinity locus of a plurisubharmonic function. It is known that a compact set K is pluripolar if and only if its transfinite diameter $d(K) = 0$ (see [6]). The transfinite diameter $d(K)$ of a compact set $K \subset \mathbb{C}^n$ will be defined in Section 2.

A compact set $K \subset \mathbb{C}^n$ is called a Markov set if it satisfies the Markov inequality:

$$||D^\alpha P||_K \leq M^{|\alpha|}(\deg P)^{r|\alpha|}||P||_K$$

for some $M > 0$, $r > 0$, for every multiindex $\alpha \in \mathbb{N}^n$ and for every polynomial P. We note that every Bernstein set is a Markov set with $r = 1$. In dimension one, Białas-Cież [1] proved that Markov sets are not pluripolar. When $n \geq 2$, nonpluripolarity of Markov sets in $\mathbb{C}^n$ is an open problem. Thus finding a lower bound for the transfinite diameter for Markov sets is an interesting and hard problem. As an approach to this

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problem, Białas-Cież and Jedrzejowski in [2] found a lower bound for the transfinite diameter of Bernstein sets. Namely, they showed that

$$d(K) \geq \frac{1}{M2^{n-1}}$$

for any Bernstein set $K \subset \mathbb{C}^n$. Their proof uses the deep result of Zaharjuta [6] which computes the transfinite diameter $d(K)$ of a compact set in $\mathbb{C}^n$ with directional Chebyshev constants. In this paper, we give a simpler proof for a similar lower bound for the transfinite diameter $d(K)$ of any Bernstein set $K \subset \mathbb{C}^n$. Our main result is the following.

Theorem 1.1 *Let $K \subset \mathbb{C}^n$ be a Bernstein set. Then*

$$d(K) \geq \frac{1}{enM}.$$

Our proof uses idea of [4] related to generalized extremal Leja points.

2. Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers and $\alpha(j) = (\alpha_1(j), \dots, \alpha_n(j))$ be a multiindex in $\mathbb{N}^n$ with the length $|\alpha(j)| = \alpha_1(j) + \cdots + \alpha_n(j)$. We denote by $e_j(z) = z^{\alpha(j)} = z_1^{\alpha_1(j)} \dots z_n^{\alpha_n(j)}$ all the monomials in $\mathbb{C}^n$ ordered by increasing degrees, that is, $|\alpha(j)| \leq |\alpha(k)|$ if $j \leq k$ and monomials of a fixed degree are ordered lexicographically. Let h_s be the number of monomials of degree s and m_s be the number of monomials of degree at most s. It is easy to check that

$$h_s = \binom{s+n-1}{s}, \ m_s = \binom{s+n}{n}.$$

Let K be a compact set in $\mathbb{C}^n$ and $w_1, \dots, w_k$ be points in K. Vandermonde determinant is defined by

$$V(w_1, \dots, w_k) := \det[e_i(w_j)]_{i,j=1,\dots k}.$$

Note that $V(w_1, \dots, w_{m_s})$ is a polynomial of degree

$$l_{m_s} = \sum_{i=1}^{m_s} \deg(e_i) = \sum_{i=0}^{s} i \cdot h_i = n\binom{s+n}{n+1}.$$

A system $\{\zeta_1, \dots \zeta_k\}$ of k points in K is called a set of Fekete points of order k if

$$|V(\zeta_1, \dots \zeta_k)| = \sup_{\{w_1,\dots,w_k\} \subset K} |V(w_1, \dots, w_k)|.$$

Using Fekete points we define

$$d_s(K) := V_s^{\frac{1}{l_s}}$$

where $V_s = V_s(K) := |V(\zeta_1, \dots, \zeta_s)|$ and $l_s = \sum_{i=1}^{s} \deg(e_i)$. Existence of the limit

$$d(K) := \lim_{s \to \infty} d_s(K)$$

was shown by Fekete [3] in dimension $n = 1$ and by Zaharjuta [6] for $n \geq 2$. The limit $d(K)$ is called the transfinite diameter of K. We should note that the set of Fekete points of order i is not necessarily a subset of the set of Fekete points of order j when $i \leq j$. In [4], Jedrzejowski generalized extremal Leja points to the case of compact sets in $\mathbb{C}^n$ with $n \geq 2$ and proved that the transfinite diameter can be computed by means of them. For Leja points (in multidimensional case as well as in the complex plane) i^{th} order extremal set is a subset of j^{th} order extremal set when $i \leq j$. The construction is inductive. Let a_1 be an arbitrary point of K and $W_1 = 1$. Given a set of points $\{a_1, \dots, a_{k-1}\}$ in K the polynomial $P_k(z)$ is defined by

$$P_k(z) := V(a_1, \dots, a_{k-1}, z) = \det \begin{bmatrix} 1 & \dots & 1 & 1 \\ e_2(a_1) & \dots & e_2(a_{k-1}) & e_2(z) \\ \vdots & \vdots & \vdots & \vdots \\ e_k(a_1) & \dots & e_k(a_{k-1}) & e_k(z) \end{bmatrix}$$

Then a_k is chosen so that

$$W_k := |P_k(a_k)| = \sup_{z \in K} |P_k(z)|. \tag{2.1}$$

Then it follows from [4] that

$$\lim_{k \to \infty} W_k^{\frac{1}{l_k}} = d(K).$$

3. Proof of the main result

We will need the following generalization of Stirling formula in the proof of the main theorem.

Lemma 3.1 *There exists a k_0 such that if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| \geq k_0$, then*

$$\alpha! > \frac{\sqrt{2\pi}|\alpha|^{|\alpha|+1/2}(en)^{-|\alpha|}}{2}$$

where $\alpha! = \alpha_1! \dots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Proof Since for any $k \in \mathbb{N}$,

$$n^k = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = k} \frac{k!}{\alpha!}$$

we have $|\alpha|! \leq \alpha! n^{|\alpha|}$ for any $\alpha \in \mathbb{N}^n$. Applying Stirling formula to $|\alpha|!$, we obtain that

$$\alpha! \geq \frac{|\alpha|!}{n^{|\alpha|}} \geq \frac{\sqrt{2\pi}|\alpha|^{|\alpha|+1/2}(en)^{-|\alpha|}}{2}$$

for all α such that $|\alpha| \geq k_0$ for some k_0. □

Proof [Proof of Theorem 1.1] Let $K \subset \mathbb{C}^n$ be a Bernstein set which satisfies inequality (1.1) and the set of order $j-1$ extremal points $\{a_1, \dots, a_{j-1}\}$ for the transfinite diameter of K be constructed as above. We define the polynomial

$$P(z) := \frac{V(a_1, \dots, a_{j-1}, z)}{V(a_1, \dots, a_{j-1})}.$$

Then $P(z)$ is of the form

$$P(z) = e_j(z) + \sum_{i=1}^{j-1} c_i e_i(z)$$

for some constants c_i and hence $D^{\alpha(j)} P = \alpha(j)!$. It follows from (1.1) that

$$\alpha(j)! \leq M^{|\alpha(j)|} |\alpha(j)|^{|\alpha(j)|} \frac{W_j}{W_{j-1}}, \tag{3.1}$$

where W_j is defined as in (2.1). Using the inequality (3.1) and Lemma 3.1 we obtain that

$$\begin{aligned} W_j &\geq \frac{\alpha(j)! W_{j-1}}{M^{|\alpha(j)|} |\alpha(j)|^{|\alpha(j)|}} \\ &\vdots \\ &\geq \frac{\Pi_{k=k_0}^{j} \alpha(k)! W_{k_0-1}}{M^{\sum_{k=k_0}^{j} |\alpha(k)|} \Pi_{k=k_0}^{j} |\alpha(k)|^{|\alpha(k)|}} \\ &\geq \frac{(\sqrt{\frac{\pi}{2}})^{j-k_0+1} (\Pi_{k=k_0}^{j} |\alpha(k)|)^{\frac{1}{2}} W_{k_0-1}}{(enM)^{\sum_{k=k_0}^{j} |\alpha(k)|}} > (enM)^{-l_j} W_{k_0-1}, \end{aligned}$$

where $l_j = \sum_{k=1}^{j} |\alpha(k)|$. Note that

$$W_{k_0-1} \geq \frac{\Pi_{k=2}^{k_0-1} \alpha(k)!}{M^{\sum_{k=2}^{k_0-1} |\alpha(k)|} \Pi_{k=2}^{k_0-1} |\alpha(k)|^{|\alpha(k)|}} > 0.$$

Hence

$$W_j^{\frac{1}{l_j}} \geq (enM)^{-1} (W_{k_0-1})^{\frac{1}{l_j}},$$

and

$$d(K) = \lim_{j \to \infty} W_j^{\frac{1}{l_j}} \geq \frac{1}{enM}.$$

□

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