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Some convergence, stability, and data dependence results for K^* iterative method of quasi-strictly contractive mappings

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Abstract: In a recent paper, Yu et al. obtained convergence and stability results of the K^* iterative method for quasi-strictly contractive mappings [An iteration process for a general class of contractive-like operators: Convergence, stability and polynomiography. AIMS Mathematics 2021; 6 (7): 6699-6714.]. To guarantee these convergence and stability results, the authors imposed some strong conditions on parametric control sequences which are used in the K^* iterative method. The aim of the presented work is twofold: (a) to recapture the aforementioned results without any restrictions imposed on the mentioned parametric control sequences (b) to complete the work of Yu et al. by adding a result regarding the data dependency of the fixed points of quasi-strictly contractive mappings. We also furnish some illustrative examples to support our results. Our work can be considered an important refinement and complement of the work of Yu et al.

Key words: Iterative methods, fixed points, convergence, stability, data dependency, quasi-strictly contractive mappings

1. Preliminaries and background

The fixed point iteration approach plays an important role in finding the solution of many problems which are not always solved with analytical methods in various fields of science. The fixed point iteration algorithms studies start with the Picard iteration [41]. The Picard iteration algorithm can often be used to approximate the fixed points of contraction mappings, whereas Krasnoselskii [32], Mann [35], and Ishikawa [27] iteration algorithms are used to approximate the fixed points of nonexpansive/noncontractive mappings. These iteration algorithms have been studied by many researchers on different subjects. Afterward, many iteration algorithms were defined and their qualifier features such as convergence, stability, data dependency, and convergence rate have been extensively studied (see [12–14, 19, 21, 25, 26, 33, 48]). For the convenience of the reader, we recall below some of the commonly used iteration algorithms and the relationships among them.

Let N be a real normed space, $\emptyset \neq S \subset N$ a convex and closed set, $T : S \to S$ a mapping and $F_T = \{x^* : x^* = Tx^*\}$ be the set of fixed points of T.

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In 2000, Noor [36] introduced a Noor iteration algorithm as follows:

$$\begin{cases} n_0^1 \in S, \\ n_{k+1}^1 = \left(1-\mu_k^1\right) n_k^1 + \mu_k^1 T n_k^2, \\ n_k^2 = \left(1-\mu_k^2\right) n_k^1 + \mu_k^2 T n_k^3, \\ n_k^3 = \left(1-\mu_k^3\right) n_k^1 + \mu_k^3 T n_k^1, \text{ for all } k \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\left\{\mu_k^i\right\}_{k=0}^{\infty} \subseteq [0,1]$ for $i = 1,2,3$ are real sequences. If (i) $\mu_k^3 = 0$ for all $k \in \mathbb{N}$, (ii) $\mu_k^2 = \mu_k^3 = 0$ for all $k \in \mathbb{N}$, (iii) $\mu_k^2 = \mu_k^3 = 0$ and $\mu_k^1 = \mu^1$ for all $k \in \mathbb{N}$, (iv) $\mu_k^1 = 1$ and $\mu_k^2 = \mu_k^3 = 0$ for all $k \in \mathbb{N}$, and (v) $\mu_k^1 = 1$ and $\mu_k^3 = 0$ for all $k \in \mathbb{N}$, then (1.1) reduces to the Ishikawa, Mann, Krasnoselskii, Picard and normal-S [46] iteration algorithms, respectively.

The following is known as S-iteration algorithm [1]:

$$\begin{cases} s_0^1 \in S, \\ s_{k+1}^1 = (1-\mu_k^1) T s_k^1 + \mu_k^1 T s_k^2, \\ s_k^2 = (1-\mu_k^2) s_k^1 + \mu_k^2 T s_k^1, \text{ for all } k \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\left\{\mu_k^i\right\}_{k=0}^{\infty} \subseteq [0,1]$ for $i = 1,2$ are real sequences. If (i) $\mu_k^1 = 1$, $\mu_k^2 = 0$ for all $k \in \mathbb{N}$ and (ii) $\mu_k^1 = 1$ for all $k \in \mathbb{N}$, then (1.2) reduces to the Picard and normal-S iteration algorithms, respectively, but it is independent from Krasnoselskii, Mann, Ishikawa and Noor iteration algorithms.

The following iteration algorithms are called SP [42] and CR [8] iteration algorithms, respectively:

$$\begin{cases} p_0^1 \in S, \\ p_{k+1}^1 = \left(1-\mu_k^1\right) p_k^2 + \mu_k^1 T p_k^2, \\ p_k^2 = \left(1-\mu_k^2\right) p_k^3 + \mu_k^2 T p_k^3, \\ p_k^3 = \left(1-\mu_k^3\right) p_k^1 + \mu_k^3 T p_k^1, \text{ for all } k \in \mathbb{N}, \end{cases} \tag{1.3}$$

and

$$\begin{cases} c_0^1 \in S, \\ c_{k+1}^1 = (1-\mu_k^1) c_k^2 + \mu_k^1 T c_k^2, \\ c_k^2 = (1-\mu_k^2) T c_k^1 + \mu_k^2 T c_k^3, \\ c_k^3 = (1-\mu_k^3) c_k^1 + \mu_k^3 T c_k^1, \text{ for all } k \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\left\{\mu_k^i\right\}_{k=0}^{\infty} \subseteq [0,1]$ for $i = 1,2$ are real sequences. If (i) $\mu_k^1 = \mu_k^3 = 0$ and $\mu_k^2 = 1$ for all $k \in \mathbb{N}$, (ii) $\mu_k^1 = 0$ for all $k \in \mathbb{N}$, and (iii) $\mu_k^1 = 0$ and $\mu_k^2 = 1$ for all $k \in \mathbb{N}$, then (1.4) reduces to the Picard, S, and normal-S iteration algorithms, respectively. If (i) $\mu_k^3 = 0$ for all $k \in \mathbb{N}$, (ii) $\mu_k^2 = \mu_k^3 = 0$ for all $k \in \mathbb{N}$, (iii) $\mu_k^2 = \mu_k^3 = 0$ and $\mu_k^1 = \mu^1$ for all $k \in \mathbb{N}$, (iv) $\mu_k^1 = 1$ and $\mu_k^2 = \mu_k^3 = 0$ for all $k \in \mathbb{N}$, and (v) $\mu_k^1 = 1$ and $\mu_k^3 = 0$ for all $k \in \mathbb{N}$, then (1.3) reduces to the Thianwan [47], Mann, Krasnoselskii, Picard, and normal-S iteration algorithms, respectively. Furthermore, if $\mu_k^1 = 1$ for all $k \in \mathbb{N}$ in (1.3), then we obtain the following iteration algorithm:

$$\begin{cases} p_0^1 \in S, \\ p_{k+1}^1 = T p_k^2, \\ p_k^2 = \left(1-\mu_k^2\right) p_k^3 + \mu_k^2 T p_k^3, \\ p_k^3 = \left(1-\mu_k^3\right) p_k^1 + \mu_k^3 T p_k^1, \text{ for all } k \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\left\{\mu_k^i\right\}_{k=0}^{\infty} \subseteq [0,1]$ for $i = 1,2$ are real sequences. This algorithm was introduced in [31]. We will hereinafter call it the SP* iteration algorithm.

Almost all of the iterations which are mentioned above can be used to solve an operator equation, which is generated by a particular contractive type mapping. In such a case, the following concepts of convergence rate and stability emerge as two very important criteria that allow us to decide which iteration algorithm is the most appropriate to solve the operator equation in question.

Definition 1.1 *(see [7]) Let* $\left\{\Theta_n^{(1)}\right\}_{n=0}^{\infty}, \left\{\Theta_n^{(2)}\right\}_{n=0}^{\infty}$ *be two sequences such that* $\lim_{n\to\infty}\Theta_n^{(1)} = \Theta_1$ *and* $\lim_{n\to\infty}\Theta_n^{(2)} = \Theta_2$. *If*

$$\lim_{n\to\infty}\frac{\left\|\Theta_n^{(1)}-\Theta_1\right\|}{\left\|\Theta_n^{(2)}-\Theta_2\right\|}=0,$$

then it is said that $\left\{\Theta_n^{(1)}\right\}_{n=0}^{\infty}$ *converges faster than* $\left\{\Theta_n^{(2)}\right\}_{n=0}^{\infty}$.

Definition 1.2 *(see [3]) Let* $\left\{\Theta_n^{(1)}\right\}_{n=0}^{\infty}, \left\{\Theta_n^{(2)}\right\}_{n=0}^{\infty}$ *be two iterative sequences such that* $\lim_{n\to\infty}\Theta_n^{(1)} = \lim_{n\to\infty}\Theta_n^{(2)} = \Theta^*$. *Suppose that the following error estimates*

$$\left\|\Theta_n^{(1)}-\Theta^*\right\|\le\kappa_n^{(1)} \text{ and } \left\|\Theta_n^{(2)}-\Theta^*\right\|\le\kappa_n^{(2)}, \text{ for all } n\in\mathbb{N}$$

are available (and these estimates are the best possible, see [4]) where $\left\{\kappa_n^{(1)}\right\}_{n=0}^{\infty}, \left\{\kappa_n^{(2)}\right\}_{n=0}^{\infty}$ *are two sequences of positive numbers (converging to zero). If* $\left\{\kappa_n^{(1)}\right\}_{n=0}^{\infty}$ *converges faster than* $\left\{\kappa_n^{(2)}\right\}_{n=0}^{\infty}$ *(in the sense of Definition 1.1), then it is said that* $\left\{\Theta_n^{(1)}\right\}_{n=0}^{\infty}$ *converges to* Θ^* *faster than* $\left\{\Theta_n^{(2)}\right\}_{n=0}^{\infty}$.

Definition 1.3 *(see [24]) Let* $T : S \to S$ *be a mapping with a fixed point* p, F *a function,* $\{\Theta_n^*\}_{n=0}^{\infty} \subset S$ *an arbitrary sequence, and* $\{\Theta_n\}_{n=0}^{\infty}$ *be an iterative sequence produced by* $\Theta_{n+1} = F(T, \Theta_n)$, $n \in \mathbb{N}$ *for an initial guess* $\Theta_0 \in S$. *Assume that* $\{\Theta_n\}_{n=0}^{\infty}$ *converges to* p. *If* $\lim_{n\to\infty}\left\|F(T,\Theta_n^*)-\Theta_{n+1}^*\right\| = 0 \Leftrightarrow \lim_{n\to\infty} y_n = p$, *then one says that* $\{\Theta_n\}_{n=0}^{\infty}$ *is stable w.r.t.* T.

Working with an iteration algorithm that has a higher convergence rate is very important to save time in solving problems that arise in applied and computational research areas. Therefore, many studies have been conducted on the comparison of convergence rates among iteration algorithms. As a result, a very large literature has emerged in this context. Rhoades [43] proved that the Ishikawa iteration algorithm converges to the fixed points of the increasing functions faster than the Mann iteration algorithm and vice versa for the decreasing functions. Agarwal et al. [1] showed that the Picard and S iteration algorithms have the same convergence rate when approximating the fixed point of contraction mappings and both are faster than the classical Mann, Ishikawa, and Noor iteration algorithms for the same class of mappings. Sahu [46] proposed the normal-S iteration algorithm, which is better (i.e. simpler and faster) than the S iteration algorithm for contraction mappings. Chugh et al. [8] introduced the CR iteration algorithm and proved that this algorithm is equivalent to the Picard, Mann, Ishikawa, Noor, SP, and S iteration algorithms for a general class of contractive mappings and faster than those algorithms for the same class of mappings. It was shown in [30] that the CR iteration algorithm is faster than an S* [28] iteration algorithm for contraction mappings and a data dependence result for the fixed point points of contractive-like operators via the CR iteration algorithm was presented in [29]. Other features, such as stability and data dependency, are important to examine in the study of iteration algorithms. They also give us important information about how accurately an iteration algorithm performs the required computations for a particular problem. In the last thirty years or so, a very large literature has emerged on the stability and data dependency for various iteration algorithms of different classes of contractive mappings, see, e.g., [5, 6, 9–11, 15–18, 20, 22, 23, 37–40, 44, 45, 50] , and references contained therein.

Inspired by the studies mentioned above, Ullah and Arshad [49] introduced a K* iteration algorithm as follows:

$$\begin{cases} u_0^1 \in S \\ u_{k+1}^1 = Tu_k^2, \\ u_k^2 = T\left(\left(1-\mu_k^1\right)u_k^3 + \mu_k^1 Tu_k^3\right), \\ u_k^3 = \left(1-\mu_k^2\right)u_k^1 + \mu_k^2 Tu_k^1, \text{ for all } k \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\left\{\mu_k^i\right\}_{k=0}^{\infty} \subseteq [0,1]$ for $i = 1,2$ are real sequences. It was shown in [49] that the iterative sequence $\left\{u_k^1\right\}_{k=0}^{\infty}$ generated by (1.6) with the control condition $\sum\limits_{k=0}^{\infty} \mu_k^1 = \infty$ strongly converges to the unique fixed point of a contraction mapping T and is $T-$stable under the same condition. In a very recent paper [51] by Yu et al., the iteration algorithm (1.6) was investigated for a general class of mappings called quasistrictly contractive mappings satisfying the following condition:

$$\|u^* - Tu\| \leq \delta \|u^* - u\|, \ \delta \in [0,1), \ u^* \in F_T, \ \forall u \in N. \tag{1.7}$$

More precisely, they proved the following results:

Theorem 1.4 *Consider a real normed* $(N, \|\cdot\|)$ *and a mapping* $T : N \to N$ *with a fixed point* x^* *satisfying (1.7). Let* $\left\{p_k^1\right\}_{k=0}^{\infty}, \left\{u_k^1\right\}_{k=0}^{\infty}$ *be defined by (1.5) and (1.6), respectively, where* $\left\{\mu_k^i\right\}_{k=0}^{\infty} \subseteq [0,1]$ *for* $i = 1,2$ *are real sequences and* $\sum\limits_{k=0}^{\infty} \mu_k^1 \mu_k^2 = \infty$. *Then, (i)* $\left\{u_k^1\right\}_{k=0}^{\infty}$ *converges strongly to* x^* *[51, Theorem 3.1], (ii)* $\left\{u_k^1\right\}_{k=0}^{\infty}$ *is* $T-$*stable [51, Theorem 3.2], (iii) the convergence of* $\left\{u_k^1\right\}_{k=0}^{\infty}$ *to* x^* *is faster than that of* $\left\{p_k^1\right\}_{k=0}^{\infty}$ *[51, Theorem 3.3].*

We will reprove the results in (i)-(iii) of Theorem 1.4 in this study by removing the condition $\sum\limits_{k=0}^{\infty} \mu_k^1 \mu_k^2 = \infty$ that was used in the proofs of the corresponding results in [21]. We will also present a result on the data dependence of the fixed point of the quasi-strictly contractive mappings satisfying (1.7) by using the iteration algorithm (1.6). Some illustrative numerical examples are also provided to support our results. Our results can be considered a significant refinement and enhancement over the corresponding results of Ullah and Arshad [49] and Yu et al. [51].

2. Main results

Theorem 2.1 *Let* $(N, \|\cdot\|)$ *be a real normed space,* $T : N \to N$ *be a mapping which satisfies (1.7) and has a fixed point* u^*, *and* $\left\{u_k^1\right\}$ *be an iterative sequence which is produced by (1.6) with real sequences* $\left\{\mu_k^1\right\}, \left\{\mu_k^2\right\} \subseteq [0,1]$. *Then,* $\left\{u_k^1\right\}$ *converges strongly to* u^*.

Proof From (1.6) and (1.7), we have

$$\begin{aligned}
\left\|u_k^3-u^*\right\| &= \left\|(1-\mu_k^2)u_k^1+\mu_k^2 Tu_k^1-u^*\right\| \\
&= \left\|(1-\mu_k^2)u_k^1+\mu_k^2 u^*-\mu_k^2 u^*\mu_k^2 Tu_k^1-u^*\right\| \\
&= \left\|(1-\mu_k^2)(u_k^1-u^*)+\mu_k^2(Tu_k^1-u^*)\right\| \\
&\leq (1-\mu_k^2)\left\|u_k^1-u^*\right\|+\mu_k^2\left\|Tu_k^1-u^*\right\| \\
&\leq (1-\mu_k^2)\left\|u_k^1-u^*\right\|+\delta\mu_k^2\left\|u_k^1-u^*\right\| \\
&\leq (1-\mu_k^2+\delta\mu_k^2)\left\|u_k^1-u^*\right\|=(1-\mu_k^2(1-\delta))\left\|u_k^1-u^*\right\|.
\end{aligned}$$

Then,

$$\begin{aligned}
\left\|u_k^2-u^*\right\| &= \left\|T((1-\mu_k^1)u_k^3+\mu_k^1 Tu_k^3)-u^*\right\| \\
&= \delta\left\|(1-\mu_k^1)u_k^3+\mu_k^1 u^*-\mu_k^1 u^*+\mu_k^1 Tu_k^3-u^*\right\| \\
&\leq \delta\left\|(1-\mu_k^1)(u_k^3-u^*)+\mu_k^1(Tu_k^3-u^*)\right\| \\
&\leq \delta(1-\mu_k^1)\left\|u_k^3-u^*\right\|+\mu_k^1\left\|Tu_k^3-u^*\right\| \\
&\leq \delta(1-\mu_k^1)\left\|u_k^3-u^*\right\|+\delta\mu_k^1\left\|u_k^3-u^*\right\| \\
&= \delta[1-\mu_k^1+\delta\mu_k^1]\left\|u_k^3-u^*\right\| \\
&= \delta[1-\mu_k^1(1-\delta)][1-\mu_k^2(1-\delta)]\left\|u_k^1-u^*\right\|.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\left\|u_{k+1}^1-u^*\right\| &= \left\|Tu_k^2-u^*\right\|\leq\delta\left\|u_k^2-u^*\right\| \\
&\leq \delta^2[1-\mu_k^1(1-\delta)][1-\mu_k^2(1-\delta)]\left\|u_k^1-u^*\right\|.
\end{aligned}$$

.

If we apply induction to this inequality, we get

$$\left\|u_{k+1}^1-u^*\right\|\leq\delta^{2(k+1)}\prod_{n=0}^{k}\left[1-\mu_n^1(1-\delta)\right]\left[1-\mu_n^2(1-\delta)\right]\left\|u_0^1-u^*\right\|. \tag{2.1}$$

Since $\delta\in[0,1)$ and $0\leq\mu_k^1,\mu_k^2\leq 1$, $\forall k\in\mathbb{N}$, $1-\mu_k^1(1-\delta)<1$ and $1-\mu_k^2(1-\delta)<1$, $\forall k\in\mathbb{N}$. Therefore, we have

$$\left\|u_{k+1}^1-u^*\right\|\leq\delta^{2(k+1)}\left\|u_0^1-u^*\right\|,\ \forall k\in\mathbb{N}.$$

If we take the limit of the both side of the above inequality, we obtain $\lim_{k\longrightarrow\infty}\left\|u_{k+1}^1-u^*\right\|=0$. □

Theorem 2.2 *Let $(N,\|\cdot\|)$ be a real normed space, $T:N\to N$ be a mapping which satisfies (1.7) and has a fixed point u^*, and $\{u_k^1\}$ be an iterative sequence which is produced by (1.6) with real sequences $\{\mu_k^1\},\{\mu_k^2\}\subseteq[0,1]$. Then, $\{u_k^1\}$ is T-stable.*

Proof Let $\{\tau_k\}$ be an arbitrary sequence in N and $\varepsilon_k \in \mathbb{R}^+$ is in a form

$$\varepsilon_k = \|\tau_{k+1} - T\varsigma_k\|, \forall k \in \mathbb{N}, \tag{2.2}$$

where $\varsigma_k = T\left[\left(1-\mu_k^1\right)\gamma_k + \mu_k^1 T\gamma_k\right]$, $\gamma_k = \left(1-\mu_k^2\right)\tau_k + \mu_k^2 T\tau_k$, for all $k \in \mathbb{N}$ and admits $\lim_{k\longrightarrow\infty}\varepsilon_k = 0$, we will demonstrate that $\lim_{k\longrightarrow\infty}\tau_k = u^*$. Using (1.6), (1.7), (2.2), $\delta \in [0,1)$, and the facts $0 \le \mu_k^1, \mu_k^2 \le 1$, $1-\mu_k^1(1-\delta) < 1$, $1-\mu_k^2(1-\delta) < 1$, $\forall k \in \mathbb{N}$, we get

$$\begin{aligned}
\|\tau_{k+1} - u^*\| &\le \left\|\tau_{k+1} - u_{k+1}^1\right\| + \left\|u_{k+1}^1 - u^*\right\| \\
&\le \|\tau_{k+1} - T\varsigma_k\| + \left\|T\varsigma_k - Tu_k^2\right\| + \left\|u_{k+1}^1 - u^*\right\| \\
&\le \varepsilon_k + \delta\left\|\varsigma_k - u_k^2\right\| + 2\delta\left\|u_k^2 - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| \\
&\le \varepsilon_n + \delta\left\|\left(1-\mu_k^1\right)\gamma_k + \mu_k^1 T\gamma_k - \left[\left(1-\mu_k^1\right)u_k^3 + \mu_k^1 Tu_k^3\right]\right\| \\
&\quad +2\delta\left\|\left(1-\mu_k^1\right)u_k^3 + \mu_k^1 Tu_k^3 - u^*\right\| + 2\delta\left\|u_k^2 - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| \\
&\le \varepsilon_n + \delta\left(1-\mu_k^1\right)\left\|\gamma_k - u_k^3\right\| + \delta\mu_k^1\left\|T\gamma_k - Tu_k^3\right\| + 2\delta\left(1-\mu_k^1\right)\left\|u_k^3 - u^*\right\| \\
&\quad +2\delta^2\mu_k^1\left\|u_k^3 - u^*\right\| + 2\delta\left\|u_k^2 - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| \\
&\le \varepsilon_n + \delta\left(1-\mu_k^1\right)\left(1-\mu_k^2\right)\left\|\tau_k - u_k^1\right\| + \delta^2\left(1-\mu_k^1\right)\mu_k^2\left\|\tau_k - u_k^1\right\| \\
&\quad +2\delta^2\left(1-\mu_k^1\right)\mu_k^2\left\|u_k^1 - u^*\right\| + \delta^2\mu_k^1\left\|\gamma_k - u_k^3\right\| + 2\delta^2\mu_k^1\left\|u_k^3 - u^*\right\| \\
&\quad +2\delta\left[1-\mu_k^1(1-\delta)\right]\left\|u_k^3 - u^*\right\| + 2\delta\left\|u_k^2 - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| \\
&\le \varepsilon_n + \delta\left(1-\mu_k^1\right)\left(1-\mu_k^2\right)\left\|\tau_k - u_k^1\right\| + \delta^2\left(1-\mu_k^1\right)\mu_k^2\left\|\tau_k - u_k^1\right\| \\
&\quad +2\delta^2\left(1-\mu_k^1\right)\mu_k^2\left\|u_k^1 - u^*\right\| + \delta^2\mu_k^1\left(1-\mu_k^2\right)\left\|\tau_k - u_k^1\right\| + \delta^3\mu_k^1\mu_k^2\left\|\tau_k - u_k^1\right\| \\
&\quad +2\delta^3\mu_k^1\mu_k^2\left\|u_k^1 - u^*\right\| + 2\delta^2\mu_k^1\left\|u_k^3 - u^*\right\| + 2\delta\left[1-\mu_k^1(1-\delta)\right]\left\|u_k^3 - u^*\right\| \\
&\quad +2\delta\left\|u_k^2 - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| \\
&\le \varepsilon_n + \delta\left[1-\mu_k^1(1-\delta)\right]\left[1-\mu_k^2(1-\delta)\right]\|\tau_k - u^*\| \\
&\quad +\delta\left[1-\mu_k^1(1-\delta)\right]\left[1-\mu_k^2(1-\delta)\right]\left\|u_k^1 - u^*\right\| + 2\delta^2\left(1-\mu_k^1\right)\mu_k^2\left\|u_k^1 - u^*\right\| \\
&\quad +2\delta^3\mu_k^1\mu_k^2\left\|u_k^1 - u^*\right\| + 2\delta^2\mu_k^1\left\|u_k^3 - u^*\right\| + 2\delta\left[1-\mu_k^1(1-\delta)\right]\left\|u_k^3 - u^*\right\| \\
&\quad +2\delta\left\|u_k^2 - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| \\
&\le \varepsilon_n + \delta\left[1-\mu_k^1(1-\delta)\right]\left[1-\mu_k^2(1-\delta)\right]\|\tau_k - u^*\| \\
&\quad +\delta\left[1-\mu_k^1(1-\delta)\right]\left[1-\mu_k^2(1-\delta)\right]\left\|u_k^1 - u^*\right\| + 2\delta^2\left(1-\mu_k^1\right)\mu_k^2\left\|u_k^1 - u^*\right\| \\
&\quad +2\delta^3\mu_k^1\mu_k^2\left\|u_k^1 - u^*\right\| + 2\delta^2\mu_k^1\left[1-\mu_k^2(1-\delta)\right]\left\|u_k^1 - u^*\right\| \\
&\quad +2\delta\left[1-\mu_k^1(1-\delta)\right]\left[1-\mu_k^2(1-\delta)\right]\left\|u_k^1 - u^*\right\| \\
&\quad +2\delta^2\left[1-\mu_k^1(1-\delta)\right]\left[1-\mu_k^2(1-\delta)\right]\left\|u_k^1 - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| \\
&\le \delta\|\tau_k - u^*\| + 11\left\|u_k^1 - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| + \varepsilon_k, \forall k \in \mathbb{N}.
\end{aligned}$$

Let us define ρ_k, σ_k, α as :

$$\begin{aligned}
\rho_k &: = 11\left\|u_k^1 - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| + \varepsilon_k \geq 0 \\
\sigma_k &: = \left\|\tau_k - u^*\right\| \geq 0 \\
\alpha &: = \delta \in [0,1), \forall k \in \mathbb{N}.
\end{aligned}$$

From Theorem 2.1, we have $\lim_{k\longrightarrow\infty}\left\|u_{k+1}^1 - u^*\right\| = \lim_{k\longrightarrow\infty}\left\|u_k^1 - u^*\right\| = 0$ and also, for $\lim_{k\longrightarrow\infty}\varepsilon_k = 0$, we find $\lim_{k\longrightarrow\infty}\rho_k = 0$. Now, we have satisfied conditions of [34, Lemma on page 302], so $\lim_{k\longrightarrow\infty}\sigma_k = \lim_{k\longrightarrow\infty}\left\|\tau_k - u^*\right\| = 0$.

Conversely, let $\lim_{k\longrightarrow\infty}\tau_k = u^*$. We will show that $\lim_{k\longrightarrow\infty}\varepsilon_k = 0$. Using (1.6), (1.7), (2.2), $\delta \in [0,1)$, and the facts $0 \leq \mu_k^1, \mu_k^2 \leq 1$, $1 - \mu_k^1(1-\delta) < 1$, $1 - \mu_k^2(1-\delta) < 1$, $\forall k \in \mathbb{N}$, we obtain

$$\begin{aligned}
\varepsilon_k &= \left\|\tau_{k+1} - T\varsigma_k\right\| \leq \left\|\tau_{k+1} - u_{k+1}^1\right\| + \left\|Tu_k^2 - T\varsigma_k\right\| \\
&\leq \left\|\tau_{k+1} - u_{k+1}^1\right\| + \delta\left\|u_k^2 - \varsigma_k\right\| + 2\delta\left\|\varsigma_k - u^*\right\| \\
&\leq \left\|\tau_{k+1} - u_{k+1}^1\right\| + \delta\left[1 - \mu_k^1(1-\delta)\right]\left\|u_k^3 - \varsigma_k\right\| \\
&\quad + 2\delta\left[1 - \mu_k^1(1-\delta) + \delta\mu_k^1\right]\left\|\gamma_k - u^*\right\| + 2\delta\left\|\varsigma_k - u^*\right\| \\
&\leq \left\|\tau_{k+1} - u_{k+1}^1\right\| + \delta\left[1 - \mu_k^1(1-\delta)\right]\left[1 - \mu_k^2(1-\delta)\right]\left\|u_k^1 - \tau_k\right\| \\
&\quad + 2\delta^2\left[1 - \mu_k^1(1-\delta)\right]\mu_k^2\left\|\tau_k - u\right\| \\
&\quad + 2\delta\left[1 - \mu_k^1(1-\delta) + \delta\mu_k^1\right]\left[1 - \mu_k^2(1-\delta)\right]\left\|\tau_k - u^*\right\| \\
&\quad + 2\delta^2\left[1 - \mu_k^1(1-\delta)\right]\left[1 - \mu_k^2(1-\delta)\right]\left\|\tau_k - u^*\right\| \\
&\leq \left\|\tau_{k+1} - u^*\right\| + \left\|u_{k+1}^1 - u^*\right\| + \left\|u_k^1 - u^*\right\| + \left\|\tau_k - u^*\right\| + \varkappa_k\left\|\tau_k - u^*\right\|, \forall k \in \mathbb{N}.
\end{aligned}$$

Here, $\varkappa_k = 2\left[3 - \mu_k^1(1-2\delta)\right] \in [2,4]$, $\forall k \in \mathbb{N}$, i.e. $\{\varkappa_k\}$ is a finite sequence of positive numbers. From the assumption $\lim_{k\longrightarrow\infty}\tau_k = u^*$ and from Theorem 2.1, we have $\lim_{k\longrightarrow\infty}u_k^1 = u^*$.

If we take the limit of both sides of the above inequality, we have $\lim_{k\longrightarrow\infty}\varepsilon_k = 0$. Hence, $\{u_k^1\}$ is T*-stable.* □

Theorem 2.3 *Let $(N, \|\cdot\|)$ be a real normed space, $T : N \to N$ be a mapping which satisfies (1.7) and has a fixed point u^*, $\{u_k^1\}$ be a sequence produced by (1.6), and $\{p_k^1\}$ be a sequence produced by*

$$\begin{cases} p_0^1 \in S, \\ p_{k+1}^1 = Tp_k^2, \\ p_k^2 = \left(1 - \mu_k^1\right)p_k^3 + \mu_k^1 Tp_k^3, \\ p_k^3 = \left(1 - \mu_k^2\right)p_k^1 + \mu_k^2 Tp_k^1, \ \textit{for all } k \in \mathbb{N}, \end{cases} \tag{2.3}$$

with real sequences $\{\mu_k^1\}, \{\mu_k^2\} \subseteq [0,1]$. If $u_0^1, p_0^1 \neq u^*$, $\{u_k^1\}$ converges to u^* faster than $\{p_k^1\}$ does.

Proof We got the following inequality in (2.1) in the proof of the Theorem 2.1

$$\left\|u_{k+1}^1 - u^*\right\| \leq \delta^{2(k+1)}\prod_{n=0}^{k}\left[1 - \mu_n^1(1-\delta)\right]\left[1 - \mu_n^2(1-\delta)\right]\left\|u_0^1 - u^*\right\| = \kappa_k^{(1)}, \forall k \in \mathbb{N}.$$

Now, from (1.7) and (2.3), we have

$$\begin{aligned}
\left\|p_{k+1}^{1}-u^{*}\right\| &\leq \delta\left\|p_{k}^{2}-u^{*}\right\| \\
&\leq \delta\left[1-\mu_{k}^{1}(1-\delta)\right]\left\|p_{k}^{3}-u^{*}\right\| \\
&\leq \delta\left[1-\mu_{k}^{1}(1-\delta)\right]\left[1-\mu_{k}^{2}(1-\delta)\right]\left\|p_{k}^{1}-u^{*}\right\| \\
&\leq \cdots \\
&\leq \delta^{k+1}\prod_{n=0}^{k}\left[1-\mu_{n}^{1}(1-\delta)\right]\left[1-\mu_{n}^{2}(1-\delta)\right]\left\|p_{0}^{1}-u^{*}\right\|=\kappa_{k}^{(2)},\ \forall k\in\mathbb{N}.
\end{aligned}$$

Observe that $\kappa_k^{(1)}$ and $\kappa_k^{(2)}$ are the best possible upper bounds for $\left\|u_{k+1}^{1}-u^{*}\right\|$ and $\left\|p_{k+1}^{1}-u^{*}\right\|$, respectively, and $\lim_{k\longrightarrow\infty}\kappa_k^{(1)}=\lim_{k\longrightarrow\infty}\kappa_k^{(2)}=0$.

Therefore, all the necessities of Definition 1.2 are satisfied.

Let us define Δ_k :

$$\Delta_k=\frac{\left\|\kappa_k^{(1)}-0\right\|}{\left\|\kappa_k^{(2)}-0\right\|}=\delta^{k+1}\frac{\left\|u_0^1-u^*\right\|}{\left\|p_0^1-u^*\right\|},\ \forall k\in\mathbb{N}.$$

Now, we can formulate the following ratio test

$$\lim_{k\to\infty}\frac{\Delta_{k+1}}{\Delta_k}=\lim_{k\to\infty}\frac{\delta^{k+2}\frac{\left\|u_0^1-u^*\right\|}{\left\|p_0^1-u^*\right\|}}{\delta^{k+1}\frac{\left\|u_0^1-u^*\right\|}{\left\|p_0^1-u^*\right\|}}=\delta<1,$$

which implies that the series $\sum_{k=0}^{\infty}\Delta_k$ is convergent. Thus, we arrive at $\lim_{k\to\infty}\Delta_k=\lim_{k\to\infty}\frac{\left\|\kappa_k^{(1)}-0\right\|}{\left\|\kappa_k^{(2)}-0\right\|}=0$. Now, by Definition 1.1, $\left\{\kappa_k^{(1)}\right\}$ converges faster than $\left\{\kappa_k^{(2)}\right\}$ and thus, by Definition 1.2, $\left\{u_k^{(1)}\right\}$ converges faster than $\left\{p_k^{(2)}\right\}$. □

Example 2.4 *Let $S=[0,1]$ be endowed with the usual metric and let us define an operator $T:[0,1]\to[0,1]$ by $T\upsilon=\frac{1}{1+\upsilon}$.*

The unique fixed point of T can easily be computed: $\upsilon=1/(1+\upsilon)$ in S at $\upsilon=(\sqrt{5}-1)/2\approx 0.618033988$. When $\upsilon_1\neq\upsilon_2$, we have $|T\upsilon_1-T\upsilon_2|=|\upsilon_1-\upsilon_2|/|(1+\upsilon_1)(1+\upsilon_2)|$ which implies $|T\upsilon_1-T\upsilon_2|/|\upsilon_1-\upsilon_2|=1/|(1+\upsilon_1)(1+\upsilon_2)|<1$. Therefore, we have $|T\upsilon_1-T\upsilon_2|<|\upsilon_1-\upsilon_2|$.

Hence, T is not a contraction mapping on S. However, Figure 1 shows that the operator T satisfies condition (1.7) for any $\delta\in[0.618034,1)$.

Take $\mu_k^1=\mu_k^2=\mu_k^3=\frac{10}{k+10\sqrt{10}}$ for all $k\in\mathbb{N}$, and choose the initial guess as 0.8 for each iteration algorithm; the convergence behaviors of the various iteration algorithms are listed in Table 1 and demonstrated in Figure 2.

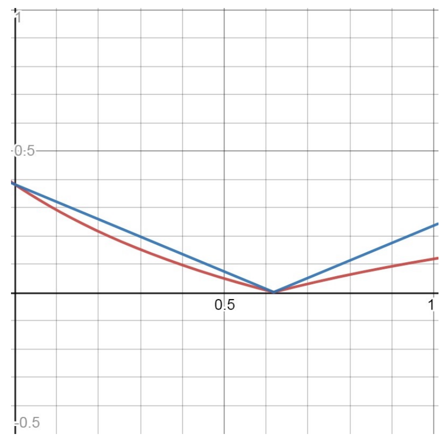

Figure 1. T satisfies condition (1.7) for Example 2.4. The blue line and red curve represent, respectively, the RHS and LHS of (1.7).

Table 1. The convergence behaviors of the various iteration algorithms for Example 2.4.

# of Iter.	S	Normal-S	CR	Picard-S	K*
0	0.8	0.8	0.8	0.8	0.8
1	0.563438674	0.580484164	0.587527981	0.639615752	0.626548978
2	0.636730509	0.626371958	0.623983442	0.620754250	0.618446952
3	0.611828608	0.616162026	0.616863223	0.618382033	0.618054890
4	0.620138126	0.618463819	0.618271872	0.618078907	0.618035088
5	0.617318635	0.617933513	0.617984439	0.618039829	0.618034049
6	0.618279171	0.618057885	0.618044559	0.618034752	0.618033992
7	0.617949493	0.618028213	0.618031684	0.618034088	0.618033988
⋮	⋮	⋮	⋮	⋮	⋮
10	0.618037558	0.618034076	0.618034016	0.618033988	
⋮	⋮	⋮	⋮	⋮	
13	0.618033832	0.618033988	0.618033988		
⋮	⋮	⋮	⋮		
19	0.618033988				
⋮	⋮				
# of Iter.	Picard	Mann	Ishikawa	Noor	SP*
0	0.8	0.8	0.8	0.8	0.8
1	0.555555556	0.722699880	0.730582998	0.729746825	0.596044416
2	0.642857141	0.679105885	0.688687472	0.687633216	0.620824628
3	0.608695653	0.654256412	0.663028749	0.662032296	0.617664249
4	0.621621622	0.639885756	0.647085147	0.646244532	0.618084977
5	0.616666667	0.631439704	0.637038324	0.636368867	0.618026693
6	0.618556700	0.626392527	0.630621554	0.630105309	0.618035069
7	0.617834395	0.623326616	0.626470258	0.626079526	0.618033824
⋮	⋮	⋮	⋮	⋮	⋮
10	0.618045111	0.619495877	0.620743540	0.620581775	0.618033988
⋮	⋮	⋮	⋮	⋮	⋮
19	0.618033988	0.618089961	0.618177720	0.618165526	
⋮	⋮	⋮	⋮	⋮	

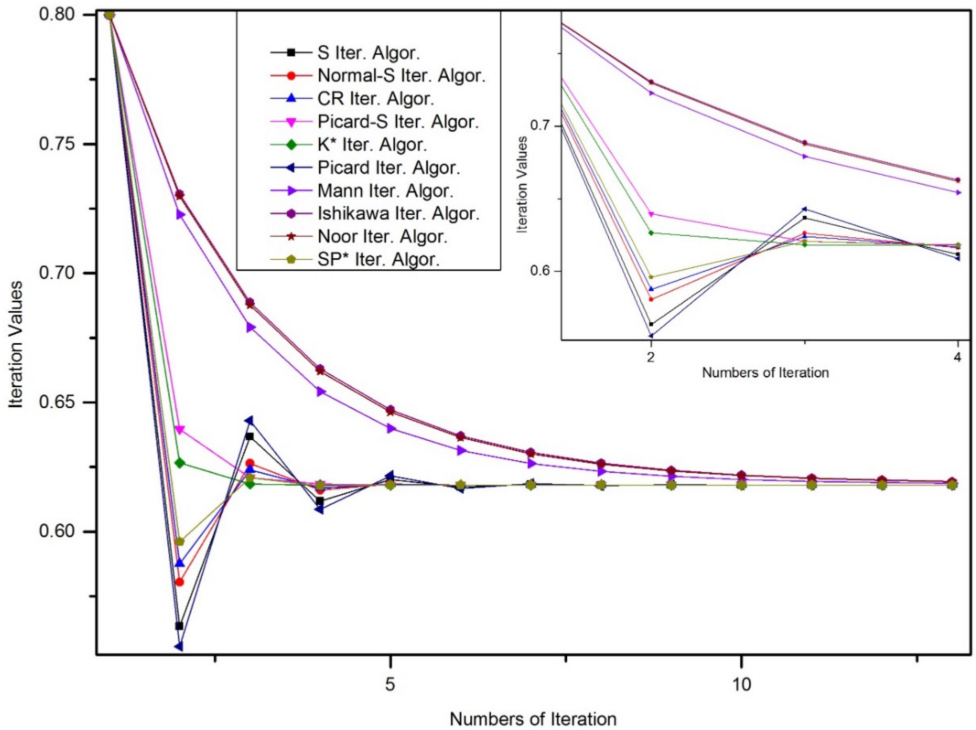

Figure 2. The convergence behaviors of the various iteration algorithms for Example 2.4.

Theorem 2.5 *Let $T : N \to N$ be a mapping satisfying condition (1.7) with a fixed point u^* and $\widetilde{T} : N \to N$ be an approximate mapping of T, that is, there exists $\epsilon > 0$ such that $\left\| Tu - \widetilde{T}u \right\| < \epsilon$ for all $u \in N$. Consider the sequence $\{u_k^1\}$ produced by (1.6) and a sequence $\{\widetilde{u}_k^1\}$ produced by*

$$\begin{cases} \widetilde{u}_0^1 \in S \\ \widetilde{u}_{k+1}^1 = \widetilde{T}\widetilde{u}_k^2, \\ \widetilde{u}_k^2 = \widetilde{T}\left[\left(1-\mu_k^1\right)\widetilde{u}_k^3 + \mu_k^1 \widetilde{T}\widetilde{u}_k^3\right], \\ \widetilde{u}_k^3 = \left(1-\mu_k^2\right)\widetilde{u}_k^1 + \mu_k^2 \widetilde{T}\widetilde{u}_k^1, \text{ for all } k \in \mathbb{N}, \end{cases} \tag{2.4}$$

where $\{\mu_k^1\}, \{\mu_k^2\} \subseteq [0,1]$ are real sequences. If $\lim_{k\longrightarrow\infty} \widetilde{u}_k^1 = \widetilde{u}^*$, then it holds that

$$\|u^* - \widetilde{u}^*\| \le \left(\frac{1+\delta^2}{1-\delta}\right)\epsilon, \text{ for } \epsilon > 0.$$

Proof Using (1.6), (1.7), and (2.4),we have

$$\begin{aligned} \left\|u_{k+1}^1 - \widetilde{u}_{k+1}^1\right\| &\le \left\|Tu_k^2 - T\widetilde{u}_k^2\right\| + \left\|T\widetilde{u}_k^2 - \widetilde{T}\widetilde{u}_k^2\right\| \\ &\le \delta\left\|u_k^2 - \widetilde{u}_k^2\right\| + 2\delta\left\|u_k^2 - u^*\right\| + \epsilon, \end{aligned} \tag{2.5}$$

$$\begin{aligned}
\left\|u_k^2-\widetilde{u}_k^2\right\| &= \left\|T\left[\left(1-\mu_k^1\right)u_k^3+\mu_k^1 T u_k^3\right]-\widetilde{T}\left[\left(1-\mu_k^1\right)\widetilde{u}_k^3+\mu_k^1\widetilde{T}\widetilde{u}_k^3\right]\right\| \\
&\leq \left\|T\left[\left(1-\mu_k^1\right)u_k^3+\mu_k^1 T u_k^3\right]-T\left[\left(1-\mu_k^1\right)\widetilde{u}_k^3+\mu_k^1\widetilde{T}\widetilde{u}_k^3\right]\right\| \\
&\quad +\left\|T\left[\left(1-\mu_k^1\right)\widetilde{u}_k^3+\mu_k^1\widetilde{T}\widetilde{u}_k^3\right]-\widetilde{T}\left[\left(1-\mu_k^1\right)\widetilde{u}_k^3+\mu_k^1\widetilde{T}\widetilde{u}_k^3\right]\right\| \\
&\leq \delta\left\|\left(1-\mu_k^1\right)u_k^3+\mu_k^1 T u_k^3-\left(1-\mu_k^1\right)\widetilde{u}_k^3-\mu_k^1\widetilde{T}\widetilde{u}_k^3\right\| \\
&\quad +2\delta\left\|\left(1-\mu_k^1\right)u_k^3+\mu_k^1 T u_k^3-u^*\right\|+\epsilon \\
&\leq \delta\left[1-\mu_k^1(1-\delta)\right]\left\|u_k^3-\widetilde{u}_k^3\right\|+2\delta^2\mu_k^1\left\|u_k^3-u^*\right\|+\delta\mu_k^1\epsilon \\
&\quad +2\delta\left[1-\mu_k^1(1-\delta)\right]\left\|u_k^3-u^*\right\|+\epsilon, \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}
\left\|u_k^2-u^*\right\| &= \left\|T\left[\left(1-\mu_k^1\right)u_k^3+\mu_k^1 T u_k^3\right]-u^*\right\| \\
&\leq \delta\left\|\left(1-\mu_k^1\right)u_k^3+\mu_k^1 T u_k^3-u^*\right\| \\
&\leq \delta\left[1-\mu_k^1(1-\delta)\right]\left\|u_k^3-u^*\right\|, \qquad (2.7)
\end{aligned}$$

$$\begin{aligned}
\left\|u_k^3-\widetilde{u}_k^3\right\| &= \left\|\left(1-\mu_k^2\right)u_k^1+\mu_k^2 T u_k^1-\left(1-\mu_k^2\right)\widetilde{u}_k^1-\mu_k^2\widetilde{T}\widetilde{u}_k^1\right\| \\
&\leq \left[1-\mu_k^2(1-\delta)\right]\left\|u_k^1-\widetilde{u}_k^1\right\|+2\delta\mu_k^2\left\|u_k^1-u^*\right\|+\mu_k^2\epsilon, \qquad (2.8)
\end{aligned}$$

and

$$\left\|u_k^3-u^*\right\|=\left[1-\mu_k^2(1-\delta)\right]\left\|u_k^1-u^*\right\|. \qquad (2.9)$$

Combining (2.5)-(2.9), we get

$$\begin{aligned}
\left\|u_{k+1}^1-\widetilde{u}_{k+1}^1\right\| &\leq \delta^2\left\{\left(1-\mu_k^2\right)\left[1-\mu_k^1(1-\delta)\right]+\delta^2\mu_k^1\mu_k^2\right\}\left\|u_k^1-\widetilde{u}_k^1\right\| \\
&\quad +2\delta^4\mu_k^1\mu_k^2\left\|u_k^1-u^*\right\|+2\delta^3\mu_k^1\left[1-\mu_k^2(1-\delta)\right]\left\|u_k^1-u^*\right\| \\
&\quad +4\delta^2\left[1-\mu_k^1(1-\delta)\right]\left[1-\mu_k^2(1-\delta)\right]\left\|u_k^1-u^*\right\| \\
&\quad +\delta^2\left(1-\mu_k^1\right)\mu_k^2\epsilon+\delta^3\mu_k^1\mu_k^2\epsilon+\delta^2\mu_k^1\epsilon+\delta\epsilon+\epsilon
\end{aligned}$$

As $\delta\in[0,1)$ and $0\leq\mu_k^1,\mu_k^2\leq 1$, $1-\mu_k^1(1-\delta)<1$, $1-\mu_k^2(1-\delta)<1$ for all $k\in\mathbb{N}$, the above inequality becomes

$$\left\|u_{k+1}^1-\widetilde{u}_{k+1}^1\right\|\leq\delta^2\left\|u_k^1-\widetilde{u}_k^1\right\|+\left(2\delta^4+2\delta^3+4\delta^2\right)\left\|u_k^1-u^*\right\|+\delta^3\epsilon+\delta^2\epsilon+\delta\epsilon+\epsilon. \qquad (2.10)$$

From Theorem 2.1, we have $\lim_{k\longrightarrow\infty}u_k^1=u^*$ and from the hypothesis of this theorem we have $\lim_{k\longrightarrow\infty}\widetilde{u}_k^1=\widetilde{u}^*$. Now, taking the limit of both side of (2.10), we obtain

$$\left\|u^*-\widetilde{u}^*\right\|\leq\delta^2\left\|u^*-\widetilde{u}^*\right\|+\delta^3\epsilon+\delta^2\epsilon+\delta\epsilon+\epsilon,$$

or

$$\left\|u^*-\widetilde{u}^*\right\|\leq\left(\frac{1+\delta^2}{1-\delta}\right)\epsilon.$$

□

In the following example, we will show data dependence of T iteration process (in Example 2.4).

Example 2.6 *Let S, T, and δ be as in Example 2.4 and define an operator $\widetilde{T}: S \to S$ with a unique fixed point $\widetilde{\upsilon}^* = 0.624321324$ in S by*

$$\widetilde{T}\upsilon = \frac{\upsilon^2 + 4.2\upsilon + 24.41}{25} - \lim arcsinh\left(\frac{\pi\upsilon}{4}\right). \tag{2.11}$$

With a calculation tool software package, it is easy to calculate that

$$\sup_{\upsilon\in[0,1]} \left|T\upsilon - \widetilde{T}\upsilon\right| = 0.0368247$$

which implies $\left|T\upsilon - \widetilde{T}\upsilon\right| \leq \epsilon = 0.0368247$ for all $\upsilon \in [0,1]$. Thence, $\widetilde{T}$ can be treated as an approximate operator of T. It has already been shown in Example 2.4 that iteration algorithm (1.6) converges to $0.618033988 = \upsilon^* = T\upsilon^*$. On the other hand, for given $\widetilde{u}_0^1 = 0.8 \in S$, iteration algorithm (2.4) of $\widetilde{T}$ in (2.11) with $\mu_k^1 = \mu_k^2 = \frac{10}{k+10\sqrt{10}}$ for all $k \in \mathbb{N}$ converges to $0.624321324 = \widetilde{\upsilon}^* = \widetilde{T}\widetilde{\upsilon}^*$; and it is presented in Table 2.

Table 2. The convergence behavior of the iteration algorithm (2.4) for Example 2.6.

# of Iter.	Iteration algorithm (2.4) of $\widetilde{T}$ in (2.11)
0	0.8
1	0.636212670
2	0.625157924
3	0.624383094
4	0.624326092
5	0.624321699
6	0.624321349
7	0.624321326
8	0.624321324
$\vdots$	$\vdots$

Now we have $|\upsilon^* - \widetilde{\upsilon}^*| = 0.00628734$. However, without knowing $\widetilde{\upsilon}^*$ and without computing it, we can find the following upper bound for $|\upsilon^* - \widetilde{\upsilon}^*|$ by using the conclusion of Theorem 2.5:

$$|\upsilon^* - \widetilde{\upsilon}^*| \leq \left(\frac{1 + (0.618034)^2}{1 - 0.618034}\right) \times 0.0368247 = 0.133233.$$

3. Conclusion

In this paper, for further examination, we reconsider the convergence and stability results obtained by Yu et al. [51]. We showed that these results can be obtained with no conditions imposed on parametric control sequences used in the K^* iteration algorithm. We also proved a data dependence result for the fixed point of quasistrictly contractive mappings. Our results considerably improve the results of Yu et al. [51]. As a future study, we will apply the K^* iteration algorithm to solve various variational inequalities and convex minimization problems.

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