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On the convergence and stability analysis of finite-difference methods for the fractional Newell-Whitehead-Segel equations

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Abstract: In this study, standard and non-standard finite-difference methods are proposed for numerical solutions of the time-spatial fractional generalized Newell-Whitehead-Segel equations describing the dynamical behavior near the bifurcation point of the Rayleigh-Benard convection of binary fluid mixtures. The numerical solutions have been found for high values of p which shows the degree of nonlinear terms in the equations. The stability and convergence conditions of the obtained difference schemes are determined for each value of p. Errors of methods for various values of p are given in tables. The compatibility of exact solutions and numerical solutions and the effectiveness of the methods are interpreted with the help of tables and graphics. It can be said that not only standard and non-standard finite-difference methods are feasible and effective methods to solve the given equation numerically but also useful in terms of computational cost and memory.

Key words: Generalized Newell-Whitehead-Segel equation (GNWS), Standard Finite-Difference method (SFDM), Non-Standard Finite-Difference method (NSFDM), CFL conditions

1. Introduction

One of the reaction-diffusion equations associated with various physical phenomena such as biology, geology, physics, ecology, and chemistry is the Newell-Whitehead-Segel (NWS) equations. The Newell-Whitehead-Segel equation has an important role in nonlinear systems used to describe the appearance of the stripe patterns such as on Zebra's skin, in human fingerprints, and in the visual cortex in two-dimensional systems. It is also the equation that models the spatial and time-dependent change in substance concentration. Recently, many researchers have been interested in both analytical and numerical solutions of nonlinear fractional NWS equations. The time-spatial fractional generalized Newell-Whitehead-Segel (NWS) equation is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = k\frac{\partial^\gamma u}{\partial x^\gamma} + cu - du^p, u(x,t_0) = \xi(x), u(a,t) = \zeta(t), u(b,t) = \Omega(t) \tag{1.1}$$

where $0 < \alpha \leq 1$, $1 < \gamma \leq 2$, $c, d, k \in \mathbb{R}^+$, $2 \leq p \in \mathbb{N}$, $t_0 \leq t \leq T$, $a \leq x \leq b$. [22].

There are many studies on numerical solutions of the NWS equation in the literature. Patade and Bhalekar [17] used the new iteration method to solve the NWS equation numerically. Latif et al. [12] used the semi-analytical iterative method to numerically solve the NWS equation. Devi and Jakhar [7] solved the

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NWS equation with the homotopy perturbation algorithm using the Elzaki transform. Saravanan and Magesh [20] compared the reduced differential transform method with the Adomian decomposition method for the numerical solution of the NWS equation. Pue-on [19] used Laplace Adomian Decomposition method to solve the NWS equation numerically. Zellal and Belghaba [21] used the homotopy perturbation transform method to numerically solve the NWS equation. Also, numerical solutions of the fractional NWS equations are studied in the literature. Alderremy et al. [1] used the modified reduced differential transform method to numerically solve the fractional NWS equation. Odibat and Momani [16] used Adomian decomposition method and variational iteration methods to solve the fractional NWS equation numerically. Also, standard and non-standard finite-difference methods were used in many fields. Ali et al. [2] used finite-difference method to numerically solve for the HIV-1 infection of CD4+ T-cells conformable fractional mathematical model. Alkhazzan et al. [3] examined a class of nonlinear fractional differential equations with singularity. Anguelov et al. [4] solved Hamilton–Jacobi equations via the nonstandard finite-difference method. Gu et al. [10] used fast implicit difference schemes for time-space fractional diffusion equations with the integral fractional Laplacian. Inan et al. [11] used the explicit exponential finite-difference method for mathematical biology models. Li and Zeng [13] studied how to construct finite-difference schemes for fractional differential equations.

There were different numerical or semi-analytic methods to solve the fractional NWS equations. In these studies, researchers illustrated their methods using $p = 2, 3, 4$ in Equation 1.1 to show the effectiveness of their methods. Also, standard and non-standard finite-difference methods were used to solve the time-fractional NWS equations with $p = 2, 3, 4$ in [5].

In this study, the NWS equation has been generalized by adding both the time-fractional derivative and the spatial-fractional derivative in all aspects. Standard and non-standard finite-difference schemes are obtained for the time-spatial fractional generalized NWS equation and the conditions for consistency, convergence, and stability are found for each scheme and for the general value of p. In addition, different time-spatial fractional generalized NWS equation types with high values of p are examined, numerical solutions are compared with the exact solutions and error analysis is utilized. In the non-standard finite-difference schemes, time and spatial step size in the classical sense and different denominator functions are used and compared with the standard finite-difference values. Error-values for $p = 5, 7, 10$ are supported with the help of tables and graphics. Also, the effects of changing spatial-fractional derivative (γ) and time-fractional derivative (α) on the solution are discussed and showed by graphically.

2. Preliminaries and notations

In this section, some basic definitions and properties of standard, non-standard discretization and fractional analysis used in this study are briefly given. In this study, the fractional derivative in the sense of Riemann-Liouville defined as follows is used. [6]

$$ {}_{a}^{RL}D_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}u(s)ds \tag{2.1}$$

where $n-1 \leq \alpha < n$, $n \in \mathbb{N}$, $t > a$.

The fractional derivative in the sense of Grunwald-Letnikov is defined as follows [18].

$$ {}_{a}^{GL}D_t^{\alpha}f(t) = \lim_{\substack{h\to 0\\ N\Delta t = t-a}} \Delta t^{-\alpha}\sum_{j=0}^{N}\omega_j^{\alpha}(t-j\Delta t) \tag{2.2}$$

Here $n-1 \leq \alpha < n, \in \mathbb{N}^{+}, t > a$ and ω_j^{α} are named the Grunwald-Letnikov coefficients and satisfy Equation (2.3).

$$\omega_j^{\alpha} = (-1)^{\alpha} \binom{\alpha}{j}. \tag{2.3}$$

Grunwald-Letnikov approach is

$${}_{t_0}^{RL}D_t^{\alpha} u(t_k) \approx {}_{t_0}^{GL}D_t^{\alpha} u(t_k) = \Delta t^{-\alpha} \sum_{j=0}^{k} \omega_j^{\alpha} u(t_{k-j}) \tag{2.4}$$

where $n-1 \leq \alpha < n,\ n \in \mathbb{N}^{+}\ ,\ t > a,\ \omega_j^{\alpha}$ satisfy Equation (2.3) [13, 18].

Since the definitions of Riemann-Liouville and Grunwald-Letnikov are equivalent, the Grunwald-Letnikov approach is used as a numerical approach in our study.[18]

2.1. Standard discretization

Consider the fractional ordinary differential equation defined as

$$D^{\alpha} y(t) = f(t, y(t)),\ y(t_0) = 0,\ t_0 \leq t \leq T. \tag{2.5}$$

where $\alpha > 0$ and D^{α} is the Riemann-Liouville fractional derivative operator [15].

For Equation (2.5) the standard finite-difference scheme is written as

$$\Delta t^{-\alpha} \sum_{j=0}^{N} \omega_j^{\alpha} y(t_{k-j}) = f(t_k, y(t_k)), k = 1, 2, 3, \dots \tag{2.6}$$

where $t_k = k\Delta t$ and ω_j^{α} are the Grunwald-Letnikov coefficients.

When the derivative approximations are substituted in Equation (1.1), following standard finite-difference scheme for time-spatial fractional generalized NWS equation is obtained.

$$\sum_{j=0}^{k+1} \omega_j^{\alpha} u_m^{k-j+1} = \frac{\Delta t^{\alpha}}{\Delta x^{\gamma}} \sum_{i=0}^{m+1} \omega_i^{\gamma} u_{m-i+1}^{k} + \Delta t^{\alpha} u_m^k - \Delta t^{\alpha} (u_m^k)^p \tag{2.7}$$

2.2. Non-standard discretization

Non-standard finite-difference rules for ODEs or PDEs were introduced by Mickens [14]. Some basic rules of non-standard finite-difference for ODEs are given in this subsection. Additionally, similar rules exist for PDEs too.

In investigating non-standard finite-difference discretization for ODE in form

$$\frac{dy}{dt} = f(t, y(t)). \tag{2.8}$$

The non-standard discrete derivative is

$$\frac{dy}{dt} = \frac{y_{k+1} - y_k}{\varphi(\Delta t, \lambda)}. \tag{2.9}$$

where φ is a function of the step size Δt, φ has the following given property:

$$\Delta t \to 0,\ \varphi(\Delta t, \lambda) = \Delta t + O(\Delta t^2). \tag{2.10}$$

$\varphi(\Delta t, \lambda)$ denominator functions such as Δt ,$\frac{\sin(\lambda \Delta t)}{\lambda}$,$\frac{e^{\lambda \Delta t}-1}{\lambda}$, $1 - e^{-\Delta t}$ are frequently used in non-standard schemes. In non-standard finite-difference schemes, the nonlinear terms are replaced by non-local representations. Some examples are given below:

$y^2 \approx y_k(\frac{y_{k-1}+y_{k+1}}{2})$, $y^2 \approx y_k y_{k+1}$, $y^3 \approx y_k^2 y_{k+1}$, $y^3 \approx y_{k-1} y_k y_{k+1}$,
$y^2 \approx y_k(\frac{y_{k+1}+y_k+y_{k-1}}{3})$, $y^3 \approx (\frac{y_k+y_{k-1}}{2})^2 y_{k+1}$.
Where $\Delta t = \frac{T}{K}$, $t_k = k\Delta t$, $k = 0, 1, \ldots,\ K \in \mathbb{Z}^+$.

Similarly, non-standard discretization of partial derivatives are given as follow

$$\frac{\partial f(x,t)}{\partial x} = \frac{f_{m+1}^k - f_{m-1}^k}{\varphi(\Delta x, \mu)} \tag{2.11}$$

$$\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{f_m^{k+1} + f_m^k - f_m^{k-1}}{\psi(\Delta t, \lambda)} \tag{2.12}$$

Here $x_m = m\Delta x$, $t_k = k\Delta t$, $f(x_m, t_k) = f_m^k$, $m = 0, 1, \ldots,\ M \in \mathbb{Z}^+$, $k = 0, 1, \ldots\ ,\ K \in \mathbb{Z}^+$.
Δx and Δt have the following features

$$\varphi(\Delta x, \mu) = \Delta x + O(\Delta x^2),\ \Delta x \to 0 \tag{2.13}$$

$$\psi(\Delta t, \lambda) = \Delta t^2 + O(\Delta t^4),\ \Delta t \to 0. \tag{2.14}$$

Nonlocal representations for the terms u, u^2 in Equation 1.1 are

$$u_m^k = 2u_m^k - u_m^k \to 2u_m^k - u_m^{k+1}, (u_m^k)^p \to (u_m^k)^{p-1} u_m^{k+1}, u_m^k = \frac{u_{m+1}^k + u_{m-1}^k}{2}. \tag{2.15}$$

The time-fractional derivative and the spatial-fractional derivative will be represented by the following difference approximations:

$$\frac{\partial^\alpha u}{\partial t^\alpha} \approx \frac{1}{\varphi(\Delta t, \lambda)^\alpha} \sum_{j=0}^{k+1} \omega_j^\alpha u_m^{k-j+1}, \frac{\partial^\gamma u}{\partial x^\gamma} \approx \frac{1}{\psi(\Delta x, \mu)^\gamma} \sum_{i=0}^{m+1} \omega_i^\gamma u_{m-i+1}^k \tag{2.16}$$

where $\varphi(\Delta t, \lambda)$, $\psi(\Delta x, \mu)$ are the denominator functions of Δt and Δx respectively. λ and μ are parameters that evaluated equilibrium points of the nonlinear terms in Equation 1.1.

Substituting nonlocal representations 2.15 and derivative approximations 2.16 in Equation 1.1, the NFSD scheme for the time-spatial fractional generalized NWS equation is obtained as follows.

$$\frac{1}{\varphi(\Delta t, \lambda)^\alpha} \sum_{j=0}^{k+1} \omega_j^\alpha u_m^{k-j+1} = \frac{1}{\psi(\Delta x, \mu)^\gamma} \sum_{i=0}^{m+1} \omega_i^\gamma u_{m-i+1}^k + 2u_m^k - u_m^{k+1} - (u_m^k)^{p-1} u_m^{k+1} \tag{2.17}$$

3. Stability and convergence conditions of SFDM and NSFDM for the time-spatial fractional generalized Newell-Whitehead-Segel equations

3.1. Stability and convergence conditions of SFDM for the time-spatial fractional generalized Newell-Whitehead-Segel equations

Remark 3.1 *Let ω_j^{α} is the Grunwald-Letnikov coefficient as given in 2.3. Then; for $0 < \alpha \leq 1$, $|\omega_j^{\alpha}| \leq \frac{1}{j}$ satisfies.*

Proof The following inequality can be written for the $\omega_j^{\alpha} = (1 - \frac{1+\alpha}{j})\omega_{j-1}^{\alpha}, \omega_0^{\alpha} = 1$ difference equation.

$$\begin{aligned}
|\omega_j^{\alpha}| &= |(1 - \frac{1+\alpha}{j})(1 - \frac{1+\alpha}{j-1})(1 - \frac{1+\alpha}{j-2}) \dots \alpha.1| \\
&= |(\frac{j-1}{j} - \frac{\alpha}{j})(\frac{j-2}{j-1} - \frac{\alpha}{j-1})(\frac{j-3}{j-2} - \frac{\alpha}{j-2}) \dots \alpha.1| \\
&\leq |\frac{j-1}{j}.\frac{j-2}{j-1}.\frac{j-3}{j-2} \dots \alpha.1| = \frac{\alpha}{j} \leq \frac{1}{j}
\end{aligned}$$

□

Remark 3.2 $|\omega_i^{\gamma}| \leq \frac{2}{i}$ *satisfies for $1 < \gamma \leq 2$ and $i = 1, 2, \dots N$.*

Proof It can be done like proof of Remark 3.1. □

To obtain stability condition for finite-difference method, firstly Equation (2.7) is edited according to unknown term. The standard finite-difference scheme (SFD scheme) can be written as:

$$u_m^{k+1} = (\alpha - \gamma R)u_m^k + R(u_{m+1}^k + \frac{(\gamma-1)\gamma}{2}u_{m-1}^k) + R\sum_{i=3}^{m+1}\omega_i^{\gamma}u_{m-i+1}^k - \sum_{j=2}^{k+1}\omega_j^{\alpha}u_m^{k-j+1} + \Delta t^{\alpha}u_m^k - \Delta t^{\alpha}(u_m^k)^p \quad (3.1)$$

here $R = \frac{\Delta t^{\alpha}}{\Delta x^{\gamma}}$. For the positivity of SFD scheme, it is enough condition that $(\alpha - \gamma R) \geq 0$ since other terms in Equation (3.1) are positive.

The SFD scheme is bounded assuming that the $R = \frac{\Delta t^{\alpha}}{\Delta x^{\gamma}} \leq \frac{\alpha}{\gamma}$ and $0 \leq u_m^k \leq \frac{1}{4}$ conditions, so the positivity condition is $= \frac{\Delta t^{\alpha}}{\Delta x^{\gamma}} \leq \frac{\alpha}{\gamma}$.

$$\begin{aligned}
|u_m^{k+1}| &\leq |(\alpha - \gamma R)u_m^k| + |R(u_{m+1}^k + \frac{(\gamma-1)\gamma}{2}u_{m-1}^k)| + |R\sum_{i=3}^{m+1}\omega_i^{\gamma}u_{m-i+1}^k| + |\sum_{j=2}^{k+1}\omega_j^{\alpha}u_m^{k-j+1}| + |\Delta t^{\alpha}u_m^k - \Delta t^{\alpha}(u_m^k)^p| \\
&\leq \frac{(\alpha - \gamma R)}{4} + \frac{R}{2} + \frac{R}{4}.\frac{2}{3} + \frac{1}{8} + \frac{\Delta t^{\alpha}}{4} \\
&\leq \frac{1}{4} - \frac{\gamma R}{4} + \frac{1}{4} + \frac{1}{12} + \frac{1}{8} + \frac{1}{4} = \frac{23}{24} - \frac{\gamma R}{4} \leq 1
\end{aligned}$$

The SFD scheme for the time-spatial fractional generalized NWS equation under the positivity and bounded conditions is consistent. When the conditions $R = \frac{\Delta t^{\alpha}}{\Delta x^{\gamma}} \leq \frac{\alpha}{\gamma}$ and $0 \leq u_m^k \leq \frac{1}{4}$ are satisfied, CFL theorems [8, 9] are provided for stability and convergence. In this case, the SFD scheme for the time-spatial fractional generalized NWS equation is both convergent and stable.

3.2. Stability and convergence conditions of NSFDM for the time-spatial fractional generalized Newell-Whitehead-Segel equations

From Equation (2.17), the non-standard finite-difference scheme (NSFD scheme) can be written as

$$u_m^{k+1} = \frac{R(u_{m+1}^k + \frac{\gamma(\gamma-1)}{2}u_{m-1}^k) + 2\varphi(\Delta t,\lambda)^\alpha u_m^k + (\alpha-\gamma R)u_m^k + R\sum_{i=3}^{m+1}\omega_i^\gamma u_{m-i+1}^k - \sum_{j=2}^{k+1}\omega_j^\alpha u_m^{k-j+1}}{1+\varphi(\Delta t,\lambda)^\alpha + \varphi(\Delta t,\lambda)^\alpha (u_m^k)^{p-1}} \tag{3.2}$$

here $R = \frac{\varphi(\Delta t,\lambda)^\alpha}{\psi(\Delta x,\mu)^\gamma}$.

For the positivity of the NSFD scheme, condition $(\alpha-\gamma R)\geq 0 \Leftrightarrow R = \frac{\varphi(\Delta t,\lambda)^\alpha}{\psi(\Delta x,\mu)^\gamma} \leq \frac{\alpha}{\gamma}$ must be satisfied. For simplicity, Equation 3.2 can be written as follows

$$u_m^{k+1} = \frac{R(u_{m+1}^k + \frac{\gamma(\gamma-1)}{2}u_{m-1}^k) + 2\varphi(\Delta t,\lambda)^\alpha u_m^k + A(\alpha,\gamma)}{1+\varphi(\Delta t,\lambda)^\alpha + \varphi(\Delta t,\lambda)^\alpha (u_m^k)^{p-1}} \tag{3.3}$$

where $A(\alpha,\gamma) = (\alpha-\gamma R)u_m^k + R\sum_{i=3}^{m+1}\omega_i^\gamma u_{m-i+1}^k - \sum_{j=2}^{k+1}\omega_j^\alpha u_m^{k-j+1}$.

Assuming these conditions $R = \frac{\varphi(\Delta t,\lambda)^\alpha}{\psi(\Delta x,\mu)^\gamma} \leq \frac{\alpha}{\gamma}$ and $0 \leq u_m^k \leq \frac{1}{2}$,

$$R(u_{m+1}^k + \frac{\gamma(\gamma-1)}{2}u_{m-1}^k) + 2\varphi(\Delta t,\lambda)^\alpha u_m^k + A(\alpha,\gamma) \leq 1+\varphi(\Delta t,\lambda)^\alpha \tag{3.4}$$

can be written.

The Inequality 3.4 is proved by the contradiction method as follows.

Since $R(u_{m+1}^k + \frac{\gamma(\gamma-1)}{2}u_{m-1}^k) \leq 1$, it is sufficient to show $2\varphi(\Delta t,\lambda)^\alpha u_m^k + A(\alpha,\gamma) \leq \varphi(\Delta t,\lambda)^\alpha$.

Suppose that $2\varphi(\Delta t,\lambda)^\alpha u_m^k + A(\alpha,\gamma) > \varphi(\Delta t,\lambda)^\alpha$, for $\forall\alpha\in(0,1], \forall\gamma\in(1,2]$. If $\alpha=1, \gamma=2, R=\frac{1}{2}$ is chosen, $A(\alpha,\gamma)=0$ is found. Then $2\varphi(\Delta t,\lambda)^\alpha u_m^k > \varphi(\Delta t,\lambda)^\alpha$ can be written. So, $u_m^k > \frac{1}{2}$ is obtained. This contradicts the condition $0\leq u_m^k \leq \frac{1}{2}$. Thus,

$$R(u_{m+1}^k + \frac{\gamma(\gamma-1)}{2}u_{m-1}^k) + 2\varphi(\Delta t,\lambda)^\alpha u_m^k + A(\alpha,\gamma) \leq 1+\varphi(\Delta t,\lambda)^\alpha \leq 1+\varphi(\Delta t,\lambda)^\alpha + \varphi(\Delta t,\lambda)^\alpha (u_m^k)^{p-1} \tag{3.5}$$

can be written. If the Inequality 3.5 is regulated, it is shown that the NSFD scheme is bounded.

$$u_m^{k+1} = \frac{R(u_{m+1}^k + \frac{\gamma(\gamma-1)}{2}u_{m-1}^k) + 2\varphi(\Delta t,\lambda)^\alpha u_m^k + A(\alpha,\gamma)}{1+\varphi(\Delta t,\lambda)^\alpha + \varphi(\Delta t,\lambda)^\alpha (u_m^k)^{p-1}} \leq 1 \tag{3.6}$$

Consequently, the NSFD scheme for the time-spatial fractional generalized NWS equation under the positivity and boundedness conditions is consistent. When the conditions $R = \frac{\varphi(\Delta t,\lambda)^\alpha}{\psi(\Delta x,\mu)^\gamma} \leq \frac{\alpha}{\gamma}$ and $0 \leq u_m^k \leq \frac{1}{2}$ are satisfied, CFL theorems [8, 9] are provided for stability and convergence. In this case, the NSFD scheme for the time-spatial fractional generalized NWS equation is both convergent and stable.

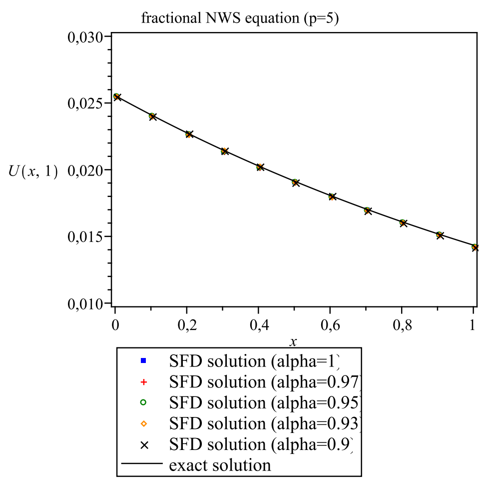
4. Applications

In this section, SFD and NSFD methods are applied to obtain numerical solutions of the time-spatial fractional generalized Newell-Whitehead-Segel equations and compared with using tables when using different denominator functions. To illustrate, three examples are solved numerically by using both SFD and NSFD methods. Also, results are compared with using tables.

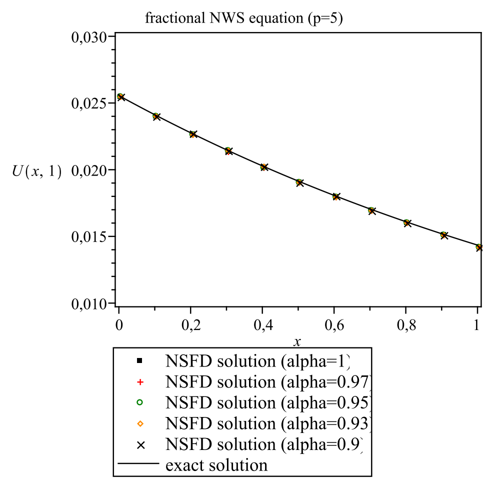
4.1. Example 1

If $0 < \alpha \leq 1, \gamma = 2, p = 5, \xi(x) = \frac{1}{(1+e^{10+\frac{2\sqrt{3}}{3}x})^{\frac{1}{2}}}$, $\zeta(t) = \frac{1}{(1+e^{10-\frac{8}{3}t})^{\frac{1}{2}}}$, $\Omega(t) = \frac{1}{(1+e^{10-\frac{8}{3}t+\frac{2\sqrt{3}}{3}})^{\frac{1}{2}}}$ is taken in Equation 1.1, exact solution of the time-fractional generalized NWS equation for $\alpha = 1$ is

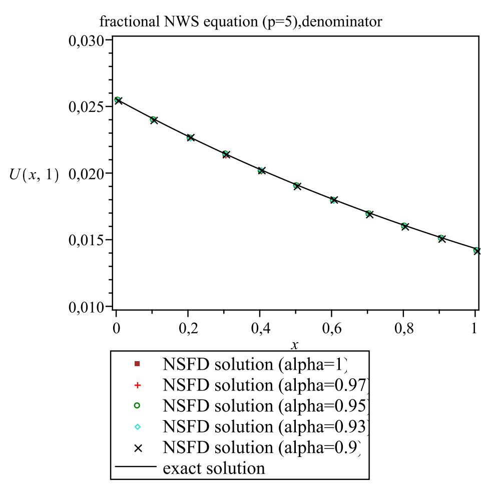
$u(x,t) = \frac{1}{(1+e^{10-\frac{8}{3}t+\frac{2\sqrt{3}}{3}x})^{\frac{1}{2}}}$. The comparison of solutions of standard and non-standard finite-difference with the exact solution for $p = 5$ is shown in Figure 1 taking $0 < x < 1, t = 1$.



(a) standard finite-difference solutions



(b) non-standard finite-difference solutions



(c) non-standard finite-difference solutions with denominator function

Figure 1. Comparison with the exact solution for $p = 5$ in Example 1

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If $\psi(\Delta x) = \Delta x, \varphi_1(\Delta t) = \Delta t, \varphi_2(\Delta t) = 1 - e^{-\Delta t}, \Delta x = \frac{1}{10}, \Delta t = \frac{1}{1000}$ values are taken in SFD scheme 3.1 and NSFD scheme 3.2, the error values for $\alpha = 1$ are given in the Table 1.

Table 1. Error values of finite-difference methods for $p = 5$ in Example 1.

x_k	$\|SFD - Exact\|$	$\|NSFD - Exact\|$ $\varphi_1(\Delta t)$,$\psi(\Delta x)$	$\|NSFD - Exact\|$ $\varphi_2(\Delta t)$,$\psi(\Delta x)$
0	0	0	0
0.1	0.00000072644	0.00000108249	0.00000048044
0.2	0.00000126311	0.00000188265	0.00000083579
0.3	0.00000162254	0.00000241901	0.00000107415
0.4	0.00000181622	0.00000270828	0.00000120283
0.5	0.00000185436	0.00000276563	0.00000122852
0.6	0.00000174617	0.00000260462	0.00000115716
0.7	0.00000149983	0.00000223742	0.00000099415
0.8	0.00000112259	0.00000167481	0.00000074423
0.9	0.00000062080	0.00000092626	0.00000041163
1	0	0	0

4.2. Example 2

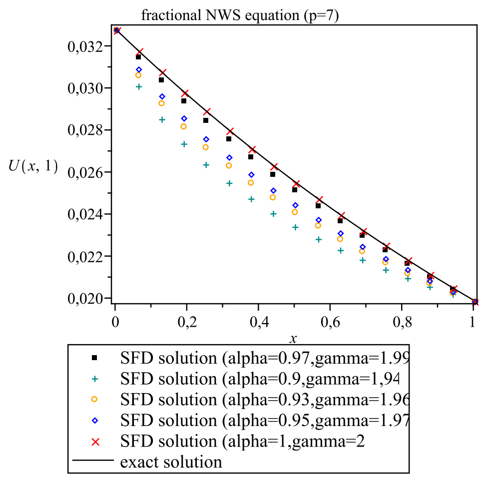
If $0 < \alpha \leq 1, 1 < \gamma \leq 2, p = 7, \xi(x) = \frac{1}{(1+e^{\frac{3x}{2}+14})^{\frac{1}{3}}}$, $\zeta(t) = \frac{1}{(1+4e^{\frac{-15}{4}t+14})^{\frac{1}{3}}}, \Omega(t) = \frac{1}{(1+e^{\frac{3}{2}-\frac{15}{4}t+14})^{\frac{1}{3}}}$ is taken in Equation 1.1, exact solution of the time-spatial fractional generalized NWS equation for $\alpha = 1, \gamma = 2$ is $u(x,t) = \frac{1}{(1+e^{\frac{3}{2}x-\frac{15}{4}t+14})^{\frac{1}{3}}}$.

When $\psi(\Delta x) = \Delta x, \varphi_1(\Delta t) = \Delta t, \varphi_2(\Delta t) = e^{\Delta t} - 1$,to satisfy the stability condition Δx and Δt are chosen $\frac{1}{16}$ and $\frac{1}{1000}$ respectively, the error values for $\alpha = 1, \gamma = 2$ are given in the Table 2.

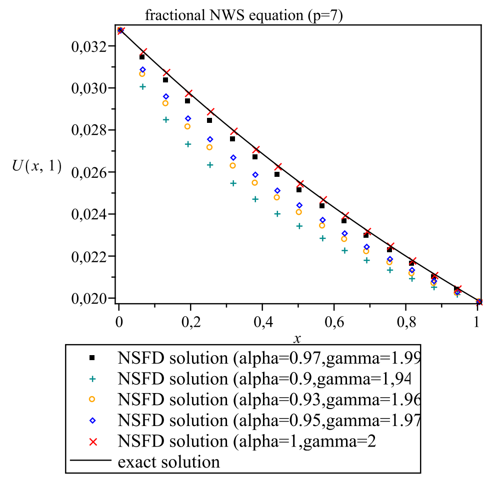
Table 2. Error values of finite-difference methods for $p = 7$ in Example 2.

x_k	$\|SFD - Exact\|$	$\|NSFD - Exact\|$ $\varphi_1(\Delta t)$,$\psi(\Delta x)$	$\|NSFD - Exact\|$ $\varphi_2(\Delta t)$,$\psi(\Delta x)$
0	0	0	0
0.125	0.00000111477	0.00000151042	0.00000059860
0.25	0.00000186702	0.00000252965	0.00000100255
0.375	0.00000228196	0.00000309183	0.00000122535
0.5	0.00000238205	0.00000322743	0.00000127908
0.625	0.00000218733	0.00000296358	0.00000117451
0.75	0.00000171547	0.00000232426	0.00000092113
0.875	0.00000098189	0.00000133035	0.00000052723
1	0	0	0

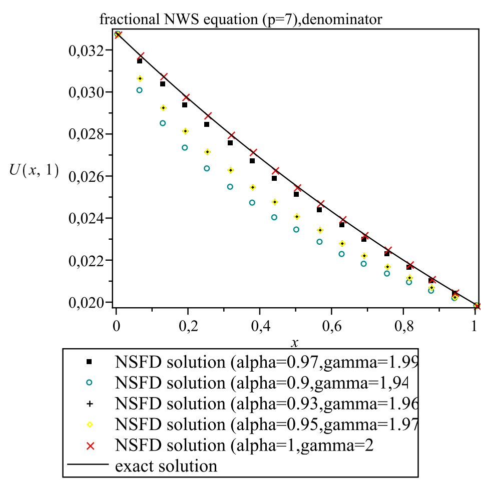
The comparison of solutions of standard and non-standard finite-difference with the exact solution for $p = 7$ is shown in Figure 2 taking $0 < x < 1, t = 1$.



(a) standard finite-difference solutions



(b) non-standard finite-difference solutions



(c) non-standard finite-difference solutions with denominator function

Figure 2. Comparison with the exact solution for $p = 7$ in Example 2.

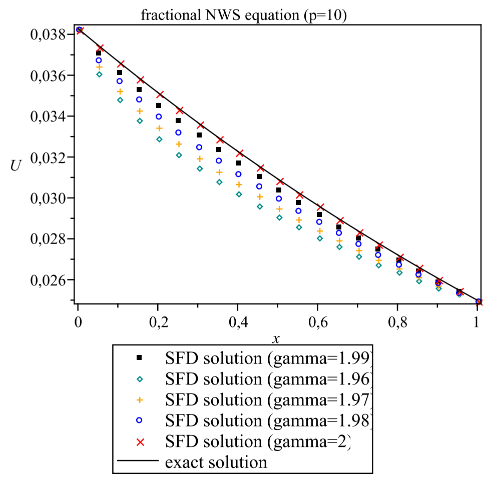
4.3. Example 3

If $\alpha = 1, 1 < \gamma \le 2$, $p = 10$,$\xi(x) = \frac{1}{(1+e^{\frac{9}{\sqrt{22}}x+20})^{\frac{2}{9}}}$, $\zeta(t) = \frac{1}{(1+e^{-\frac{117}{22}t+20})^{\frac{2}{9}}}$, $\Omega(t) = \frac{1}{(1+e^{\frac{9}{\sqrt{22}}-\frac{117}{22}t+20})^{\frac{2}{9}}}$ is taken in Equation 1.1, exact solution of the spatial-fractional generalized NWS equation for $\gamma = 2$ is

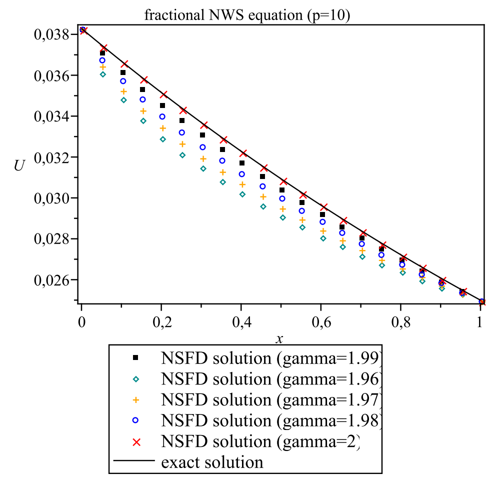
$$u(x,t) = \frac{1}{(1+e^{\frac{9}{\sqrt{22}}x-\frac{117}{22}t+20})^{\frac{2}{9}}} .$$

The comparison of solutions of standard and non-standard finite-difference with the exact solution for $p = 10$ is shown in Figure 3 taking $0 < x < 1, t = 1$.

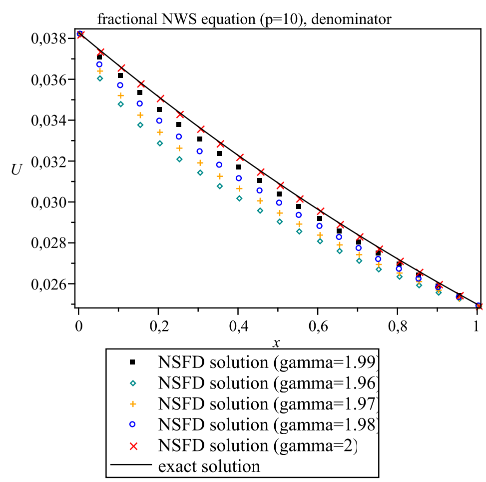
As denominator functions, $\psi_1(\Delta x) = \Delta x, \psi_2(\Delta x) = \sin(\Delta x), \varphi_1(\Delta t) = \Delta t, \varphi_2(\Delta t) = e^{\Delta t} - 1$ are taken, $\Delta x = \frac{1}{20}$ and $\Delta t = \frac{1}{1000}$ are chosen in SFD scheme 3.1 and NSFD scheme 3.2. The comparison of the numerical solutions of example 3 for $\gamma = 2$ is shown in Table 3.



(a) standard finite-difference solutions



(b) non-standard finite-difference solutions



(c) non-standard finite-difference solutions with denominator function

Figure 3. Comparison with the exact solution for $p = 10$ in Example 3.

5. Conclusions and recommendations

In this study, the stability and convergence conditions of the finite-difference methods proposed for the time-spatial fractional generalized NWS equation are determined using CFL theorems. Then, the $p = 5, 7, 10$ cases of the given equation are examined to show the solution procedures.

For $p = 5$, taking $\gamma = 2$, the exact solution of the time-fractional NWS equation when $\alpha = 1$ is compared with the proposed finite-difference solutions, and error values are given in Table 1.

For $p = 7$, the exact solution of the time-spatial fractional NWS equation, when $\alpha = 1, \gamma = 2$ is compared with the proposed finite-difference solutions and the error values are given in Table 2.

For $p = 10$, taking $\alpha = 1$, the exact solution of the spatial-fractional NWS equation, when $\gamma = 2$ is compared with the proposed finite-difference solutions and the error values are given in Table 3.

From the tables, although high values of p are used, it is observed that the maximum errors of SFDM and NSFDM are around 10^{-6}.

Table 3. Error values of finite-difference methods for $p = 10$ in Example 3.

x_k	$\lvert SFD - Exact \rvert$	$\lvert NSFD - Exact \rvert$ $\varphi_1(\Delta t)$,$\psi(\Delta x)$	$\lvert NSFD - Exact \rvert$ $\varphi_2(\Delta t)$,$\psi(\Delta x)$
0	0	0	0
0.1	0.00000100791	0.00000206326	0.00000098334
0.2	0.00000176374	0.00000361054	0.00000172075
0.3	0.00000227959	0.00000466650	0.00000222404
0.4	0.00000256644	0.00000525374	0.00000250390
0.5	0.00000263470	0.00000539343	0.00000257050
0.6	0.00000249365	0.00000510476	0.00000243290
0.7	0.00000215213	0.00000440561	0.00000209971
0.8	0.00000161794	0.00000331209	0.00000157852
0.9	0.00000089842	0.00000183913	0.00000087653
1	0	0	0

Respectively, Graph 1,2,3 show the comparison of the numerical solutions of the Examples 1, 2, 3 with the exact solutions. In each case $p = 5, 7, 10$, it can be seen from both tables and graphs that the numerical results for different α and γ values are compatible with the accuracy of the stability and the convergence conditions obtained. Also, it has been determined that the numerical solutions of the time-fractional NWS equation are located very close to the exact solution for $\gamma = 2$. In addition, the numerical results for the spatial-fractional NWS and the time-spatial fractional NWS equation are not located near the exact solution and show a stable spread. It is seen that the NWS equation is quite compatible with the time-fractional derivative, but not with the spatial-fractional derivative even if it shows a stable spread. The behaviour of the solution gives a stable spread when changing the spatial-fractional derivative and is different from the effect of the time-fractional derivative. Although the p values increased for the time-spatial fractional NWS equation, it is determined that the numerical solutions obtained a stable and consistent order around the exact solution using both methods. The standard finite-difference method gives better results when the denominator function is used in the classical sense in the non-standard finite-difference method. Additionally, it is concluded that the non-standard finite-difference method gives better quality results when using trigonometric and exponential functions as denominator functions. Therefore, the non-standard finite-difference method with various denominator functions is preferable and easy to implement for solving the time-spatial fractional generalized NWS equation.

Consequently, using different denominator functions shows the effectiveness of the proposed NSFD method for the time-spatial fractional generalized NWS equation. Also, the stability and convergence conditions obtained for SFDM and NSFDM show that the methods are conditionally stable for the related equation in general form and it has been determined that the numerical results and conditional stability for the time-spatial fractional generalized NWS equation overlap with each other. The study can be carried to a more general form by taking different values for the coefficients of the time-spatial fractional generalized NWS equation.

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