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Adjunction identity to hypersemigroups

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Abstract: It is shown that some embedding problems on hypersemigroups are actually problems of adjunction. According to the theorem of this paper, for every hypersemigroup S which does not have identity element, an hypersemigroup T having identity element can be constructed in such a way that S is an ideal of T. Moreover, if S is regular, intra-regular, right (left) regular, right (left) quasi-regular or semisimple, then so is T. If A is an ideal, subidempotent bi-ideal or quasi-ideal of S, then it is an ideal, bi-ideal, quasi-ideal of T as well. Illustrative examples are given.

Key words: Hypersemigroup, identity element, ideal, adjunction, embedding, isomorphism

1. Introduction and prerequisites

The embedding of a regular ring in a regular ring with identity has been published by L. Fuchs and I. Halperin in 1964 in Fundamenta Mathematicae [2]. A related problem has been previously established by Kohls [13] (MR0081267; Reviewer: Melvin Henriksen). Also Johnson [3] has shown that, for a certain class of rings which includes all regular rings, each of the rings is isomorphic to a subring of a regular ring with identity. We have seen in [5] that embedding problems are actually problems of adjunction by showing that for every semigroup not containing identity, there exists a semigroup T with identity such that S is an ideal of T. The aim is to show that for every hypersemigroup S which does not have identity we can construct an hypersemigroup T with identity in such a way that S is an ideal of T. It is also shown that if S is regular, quasi-regular, right (resp. left) regular, right (resp. left) quasi-regular or semisimple, then T is also so. Furthermore, if A is an ideal, subidempotent bi-ideal or quasi-ideal of $(S,\circ)$, then it is an ideal, bi-ideal, quasi-ideal of T. Examples illustrate the results.

Denote by $\mathcal{P}^*(S)$ the set of all nonempty subsets of S. An hypersemigroup is a nonempty set S with an hyperoperation $\circ : S\times S \to \mathcal{P}^*(S) \mid (a,b) \to a\circ b$ on S and an operation $* : \mathcal{P}^*(S)\times\mathcal{P}^*(S) \to \mathcal{P}^*(S) \mid (A,B) \to A*B := \bigcup\limits_{a\in A, b\in B} a\circ b$ on $\mathcal{P}^*(S)$ such that $(a\circ b)*\{c\} = \{a\}*(b\circ c)$ for every $a,b,c\in S$. We denote it by $(S,\circ,*)$ or, for short, just by $(S,\circ)$. It is not necessary $\circ$ and $*$ to be on (the same set) S to write $(S,\circ,*)$. For an hypersemigroup, we can write $(S,\circ,*)$ though $\circ$ is an operation between elements of S and $*$ an operation between subsets of S. Let $(S,\circ,*)$ be an hypersemigroup. For any $x,y\in S$, we have $\{x\}*\{y\} = x\circ y$. From the definition of $*$, we have the following: (1) if $x\in A*B$, then $x\in a\circ b$ for some $a\in A$, $b\in B$ and (2) if $a\in A$ and $b\in B$, then $a\circ b\subseteq A*B$. If A,B,C,D are subsets of S such that $A\neq\emptyset$, $C\neq\emptyset$ and $A\subseteq B$, $C\subseteq D$, then $A*C\subseteq B*D$ and $C*A\subseteq D*B$. If $(S,\circ)$ is an hypersemigroup then, for

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any nonempty subsets A, B, C of S, we have $(A * B) * C = A * (B * C)$ [7, 9] so the operation $*$ is associative and we can write $A * B * C$ without using parentheses.

A nonempty subset A of S is called *subidempotent* if $A * A \subseteq A$. That is, if $x \in a \circ b$ for some $a, b \in A$, then $x \in A$; which is equivalent to $a \circ b \subseteq A$ for every $a, b \in A$. $\big(\mathcal{P}^*(S), *\big)$ is a semigroup. An element A of $\mathcal{P}^*(S)$ is called a *subsemigroup* of $(S, \circ)$ if $A * A \subseteq A$ [9]. So the concepts "subidempotent subset of S" and "subsemigroup of S" are identical. The notion "subidempotent" comes from Birkhoff (subidempotent elements). A nonempty subset A of S is called an *ideal* of S [7] if (1) $A * S \subseteq A$ and (2) if $S * A \subseteq A$. Property (1) is equivalent to the following "if $a \in A$ and $b \in S$, then $a \circ b \subseteq A$". Property (2) is equivalent to "if $a \in S$ and $b \in A$, then $a \circ b \subseteq A$". A nonempty subset of S satisfying only the property (1) (resp. property (2)) is called a *right* (resp. *left*) *ideal* of S. Every right ideal or left ideal of $(S, \circ)$, is a subidempotent subset of $(S, \circ)$. A nonempty subset A of S is called a *bi-ideal* of S is $A * S * A \subseteq A$. This is equivalent to saying that if $x \in u \circ a$ and $u \in b \circ s$ for some $a, b \in A$, $s \in S$, then $x \in A$. A nonempty subset Q of S is called a *quasi-ideal* of S if $(Q * S) \cap (S * Q) \subseteq Q$. Equivalently, if $t \in a \circ b$ for some $a \in Q$, $b \in S$ and $t \in c \circ d$ for some $c \in S$, $d \in Q$, then $t \in Q$.

Every ideal of $(S, \circ)$ is a quasi-ideal of $(S, \circ)$ and every quasi-ideal of $(S, \circ)$ is a bi-ideal of $(S, \circ)$. Indeed, if A is an ideal of S, then A is nonempty, $A * S \subseteq A$ and $S * A \subseteq A$, then $(A * S) \cap (S * A) \subseteq A$ and so A is a quasi-ideal of S. If A is a quasi-ideal of S, then A is nonempty and $A * S * A \subseteq (A * S) \cap (S * A) \subseteq A$ and so A is a bi-ideal of S.

An hypersemigroup $(S, \circ)$ is called *regular* [7] (also [9, Definition 11]) if

$$\text{for every } a \in S \text{ there exists } x \in S \text{ such that } a \in (a \circ x) * \{a\}.$$

An hypersemigroup $(S, \circ)$ is called *intra-regular* [9, Definition 25] if

$$\text{for every } a \in S \text{ there exist } x, y \in S \text{ such that } a \in (x \circ a) * (a \circ y).$$

$(S, \circ)$ is called *right regular* (see also [4, 10]) if

$$\text{for every } a \in S \text{ there exists } x \in S \text{ such that } a \in (a \circ a) * \{x\}.$$

$(S, \circ)$ is called *left regular* if

$$\text{for every } a \in S \text{ there exists } x \in S \text{ such that } a \in \{x\} * (a \circ a).$$

$(S, \circ)$ is called *right quasi-regular* [8, Definition 2.15] if

$$\text{for every } a \in S \text{ there exist } x, y \in S \text{ such that } a \in (a \circ x) * (a \circ y).$$

$(S, \circ)$ is called *left quasi-regular* [8, Definition 2.7] if

$$\text{for every } a \in S \text{ there exist } x, y \in S \text{ such that } a \in (x \circ a) * (y \circ a).$$

$(S, \circ)$ is called *semisimple* [8, Definition 2.23] if

$$\text{for every } a \in S \text{ there exist } x, y, z \in S \text{ such that } a \in (x \circ a) * (y \circ a) * \{z\}.$$

2. Main result

Definition 2.1 *If $(S,\circ)$ is an hypersemigroup, an element e of S is called identity element if $a\circ e = e\circ a = \{a\}$ for every $a\in S$.*

Theorem 2.2 *Let $(S,\circ,*)$ be an hypersemigroup which does not have identity element. Then there exists an hypersemigroup T having identity element such that S is an ideal of T.*

Proof Take an element e such that $e\notin S$ (let $x\in S$ $(S\neq\emptyset)$; the element (x,x) is, for example, such an element) and consider the set $S\cup\{e\}$. Define an hyperoperation " $\overline{\circ}$" on $S\cup\{e\}$ and an operation "$\overline{*}$" on $\mathcal{P}^*\big(S\cup\{e\}\big)$ as follows:

$$\overline{\circ}:\ \big(S\cup\{e\}\big)\times\big(S\cup\{e\}\big)\to\mathcal{P}^*\big(S\cup\{e\}\big)\mid(x,y)\to x\,\overline{\circ}\,y$$

where

$$x\,\overline{\circ}\,y=\begin{cases}x\circ y & if\ x,y\in S\\ \{x\} & if\ x\in S,\ y=e\\ \{y\} & if\ x=e,\ y\in S\\ \{e\} & if\ x=y=e\end{cases}$$

$$\overline{*}:\ \mathcal{P}^*\big(S\cup\{e\}\big)\times\mathcal{P}^*\big(S\cup\{e\}\big)\to\mathcal{P}^*\big(S\cup\{e\}\big)\mid(A,B)\to A\,\overline{*}\,B$$

where

$$A\,\overline{*}\,B=\bigcup_{a\in A,\,b\in B}a\,\overline{\circ}\,b.$$

Then $\big(S\cup\{e\},\overline{\circ},\overline{*}\big)$ is an hypersemigroup. In fact:

(A) The hyperoperation $\overline{\circ}$ is well defined. Indeed: We have $x\in S$ or $x=e$; $y\in S$ or $y=e$.

If $x,y\in S$, then $x\,\overline{\circ}\,y=x\circ y\subseteq S\subseteq S\cup\{e\}$.

If $x\in S,\ y=e$, then $x\,\overline{\circ}\,y=\{x\}\subseteq S\subseteq S\cup\{e\}$.

If $x=e,\ y\in S$, then $x\,\overline{\circ}\,y=\{y\}\subseteq S\subseteq S\cup\{e\}$.

If $x=y=e$, then $x\,\overline{\circ}\,y=\{e\}\subseteq S\cup\{e\}$.

Let $(x,y),(z,t)\in\big(S\cup\{e\}\big)\times\big(S\cup\{e\}\big)$, $(x,y)=(z,t)$. Then $x\,\overline{\circ}\,y=z\,\overline{\circ}\,t$. Indeed:

We have $x\in S\cup\{e\}$, $y\in S\cup\{e\}$, $z\in S\cup\{e\}$, $t\in S\cup\{e\}$, $x=z$, $y=t$; that is
$x\in S$ or $x=e$
$y\in S$ or $y=e$
$z\in S$ or $z=e$
$t\in S$ or $t=e$.

Thus we have

$x\in S$, $\big(y\in S$ or $y=e\big)$, $\big(z\in S$ or $z=e\big)$, $\big(t\in S$ or $t=e\big)$

$x=e$, $\big(y\in S$ or $y=e\big)$, $\big(z\in S$ or $z=e\big)$, $\big(t\in S$ or $t=e\big)$.

So we have the following cases:

$x\in S$, $y\in S$, $\big(z\in S$ or $z=e\big)$, $\big(t\in S$ or $t=e\big)$

$x \in S,\ y = e,\ \big(z \in S \text{ or } z = e\big),\ \big(t \in S \text{ or } t = e\big)$

$x = e,\ y \in S,\ \big(z \in S \text{ or } z = e\big),\ \big(t \in S \text{ or } t = e\big)$

$x = e,\ y = e,\ \big(z \in S \text{ or } z = e\big),\ \big(t \in S \text{ or } t = e\big).$

In other words, we have to check the following cases:

(1) $x \in S,\ y \in S,\ z \in S,\ t \in S$

(2) $x \in S,\ y \in S,\ z \in S,\ t = e$

(3) $x \in S,\ y \in S,\ z = e,\ t \in S$

(4) $x \in S,\ y \in S,\ z = e,\ t = e$

(5) $x \in S,\ y = e,\ z \in S,\ t \in S$

(6) $x \in S,\ y = e,\ z \in S,\ t = e$

(7) $x \in S,\ y = e,\ z = e,\ t \in S$

(8) $x \in S,\ y = e,\ z = e,\ t = e$

(9) $x = e,\ y \in S,\ z \in S,\ t \in S$

(10) $x = e,\ y \in S,\ z \in S,\ t = e$

(11) $x = e,\ y \in S,\ z = e,\ t \in S$

(12) $x = e,\ y \in S,\ z = e,\ t = e$

(13) $x = e,\ y = e,\ z \in S,\ t \in S$

(14) $x = e,\ y = e,\ z \in S,\ t = e$

(15) $x = e,\ y = e,\ z = e,\ t \in S$

(16) $x = e,\ y = e,\ z = e,\ t = e.$

We check each of the 16 cases given above.

We have $x = z,\ y = t$.

(1) Let $x \in S,\ y \in S,\ z \in S,\ t \in S$. Then $x \,\bar{\circ}\, y = z \,\bar{\circ}\, t$. Indeed: Since $x, y \in S$, we have $x \,\bar{\circ}\, y = x \circ y$. Since $z, t \in S$, we have $z \,\bar{\circ}\, t = z \circ t$. Since $x = z,\ y = t$, we have $x \circ y = z \circ t$. Thus we have $x \,\bar{\circ}\, y = x \circ y = z \circ t = z \,\bar{\circ}\, t$.

(2) Let $x \in S,\ y \in S,\ z \in S,\ t = e$. Since $y = t$, we have $y = e$. Thus $S \ni y = e$ and so $e \in S$. The case is impossible.

(3) Let $x \in S,\ y \in S,\ z = e,\ t \in S$. Since $x = z$, we have $x = e$. Then $S \ni x = e$ i.e. $e \in S$. The case is impossible.

(4) Let $x \in S,\ y \in S,\ z = e,\ t = e$. Since $x = z$, we have $S \ni x = e$ i.e. $e \in S$. The case is impossible

(5) Let $x \in S,\ y = e,\ z \in S,\ t \in S$. Since $y = t$, we have $S \ni t = e$. The case is impossible.

(6) Let $x \in S,\ y = e,\ z \in S,\ t = e$. Since $x \in S,\ y = e$, we have $x \,\bar{\circ}\, y = \{x\}$. Since $z \in S,\ t = e$, we have $z \,\bar{\circ}\, t = \{z\}$. Since $x = z$, we have $x \,\bar{\circ}\, y = z \,\bar{\circ}\, t$.

(7) Let $x \in S,\ y = e,\ z = e,\ t \in S$. We have $S \ni x = z = e$ i.e. $e \in S$. The case is impossible.

(8) Let $x \in S,\ y = e,\ z = e,\ t = e$. We have $S \ni x = z = e$. The case is impossible.

(9) Let $x = e,\ y \in S,\ z \in S,\ t \in S$. Since $x = z$, we have $e = z \in S$. The case is impossible.

(10) Let $x = e,\ y \in S,\ z \in S,\ t = e$. Since $x = z$, we have $e = z \in S$. The case is impossible.

(11) Let $x = e,\ y \in S,\ z = e,\ t \in S$. Since $x = e,\ y \in S$, we have $x \,\bar{\circ}\, y = \{y\}$. Since $z = e,\ t \in S$, we have $z \,\bar{\circ}\, t = \{t\}$. We have $y = t$ and so $x \,\bar{\circ}\, y = z \,\bar{\circ}\, t$.

(12) $x = e,\ y \in S,\ z = e,\ t = e$. We have $S \ni y = t = e$. The case is impossible.

(13) Let $x = e,\ y = e,\ z \in S,\ t \in S$. We have $e = x = z \in S$. The case is impossible.

(14) Let $x = e,\ y = e,\ z \in S,\ t = e$. We have $S \ni z = x = e$. The case is impossible.

(15) Let $x = e,\ y = e,\ z = e,\ t \in S$. We have $S \ni t = y = e$. The case is impossible.

(16) Let $x = e,\ y = e,\ z = e,\ t = e$. Since $x = e,\ y = e$, we have $x \,\bar{\circ}\, y = \{e\}$. Since $z = e,\ t = e$, we have $z \,\bar{\circ}\, t = \{e\}$. Thus we have $x \,\bar{\circ}\, y = z \,\bar{\circ}\, t$.

(B) The operation $\bar{*}$ is well defined. Indeed: if $A, B \in \mathcal{P}^*\big(S \cup \{e\}\big)$, then $A \,\bar{*}\, B := \bigcup\limits_{a \in A,\, b \in B} a \,\bar{\circ}\, b$. Since $\emptyset \neq a \,\bar{\circ}\, b \subseteq S \cup \{e\}$ for every $a \in A$ and every $b \in B$, we have $\emptyset \neq A \,\bar{\circ}\, B \subseteq S \cup \{e\}$. Let $(A, B), (C, D) \in \mathcal{P}^*\big(S \cup \{e\}\big) \times \mathcal{P}^*\big(S \cup \{e\}\big)$ such that $(A, B) = (C, D)$. Then $A \,\bar{*}\, B = \bigcup\limits_{a \in A,\, b \in B} a \,\bar{\circ}\, b = \bigcup\limits_{a \in C,\, b \in D} a \,\bar{\circ}\, b = C \,\bar{\circ}\, D$.

(C) We have $\{x\} \,\bar{*}\, (y \,\bar{\circ}\, z) = (x \,\bar{\circ}\, y) \,\bar{*}\, \{z\}$ for every $x, y, z \in S \cup \{e\}$. Indeed: We have to check the cases:

(a) $x \in S,\ \big(y \in S \text{ or } y = e\big),\ \big(z \in S \text{ or } z = e\big)$ and

(b) $x = e,\ \big(y \in S \text{ or } y = e\big),\ \big(z \in S \text{ or } z = e\big)$.

(1) Let $x, y, z \in S$. Then $(x \,\bar{\circ}\, y) \,\bar{*}\, \{z\} = \{x\} \,\bar{*}\, (y \,\bar{\circ}\, z)$. Indeed:
Let $t \in (x \,\bar{\circ}\, y) \,\bar{*}\, \{z\}$. Then $t \in u \,\bar{\circ}\, z$ for some $u \in x \,\bar{\circ}\, y$. Since $x, y \in S$, we have $x \,\bar{\circ}\, y = x \circ y$ and so $u \in x \circ y \subseteq S$. Since $u, z \in S$, we have $u \,\bar{\circ}\, z = u \circ z$. Then we have

$$t \in u \circ z = \{u\} * \{z\} \subseteq (x \circ y) * \{z\} = \{x\} * (y \circ z).$$

Then $t \in x \circ v$ for some $v \in y \circ z \subseteq S$. Since $x, v \in S$, we have $x \circ v = x \,\bar{\circ}\, v$. Since $y, z \in S$, we have $y \circ z = y \,\bar{\circ}\, z$ and so $\{v\} \subseteq y \,\bar{\circ}\, z$. Then we have

$$t \in x \,\bar{\circ}\, v = \{x\} \,\bar{*}\, \{v\} \subseteq \{x\} \,\bar{*}\, (y \,\bar{\circ}\, z)$$

and so $(x \,\bar{\circ}\, y) \,\bar{*}\, \{z\} \subseteq \{x\} \,\bar{*}\, (y \,\bar{\circ}\, z)$. Let now $t \in \{x\} \,\bar{*}\, (y \,\bar{\circ}\, z)$. Then $t \in x \,\bar{\circ}\, u$ for some $u \in y \,\bar{\circ}\, z$. Since $y, z \in S$, we have $u \in y \,\bar{\circ}\, z = y \circ z \subseteq S$. Since $x, u \in S$, we have $x \,\bar{\circ}\, u = x \circ u$. Then we have

$$t \in x \circ u = \{x\} * \{u\} \subseteq \{x\} * (y \circ z) = (x \circ y) * \{z\} = (x \,\bar{\circ}\, y) * \{z\} \text{ (since } x, y \in S).$$

Then $t \in v \circ z$ for some $v \in x \,\bar{\circ}\, y = x \circ y \subseteq S$. Since $v, z \in S$, we have $v \circ z = v \,\bar{\circ}\, z$. Thus we have $t \in v \,\bar{\circ}\, z = \{v\} \,\bar{*}\, \{z\} \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{z\}$ and so $\{x\} \,\bar{*}\, (y \,\bar{\circ}\, z) \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{z\}$.

(2) Let $x, y \in S,\ z = e$. Then $(x \,\bar{\circ}\, y) \,\bar{*}\, \{z\} = \{x\} \,\bar{*}\, (y \,\bar{\circ}\, z)$. Indeed: Since $z = e$, we have to prove that $(x \,\bar{\circ}\, y) \,\bar{*}\, \{e\} = \{x\} \,\bar{*}\, \{y\}$. Let $t \in (x \,\bar{\circ}\, y) \,\bar{*}\, \{e\}$. Then $t \in u \,\bar{\circ}\, e$ for some $u \in x \,\bar{\circ}\, y$. Since $x, y \in S$, we have $x \,\bar{\circ}\, y = x \circ y \subseteq S$ and so $u \in S$. Since $u \in S$, we have $u \,\bar{\circ}\, e = \{u\}$. Then $t \in \{u\} \subseteq x \,\bar{\circ}\, y = \{x\} \,\bar{*}\, \{y\}$ and so $(x \,\bar{\circ}\, y) \,\bar{*}\, \{e\} \subseteq \{x\} \,\bar{*}\, \{y\}$. Let now $t \in \{x\} \,\bar{*}\, \{y\} = x \,\bar{\circ}\, y$. We have $x \,\bar{\circ}\, y \subseteq (x \,\bar{\circ}\, y) \,\bar{*}\, \{e\}$. Indeed:

Let $u \in x \,\bar{\circ}\, y$ ($\Rightarrow \exists\ v \in x \,\bar{\circ}\, y$ such that $u \in v \,\bar{\circ}\, e$?)

Since $x, y \in S$, we have $x \,\bar{\circ}\, y = x \circ y \subseteq S$ and so $u \in S$. For the element $u \in x \,\bar{\circ}\, y$, we have $u \,\bar{\circ}\, e = \{u\}$ and so $u \in u \,\bar{\circ}\, e$.

(3) Let $x \in S,\ y = e,\ z \in S$. Then $(x \,\bar{\circ}\, y) \,\bar{*}\, \{z\} = \{x\} \,\bar{*}\, (y \,\bar{\circ}\, z)$. Indeed: Since $y = e$, we have $x \,\bar{\circ}\, y = \{x\}$ and $y \,\bar{\circ}\, z = \{z\}$. Then we have $(x \,\bar{\circ}\, y) \,\bar{*}\, \{z\} = \{x\} \,\bar{*}\, \{z\} = \{x\} \,\bar{*}\, (y \,\bar{\circ}\, z)$.

(4) Let $x \in S$, $y = z = e$. Then $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$. Indeed: Since $x \in S$, $y = e$, we have $x \mathbin{\overline{\circ}} y = \{x\}$, then $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{x\} \mathbin{\overline{*}} \{z\} = x \mathbin{\overline{\circ}} z$. Since $x \in S$, $z = e$, we have $x \mathbin{\overline{\circ}} z = \{x\}$. Thus we have $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{x\}$. Since $y = z = e$, we have $y \mathbin{\overline{\circ}} z = \{e\}$. Then $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{x\} \mathbin{\overline{*}} \{e\} = x \mathbin{\overline{\circ}} e$. Since $x \in S$, we have $x \mathbin{\overline{\circ}} e = \{x\}$ and so $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{x\}$.

(5) Let $x = e$, $y, z \in S$. Then $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$. Indeed: We have

$(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{y\} \mathbin{\overline{*}} \{z\} = y \mathbin{\overline{\circ}} z$ and $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$.

On the other hand, $y \mathbin{\overline{\circ}} z = \{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$. Indeed: if $u \in y \mathbin{\overline{\circ}} z$ ($\Rightarrow \exists\ v \in y \mathbin{\overline{\circ}} z\ :\ u \in e \mathbin{\overline{\circ}} v$?)

For the element $v := u \in y \mathbin{\overline{\circ}} z$, we have $v \in S$, then $e \mathbin{\overline{\circ}} v = \{v\} = \{u\}$ and so $u \in e \mathbin{\overline{\circ}} v$. Thus we have $y \mathbin{\overline{\circ}} z \subseteq \{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$. If now $u \in \{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$. Then $u \in e \mathbin{\overline{\circ}} v$ for some $v \in y \mathbin{\overline{\circ}} z$. Since $y, z \in S$, we have $v \in y \mathbin{\overline{\circ}} z = y \circ z \subseteq S$. Since $v \in S$, we have $e \mathbin{\overline{\circ}} v = \{v\}$. Then $u \in \{v\} \subseteq y \mathbin{\overline{\circ}} z$ and so $\{e\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) \subseteq y \mathbin{\overline{\circ}} z$.

(6) Let $x = e$, $y \in S$, $z = e$. Then $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$. Indeed: Since $x = e$, we have $x \mathbin{\overline{\circ}} y = e \mathbin{\overline{\circ}} y = \{y\}$. Since $z = e$, we have $y \mathbin{\overline{\circ}} z = y \mathbin{\overline{\circ}} e = \{y\}$. Thus we have

$(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{y\} \mathbin{\overline{*}} \{z\} = y \mathbin{\overline{\circ}} z = y \mathbin{\overline{\circ}} e = \{y\}$ and $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{x\} \mathbin{\overline{*}} \{y\} = x \mathbin{\overline{\circ}} y = e \mathbin{\overline{\circ}} y = \{y\}$.

(7) Let $x = e$, $y = e$, $z \in S$. Then $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$. Indeed: Since $x = y = e$, we have $x \mathbin{\overline{\circ}} y = \{e\}$, then $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{e\} \mathbin{\overline{*}} \{z\} = e \mathbin{\overline{\circ}} z = \{z\}$. Since $y = e$, $z \in S$, we have $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{x\} \mathbin{\overline{*}} \{z\} = \{e\} \mathbin{\overline{*}} \{z\} = e \mathbin{\overline{\circ}} z = \{z\}$.

(8) Let $x = y = z = e$. Then $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z)$. Indeed: Since $x = e$, $y = e$, we have $x \mathbin{\overline{\circ}} y = e \mathbin{\overline{\circ}} e = \{e\}$. Since $z = e$, we have $(x \mathbin{\overline{\circ}} y) \mathbin{\overline{*}} \{z\} = \{e\} \mathbin{\overline{*}} \{e\} = e \mathbin{\overline{\circ}} e = \{e\}$. Since $y = e$, $z = e$, we have $y \mathbin{\overline{\circ}} z = e \mathbin{\overline{\circ}} e = \{e\}$. Since $x = e$, we have $\{x\} \mathbin{\overline{*}} (y \mathbin{\overline{\circ}} z) = \{e\} \mathbin{\overline{\circ}} \{e\} = \{e\}$.

(D) The element e is the identity element of $S \cup \{e\}$; that is, $a \mathbin{\overline{\circ}} e = e \mathbin{\overline{\circ}} a = \{a\}$ for every $a \in S \cup \{e\}$. Indeed: Let $a \in S \cup \{e\}$. If $a \in S$, then $a \mathbin{\overline{\circ}} e = e \mathbin{\overline{\circ}} a = \{a\}$. If $a = e$, then $a \mathbin{\overline{\circ}} e = e \mathbin{\overline{\circ}} e = \{e\} = \{a\}$ and $e \mathbin{\overline{\circ}} a = e \mathbin{\overline{\circ}} e = \{e\} = \{a\}$.

(E) The set S is an ideal of $S \cup \{e\}$; that is, $S \mathbin{\overline{*}} \big(S \cup \{e\}\big) \subseteq S$ and $\big(S \cup \{e\}\big) \mathbin{\overline{*}} S \subseteq S$. Indeed: Let $a \in S$ and $b \in S \cup \{e\}$. If $b \in S$, then $a \mathbin{\overline{\circ}} b = a \circ b \subseteq S$. If $b = e$, then $a \mathbin{\overline{\circ}} b = a \mathbin{\overline{\circ}} e = \{a\} \subseteq S$. Let $a \in S \cup \{e\}$ and $b \in S$. If $a \in S$, then $a \mathbin{\overline{\circ}} b = a \circ b \subseteq S$. If $a = e$, than $a \mathbin{\overline{\circ}} b = \{b\} \subseteq S$. □

As $\big(S \cup \{e\}, \overline{\circ}\,\big)$ is an hypersemigroup, for any nonempty subsets A, B of $S \cup \{e\}$, we have

$$(A \mathbin{\overline{*}} B) \mathbin{\overline{*}} C = A \mathbin{\overline{*}} (B \mathbin{\overline{*}} C) := A \mathbin{\overline{*}} B \mathbin{\overline{*}} C.$$

Also if A, B, C, D subsets of $S, A \neq \emptyset, B \neq \emptyset, A \subseteq B$ and $C \subseteq D$, then $A \mathbin{\overline{*}} C \subseteq B \mathbin{\overline{*}} D$ and $C \mathbin{\overline{*}} A \subseteq D \mathbin{\overline{*}} B$.

We apply Theorem 2.2 to the following example.

Example 2.3 We consider the ordered semigroup $(S, \cdot)$ given by Table 1 and Figure 1. From this semigroup, in the way indicated in [10], the hypersemigroup $(S, \circ)$ given by Table 2 can be obtained (see also [1]). Taking an element t not included in $(S, \circ)$, we consider the set $S \cup \{t\}$. According to Theorem 2.2, the set $S \cup \{t\}$ with the hyperoperation $\overline{\circ}$ given by Table 3 is an hypersemigroup; and t is the identity element of $\big(S \cup \{t\}, \overline{\circ}\big)$ (as Table 3 also shows).

Table 1. The multiplication table of the semigroup of Example 2.3.

$\cdot$	a	b	c	d	e	f
a	a	a	c	c	e	f
b	a	a	c	c	e	f
c	a	a	c	c	e	f
d	a	a	c	c	e	f
e	a	a	c	c	e	f
f	a	a	c	c	e	f

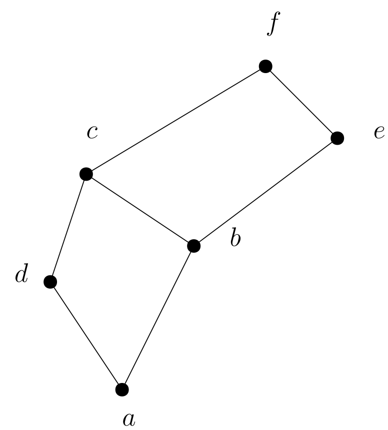

Figure 1. The order of Example 2.3.

Table 2. The hyperoperation $\circ$ of Example 2.3.

$\circ$	a	b	c	d	e	f
a	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S
b	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S
c	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S
d	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S
e	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S
f	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S

Table 3. The hyperoperation $\overline{\circ}$ of Example 2.3.

$\overline{\circ}$	a	b	c	d	e	f	t
a	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S	$\{b\}$
c	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S	$\{c\}$
d	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S	$\{d\}$
e	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S	$\{e\}$
f	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,e\}$	S	$\{f\}$
t	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$	$\{t\}$

3. If $(S,\circ)$ is regular, intra-regular etc., then so is $(S\cup\{e\},\bar{\circ})$

The hypersemigroup $(S,\circ)$ of the Example 2.3 is not regular, as $\nexists\ x\in S$ such that $b\in(b\circ x)*\{b\}$.
It is not left regular as $\nexists\ x\in S$ such that $b\in\{x\}*(b\circ b)$.
It is not left quasi-regular as there are no $x,y\in S$ such that $b\in(x\circ b)*(y\circ b)$.

$(S,\circ)$ is right regular, that is for every $a\in S$ there exists $x\in S$ such that $a\in(a\circ a)*\{x\}$. Indeed, $a\in(a\circ a)*\{c\}$, $b\in(b\circ b)*\{c\}$, $c\in(c\circ c)*\{c\}$, $d\in(d\circ d)*\{d\}$, $e\in(e\circ e)*\{e\}$, $f\in(f\circ f)*\{f\}$.

$(S,\circ)$ is intra-regular, that is for every $a\in S$ there exist $x,y\in S$ such that $a\in(x\circ a)*(a\circ y)$. Indeed, $a\in(a\circ a)*(a\circ a)$, $b\in(c\circ b)*(b\circ c)$, $c\in(c\circ c)*(c\circ c)$, $d\in(d\circ d)*(d\circ d)$, $e\in(e\circ e)*(e\circ e)$, $f\in(f\circ f)*(f\circ f)$.

$(S,\circ)$ is right quasi-regular, that is for every $a\in S$ there exist $x,y\in S$ such that $a\in(a\circ x)*(a\circ y)$. Indeed, $a\in(a\circ a)*(a\circ a)$, $b\in(b\circ f)*(b\circ f)$, $c\in(c\circ f)*(c\circ f)$, $d\in(d\circ f)*(d\circ f)$, $e\in(e\circ f)*(e\circ f)$, $f\in(f\circ f)*(f\circ f)$.

$(S,\circ)$ is semisimple, that is for every $a\in S$ there exist $x,y,z\in S$ such that $a\in(x\circ a)*(y\circ a)*\{z\}$. Indeed, $a\in(a\circ a)*(a\circ a)*\{a\}$, $b\in(b\circ b)*(b\circ b)*\{f\}$, $c\in(a\circ c)*(a\circ c)*\{f\}$, $d\in(a\circ d)*(a\circ d)*\{f\}$, $e\in(e\circ e)*(e\circ e)*\{e\}$, $f\in(a\circ f)*(b\circ f)*\{f\}$.

Theorem 3.1 *If $(S,\circ)$ is a regular hypersemigroup, then the hypersemigroup $\big(S\cup\{e\},\bar{\circ}\big)$ is also regular.*

Proof Let $a\in S\cup\{e\}$. If $a\in S$ then, since $(S,\circ)$ is regular, there exists $x\in S$ such that $a\in(a\circ x)*\{a\}$. Since $a,x\in S$, we have $a\circ x=a\,\bar{\circ}\,x$ and so $a\in(a\,\bar{\circ}\,x)*\{a\}$. On the other hand,

$$(a\,\bar{\circ}\,x)*\{a\}\subseteq(a\,\bar{\circ}\,x)\,\bar{*}\,\{a\}.$$

Indeed: if $t\in(a\,\bar{\circ}\,x)*\{a\}$, then $t\in u\circ a$ for some $u\in a\,\bar{\circ}\,x=a\circ x\subseteq S$. Since $u,a\in S$, we have $u\circ a=u\,\bar{\circ}\,a=\{u\}\,\bar{*}\,\{a\}$. Then $t\in\{u\}\,\bar{*}\,\{a\}\subseteq(a\,\bar{\circ}\,x)\,\bar{*}\,\{a\}$. So: If $a\in S$, then there exists $x\in S\cup\{e\}$ such that $a\in(a\,\bar{\circ}\,x)*\{a\}$. If $a=e$ then, for the element $x:=e\in S\cup\{e\}$, we have $e\in(e\,\bar{\circ}\,e)\,\bar{*}\,\{e\}$. Indeed: $e\,\bar{\circ}\,e=\{e\}$, $(e\,\bar{\circ}\,e)\,\bar{*}\,\{e\}=\{e\}\,\bar{*}\,\{e\}=e\,\bar{\circ}\,e=\{e\}$ and so $e\in(e\,\bar{\circ}\,e)\,\bar{*}\,\{e\}$. □

Theorem 3.2 *If $(S,\circ)$ is an intra-regular hypersemigroup, then the hypersemigroup $\big(S\cup\{e\},\bar{\circ}\big)$ is also intra-regular.*

Proof Let $a\in S\cup\{e\}$. Then there exist $x,y\in S\cup\{e\}$ such that $a\in(x\,\bar{\circ}\,a)\,\bar{*}\,(a\,\bar{\circ}\,y)$. Indeed: if $a\in S$ then, there exist $x,y\in S$ such that $a\in(x\circ a)*(a\circ y)=(x\,\bar{\circ}\,a)*(a\,\bar{\circ}\,y)$. On the other hand,

$$(x\,\bar{\circ}\,a)*(a\,\bar{\circ}\,y)\subseteq(x\,\bar{\circ}\,a)\,\bar{*}\,(a\,\bar{\circ}\,y).$$

Indeed: if $t\in(x\,\bar{\circ}\,a)*(a\,\bar{\circ}\,y)$, then $t\in u\circ v$ for some $u\in x\,\bar{\circ}\,a=x\circ a\subseteq S$, $v\in a\,\bar{\circ}\,y=a\circ y\subseteq S$. Since $u,v\in S$, we have $u\circ v=u\,\bar{\circ}\,v=\{u\}\,\bar{*}\,\{v\}$. Then we have $t\in\{u\}\,\bar{*}\,\{v\}\subseteq(x\,\bar{\circ}\,a)\,\bar{*}\,(a\,\bar{\circ}\,y)$. So: If $a\in S$, then there $x,y\in S\cup\{e\}$ such that $a\in(x\,\bar{\circ}\,a)\,\bar{*}\,(a\,\bar{\circ}\,y)$. If $a=e$ then, for the elements $x=y:=e\in S\cup\{e\}$, we have $e\in(e\,\bar{\circ}\,e)\,\bar{*}\,(e\,\bar{\circ}\,e)$. Indeed: $e\,\bar{\circ}\,e=\{e\}$, $(e\,\bar{\circ}\,e)\,\bar{*}\,(e\,\bar{\circ}\,e)=\{e\}\,\bar{*}\,\{e\}=e\,\bar{\circ}\,e=\{e\}$ and so $e\in(e\,\bar{\circ}\,e)\,\bar{*}\,(e\,\bar{\circ}\,e)$. □

Theorem 3.3 *if $(S,\circ)$ is a right (resp. left) regular hypersemigroup, then then the hypersemigroup $\big(S\cup\{e\},\bar{\circ}\big)$ is right (resp. left) regular.*

Proof Let $(S,\circ)$ be right regular and $a\in S\cup\{e\}$. Then there exists $x\in S\cup\{e\}$ such that $a\in(a\,\bar{\circ}\,a)\,\bar{*}\,\{x\}$. In fact: if $a\in S$ then, since S is right regular, there exists $x\in S$ such that $a\in(a\circ a)*\{x\}=(a\,\bar{\circ}\,a)*\{x\}$. On the other hand,

$$(a\,\bar{\circ}\,a)*\{x\}\subseteq(a\,\bar{\circ}\,a)\,\bar{*}\,\{x\}.$$

Indeed: Let $t\in(a\,\bar{\circ}\,a)*\{x\}$. Then $t\in u\circ x$ for some $u\in a\,\bar{\circ}\,a=a\circ a\subseteq S$. Since $u,x\in S$, we have $u\circ x=u\,\bar{\circ}\,x=\{u\}\,\bar{*}\,\{x\}$. Then $t\in\{u\}\,\bar{*}\,\{x\}\subseteq(a\,\bar{\circ}\,a)\,\bar{*}\,\{x\}$. If $a=e$ then, for the element $x:=e\in S\cup\{e\}$, we have $(e\,\bar{\circ}\,e)\,\bar{*}\,\{e\}=\{e\}\,\bar{*}\,\{e\}=e\,\bar{\circ}\,e=\{e\}$ and so $e\in(e\,\bar{\circ}\,e)\,\bar{*}\,\{e\}$.

Let now $(S,\circ)$ be left regular and $a\in S\cup\{e\}$. Then there exists $x\in S\cup\{e\}$ such that $a\in\{x\}\,\bar{*}\,(a\,\bar{\circ}\,a)$. In fact: if $a\in S$, then there exists $x\in S$ such that $a\in\{x\}*(a\circ a)=\{x\}*(a\,\bar{\circ}\,a)$. On the other hand,

$$\{x\}*(a\,\bar{\circ}\,a)\subseteq\{x\}\,\bar{*}\,(a\,\bar{\circ}\,a).$$

Indeed: Let $t\in\{x\}*(a\,\bar{\circ}\,a)$. Then $t\in x\circ u$ for some $u\in a\,\bar{\circ}\,a=a\circ a\subseteq S$. Since $u,x\in S$, we have $x\circ u=x\,\bar{\circ}\,u=\{x\}\,\bar{*}\,\{u\}$. Then $t\in\{x\}\,\bar{*}\,\{u\}\subseteq\{x\}\,\bar{*}\,(a\,\bar{\circ}\,a)$. If $a=e$ then, for the element $x:=e\in S\cup\{e\}$, we have $e\in\{e\}\,\bar{*}\,(e\,\bar{\circ}\,e)$. □

Theorem 3.4 *If $(S,\circ)$ is a right (resp. left) quasi-regular hypersemigroup, then the hypersemigroup $\big(S\cup\{e\},\bar{\circ}\big)$ is right (resp. left) quasi-regular as well.*

Proof Let $(S,\circ)$ be right quasi-regular and $a\in S\cup\{e\}$. Then there exist $x,y\in S\cup\{e\}$ such that $a\in(a\,\bar{\circ}\,x)\,\bar{*}\,(a\,\bar{\circ}\,y)$. Indeed: if $a\in S$, then there exist $x,y\in S$ such that $a\in(a\circ x)*(a\circ y)=(a\,\bar{\circ}\,x)*(a\,\bar{\circ}\,y)$. On the other hand,

$$(a\,\bar{\circ}\,x)*(a\,\bar{\circ}\,y)\subseteq(a\,\bar{\circ}\,x)\,\bar{*}\,(a\,\bar{\circ}\,y).$$

Indeed: if $t\in(a\,\bar{\circ}\,x)*(a\,\bar{\circ}\,y)$, then $t\in u\circ v$ for some $u\in a\,\bar{\circ}\,x=a\circ x\subseteq S$, $v\in a\,\bar{\circ}\,y=a\circ y\subseteq S$. Since $u,v\in S$, we have $u\circ v=u\,\bar{\circ}\,v=\{u\}\,\bar{*}\,\{v\}$. Then we have $t\in\{u\}\,\bar{*}\,\{v\}\subseteq(a\,\bar{\circ}\,x)\,\bar{*}\,(a\,\bar{\circ}\,y)$. So: For $a\in S$, there exist $x,y\in S\cup\{e\}$ such that $a\in(a\,\bar{\circ}\,x)\,\bar{*}\,(a\,\bar{\circ}\,y)$. If $a=e$ then, for the elements $x=y:=e$, we have $(e\,\bar{\circ}\,e)\,\bar{*}\,(e\,\bar{\circ}\,e)=\{e\}\,\bar{*}\,\{e\}=e\,\bar{\circ}\,e=\{e\}$ and so $e\in(e\,\bar{\circ}\,e)\,\bar{*}\,(e\,\bar{\circ}\,e)$.

Let now $(S,\circ)$ be left quasi-regular and $a\in S\cup\{e\}$. Then there exist $x,y\in S\cup\{e\}$ such that $a\in(x\,\bar{\circ}\,a)\,\bar{*}\,(y\,\bar{\circ}\,a)$. Indeed: if $a\in S$, then there exist $x,y\in S$ such that $a\in(x\circ a)*(y\circ a)=(x\,\bar{\circ}\,a)*(y\,\bar{\circ}\,a)$. On the other hand,

$$(x\,\bar{\circ}\,a)*(y\,\bar{\circ}\,a)\subseteq(x\,\bar{\circ}\,a)\,\bar{*}\,(y\,\bar{\circ}\,a).$$

Indeed: if $t\in(x\,\bar{\circ}\,a)*(y\,\bar{\circ}\,a)$, then $t\in u\circ v$ for some $u\in x\,\bar{\circ}\,a=x\circ a\subseteq S$, $v\in y\,\bar{\circ}\,a=y\circ a\subseteq S$. Since $u,v\in S$, we have $u\circ v=u\,\bar{\circ}\,v=\{u\}\,\bar{*}\,\{v\}\subseteq(x\,\bar{\circ}\,a)\,\bar{*}\,(y\circ a)$ and so $t\in(x\,\bar{\circ}\,a)\,\bar{*}\,(y\,\bar{\circ}\,a)$. If $a=e$ then, for the element $x:=e\in S\cup\{e\}$, we have $e\in\{e\}\,\bar{*}\,(e\,\bar{\circ}\,e)$. □

Theorem 3.5 *If $(S,\circ)$ is semisimple hypersemigroup, then the hypersemigroup $\big(S\cup\{e\},\bar{\circ}\big)$ is semisimple as well.*

Proof Let $a\in S\cup\{e\}$. Then there exist $x,y,z\in S\cup\{e\}$ such that $a\in(x\,\bar{\circ}\,a)\,\bar{*}\,(y\,\bar{\circ}\,a)\,\bar{*}\,\{z\}$. Indeed: if $a\in S$, then there exist $x,y,z\in S$ such that

$$a\in(x\circ a)*(y\circ a)*\{z\}=(x\,\bar{\circ}\,a)*(y\,\bar{\circ}\,a)*\{z\}\tag{3.1}$$

On the other hand, we have the following:

(A) $(x \,\bar{\circ}\, a) * (y \,\bar{\circ}\, a) * \{z\} \subseteq (x \,\bar{\circ}\, a) \,\overline{*}\, (y \,\bar{\circ}\, a) * \{z\}$

and

(B) $(x \,\bar{\circ}\, a) \,\overline{*}\, (y \,\bar{\circ}\, a) * \{z\} \subseteq (x \,\bar{\circ}\, a) \,\overline{*}\, (y \,\bar{\circ}\, a) \,\overline{*}\, \{z\}$.

Indeed: (A) Let $t \in (x \,\bar{\circ}\, a) * (y \,\bar{\circ}\, a) * \{z\}$. Then $t \in u \circ z$ for some $u \in (x \,\bar{\circ}\, a) * (y \,\bar{\circ}\, a)$, $u \in v \circ w$ for some $v \in x \,\bar{\circ}\, a = x \circ a \subseteq S$, $w \in y \,\bar{\circ}\, a = y \circ a \subseteq S$. Since $v, w \in S$, we have

$$v \circ w = v \,\bar{\circ}\, w = \{v\} \,\overline{*}\, \{w\} \subseteq (x \,\bar{\circ}\, a) \,\overline{*}\, (y \,\bar{\circ}\, a).$$

Thus we have $u \in (x \,\bar{\circ}\, a) \,\overline{*}\, (y \,\bar{\circ}\, a)$. Then $t \in u \circ z = \{u\} * \{z\} \subseteq (x \,\bar{\circ}\, a) \,\overline{*}\, (y \,\bar{\circ}\, a) * \{z\}$.

(B) Let $t \in (x \,\bar{\circ}\, a) \,\overline{*}\, (y \,\bar{\circ}\, a) * \{z\}$. Then $t \in u \circ z$ for some $u \in (x \,\bar{\circ}\, a) \,\overline{*}\, (y \,\bar{\circ}\, a)$, $u \in v \,\bar{\circ}\, w$ for some $v \in x \,\bar{\circ}\, a = x \circ a \subseteq S$, $w \in y \,\bar{\circ}\, a = y \circ a \subseteq S$. Since $v, w \in S$, we have $v \,\bar{\circ}\, w = v \circ w \subseteq S$ and so $u \in S$. Since $u, z \in S$, we have

$$t \in u \circ z = u \,\bar{\circ}\, z = \{u\} \,\overline{*}\, \{z\} \subseteq (x \,\bar{\circ}\, a) \,\overline{*}\, (y \,\bar{\circ}\, a) \,\overline{*}\, \{z\}.$$

Property (3.1) follows immediately by (A) and (B). □

We apply Theorems 3.1, 3.2, 3.3, 3.4, and 3.5 to the following example.

Example 3.6 We consider the ordered semigroup $(S, \cdot)$ given by Table 4 and Figure 2.

Table 4. The multiplication table of the semigroup of the Example 3.6.

$\cdot$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

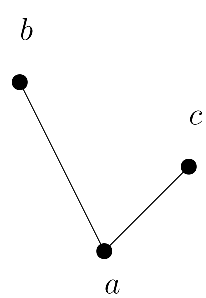

Figure 2. The order of Example 3.6.

Using the Light's associativity test we immediately see that this is really an ordered semigroup. Using the methodology described in [10], we get the hypersemigroup $(S, \circ)$ given by Table 5. If we want to prove independently that $(S, \circ)$ is an hypersemigroup, then we have to check 27 cases as, for example, $\{c\} * (b \circ a) = \{c\} * \{a\} = c \circ a = \{a\}$, $(c \circ b) * \{a\} = \{a, b\} * \{a\} = (a \circ a) \cup (b \circ a) = \{a\} \cup \{a\} = \{a\}$ and so $\{c\} * (b \circ a) = (c \circ b) * \{a\}$ and so on. But still, there is a method like the Light's associativity test to check it immediately using tables. We take an element e not included in S and, according to Theorem 2.2, the set $S \cup \{e\}$ with the hyperoperation $\bar{\circ}$ given by Table 6 is an hypersemigroup having the element e as the identity element.

Table 5. The hyperoperation $\circ$ of Example 3.6.

$\circ$	a	b	c
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$
b	$\{a\}$	$\{a,b\}$	$\{a,c\}$
c	$\{a\}$	$\{a,b\}$	$\{a,c\}$

Table 6. The hyperoperation $\overline{\circ}$ of Example 3.6.

$\overline{\circ}$	a	b	c	e
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{b\}$
c	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{c\}$
e	$\{a\}$	$\{b\}$	$\{c\}$	$\{e\}$

The hypersemigroup $(S,\circ)$ is regular, that is for every $a \in S$ there exists $x \in S$ such that $a \in (a \circ x) * \{a\}$. Indeed, $a \in (a \circ a) * \{a\}$, $b \in (b \circ b) * \{b\}$ and $c \in (c \circ c) * \{c\}$.

The hypersemigroup $(S,\circ)$ is intra-regular, that is for every $a \in S$ there exist $x, y \in S$ such that $a \in (x \circ a) * (a \circ y)$. Indeed, $a \in (a \circ a) * (a \circ a)$, $b \in (b \circ b) * (b \circ b)$ and $c \in (c \circ c) * (c \circ c)$.

$(S,\circ)$ is right regular, that is for every $a \in S$ there exists $x \in S$ such that $a \in (a \circ a) * \{x\}$. Indeed, $a \in (a \circ a) * \{a\}$, $b \in (b \circ b) * \{b\}$, $c \in (c \circ c) * \{c\}$.

$(S,\circ)$ is left regular, that is for every $a \in S$ there exists $x \in S$ such that $a \in \{x\} * (a \circ a)$. Indeed, $a \in \{a\} * (a \circ a)$, $b \in \{b\} * (b \circ b)$, $a \in \{c\} * (c \circ c)$.

$(S,\circ)$ is right quasi-regular, that is for every $a \in S$ there exist $x, y \in S$ such that $a \in (a \circ x) * (a \circ y)$; as $a \in (a \circ a) * (a \circ a)$, $b \in (b \circ b) * (b \circ b)$ and $c \in (c \circ c) * (c \circ c)$.

$(S,\circ)$ is left quasi-regular, that is for every $a \in S$ there exist $x, y \in S$ such that $a \in (x \circ a) * (y \circ a)$; as $a \in (a \circ a) * (a \circ a)$, $b \in (b \circ b) * (b \circ b)$ and $c \in (c \circ c) * (c \circ c)$.

$(S,\circ)$ is semisimple, that is for every $a \in S$ there exist $x, y, z \in S$ such that $a \in (x \circ a) * (y \circ a) * \{z\}$. Indeed, $a \in (a \circ a) * (a \circ a) * \{a\} = \{a\} * \{a\} * \{a\} = (a \circ a) * \{a\} = \{a\} * \{a\} = a \circ a = \{a\}$.

$b \in (b \circ b) * (b \circ b) * \{b\} = \{a,b\} * \{a,b\} * \{b\} = \{a,b\} * \{b\} = (a \circ b) \cup (b \circ b) = \{a,b\}$.

$c \in (c \circ c) * (c \circ c) * \{c\} = \{a,c\} * \{a,c\} * \{c\} = \{a,c\} * \{c\} = (a \circ c) \cup (c \circ c) = \{a,c\}$.

According to Theorems 3.1, 3.2, 3.3, 3.4, and 3.5, the hypersemigroup $\big(S \circ \{e\}, \overline{\circ}\big)$ is also regular, intra-regular, right regular, left regular, right quasi-regular, left quasi-regular and semisimple.

Independently, $\big(S \circ \{e\}, \overline{\circ}\big)$ is regular, that is for every $a \in S \cup \{e\}$ there exists $x \in S \cup \{e\}$ such that $a \in (a \,\overline{\circ}\, x) \,\overline{*}\, \{a\}$. Indeed,

$a \in (a \,\overline{\circ}\, a) \,\overline{*}\, \{a\} = \{a\} \,\overline{*}\, \{a\} = a \,\overline{\circ}\, a = \{a\}$,

$b \in (b \,\overline{\circ}\, b) \,\overline{*}\, \{b\} = \{a,b\} \,\overline{*}\, \{b\} = (a \,\overline{\circ}\, b) \cup (b \,\overline{\circ}\, b) = \{a,b\} \cup \{a,b\} = \{a,b\}$,

$c \in (c \,\overline{\circ}\, c) \,\overline{*}\, \{c\} = \{a,c\} \,\overline{*}\, \{c\} = (a \,\overline{\circ}\, c) \cup (c \,\overline{\circ}\, c) = \{a,c\} \cup \{a,c\} = \{a,c\}$,

$e \in (e \,\overline{\circ}\, e) \,\overline{*}\, \{e\} = \{e\} \,\overline{*}\, \{e\} = e \,\overline{\circ}\, e = \{e\}$.

$\big(S \circ \{e\}, \overline{\circ}\big)$ is intra-regular, that is for every $a \in S \cup \{e\}$ there exist $x, y \in S \cup \{e\}$ such that

$a \in (x \,\bar{\circ}\, a) \,\bar{*}\, (a \,\bar{\circ}\, y)$. Indeed,

$a \in (a \,\bar{\circ}\, a) \,\bar{*}\, (a \,\bar{\circ}\, a) = \{a\} \,\bar{*}\, \{a\} = \{a\}$,

$b \in (b \,\bar{\circ}\, b) \,\bar{*}\, (b \,\bar{\circ}\, b) = \{a,b\} \,\bar{*}\, \{a,b\} = (a \,\bar{\circ}\, a) \cup (b \,\bar{\circ}\, a) \cup (a \,\bar{\circ}\, b) \cup (b \,\bar{\circ}\, b) = \{a\} \cup \{a,b\} = \{a,b\}$,

$c \in (c \,\bar{\circ}\, c) \,\bar{*}\, (c \,\bar{\circ}\, c) = \{a,c\} \,\bar{*}\, \{a,c\} = (a \,\bar{\circ}\, a) \cup (c \,\bar{\circ}\, a) \cup (a \,\bar{\circ}\, c) \cup (c \,\bar{\circ}\, c) = \{a\} \cup \{a,c\} = \{a,c\}$,

$e \in (e \,\bar{\circ}\, e) \,\bar{*}\, (e \,\bar{\circ}\, e) = \{e\} \,\bar{*}\, \{e\} = \{e\}$.

$\big(S \circ \{e\}, \bar{\circ}\big)$ is right regular, that is for every $a \in S \cup \{e\}$ there exists $x \in S \cup \{e\}$ such that $a \in (a \,\bar{\circ}\, a) \,\bar{*} \{x\}$. Indeed,

$a \in (a \,\bar{\circ}\, a) \,\bar{*} \{a\} = \{a\} \,\bar{*}\, \{a\} = \{a\}$,

$b \in (b \,\bar{\circ}\, b) \,\bar{*}\, \{b\} = \{a,b\} \,\bar{\circ}\, \{b\} = (a \,\bar{\circ}\, b) \cup (b \,\bar{\circ}\, b) = \{a,b\} \cup \{a,b\} = \{a,b\}$,

$c \in (c \,\bar{\circ}\, c) \,\bar{*}\, \{c\} = \{a,c\} \,\bar{*}\, \{c\} = (a \,\bar{\circ}\, c) \cup (c \,\bar{\circ}\, c) = \{a,c\} \cup \{a,c\} = \{a,c\}$,

$e \in (e \,\bar{\circ}\, e) \,\bar{*}\, \{e\} = \{e\} \,\bar{*}\, \{e\} = \{e\}$.

$\big(S \circ \{e\}, \bar{\circ}\big)$ is left regular, that is for every $a \in S \cup \{e\}$ there exists $x \in S \cup \{e\}$ such that $a \in \{x\} \,\bar{*}\, (a \,\bar{\circ}\, a)$. Indeed,

$a \in \{a\} \,\bar{*}\, (a \,\bar{\circ}\, a) = \{a\} \,\bar{*}\, \{a\} = \{a\}$,

$b \in \{b\} \,\bar{*}\, (b \,\bar{\circ}\, b) = \{b\} \,\bar{*}\, \{a,b\} = (b \,\bar{\circ}\, a) \cup (b \,\bar{\circ}\, b) = \{a\} \cup \{a,b\} = \{a,b\}$,

$c \in \{c\} \,\bar{*}\, (c \,\bar{\circ}\, c) = \{c\} \,\bar{*}\, \{a,c\} = (c \,\bar{\circ}\, a) \cup (c \,\bar{\circ}\, c) = \{a\} \cup \{a,c\} = \{a,c\}$,

$e \in \{e\} \,\bar{*}\, (e \,\bar{\circ}\, e) = \{e\} \,\bar{*}\, \{e\} = e \,\bar{\circ}\, e = \{e\}$.

$\big(S \circ \{e\}, \bar{\circ}\big)$ is right quasi-regular, that is for every $a \in S \cup \{e\}$ there exist $x, y \in S \cup \{e\}$ such that $a \in (a \,\bar{\circ}\, x) \,\bar{*}\, (a \,\bar{\circ}\, y)$. Indeed,

$a \in (a \,\bar{\circ}\, a) \,\bar{*}\, (a \,\bar{\circ}\, a) = \{a\}$, $b \in (b \,\bar{\circ}\, b) \,\bar{*}\, (b \,\bar{\circ}\, b) = \{a,b\}$,

$c \in (c \,\bar{\circ}\, c) \,\bar{*}\, (c \,\bar{\circ}\, c) = \{a,c\}$, $e \in (e \,\bar{\circ}\, e) \,\bar{*}\, (e \,\bar{\circ}\, e) = \{e\}$.

$\big(S \circ \{e\}, \bar{\circ}\big)$ is left quasi-regular, that is for every $a \in S \cup \{e\}$ there exist $x, y \in S \cup \{e\}$ such that $a \in (x \,\bar{\circ}\, a) \,\bar{*}\, (y \,\bar{\circ}\, a)$. Indeed,

$a \in (a \,\bar{\circ}\, a) \,\bar{*}\, (a \,\bar{\circ}\, a)$, $b \in (b \,\bar{\circ}\, b) \,\bar{*}\, (b \,\bar{\circ}\, b)$, $c \in (c \,\bar{\circ}\, c) \,\bar{*}\, (c \,\bar{\circ}\, c)$, $e \in (e \,\bar{\circ}\, e) \,\bar{*}\, (e \,\bar{\circ}\, e)$.

$\big(S \circ \{e\}, \bar{\circ}\big)$ is semisimple, that is for every $a \in S \cup \{e\}$ there exist $x, y, z \in S \cup \{e\}$ such that $a \in (x \,\bar{\circ}\, a) \,\bar{*}\, (y \,\bar{*}\, a) \,\bar{*}\, \{z\}$. Indeed,

$a \in (a \,\bar{\circ}\, a) \,\bar{*}\, (a \,\bar{*}\, a) \,\bar{*}\, \{a\} = \{a\} \,\bar{*}\, \{a\} \,\bar{*}\, \{a\} = (a \,\bar{\circ}\, a) \,\bar{*}\, \{a\} = \{a\} \,\bar{*}\, \{a\} = a \,\bar{\circ}\, a = \{a\}$.

$b \in (b \,\bar{\circ}\, b) \,\bar{*}\, (b \,\bar{\circ}\, b) \,\bar{*}\, \{b\} = \{a,b\} \,\bar{*}\, \{a,b\} \,\bar{*}\, \{b\} = (a \,\bar{\circ}\, b) \,\bar{*}\, \{b\} = (a \,\bar{\circ}\, b) \cup (b \,\bar{\circ}\, b) = \{a,b\} \cup \{a,b\} = \{a,b\}$.

$c \in (c \,\bar{\circ}\, c) \,\bar{*}\, (c \,\bar{\circ}\, c) \,\bar{*}\, \{c\} = \{a,c\} \,\bar{*}\, \{a,c\} \,\bar{*}\, \{c\} = \{a,c\} \,\bar{*}\, \{c\} = (a \,\bar{\circ}\, c) \cup (c \,\bar{\circ}\, c) = \{a,c\}$.

$e \in (e \,\bar{\circ}\, e) \,\bar{*}\, (e \,\bar{\circ}\, e) \,\bar{*}\, \{e\} = \{e\} \,\bar{*}\, \{e\} = \{e\}$.

If we consider the semigroup given by Table 4 and change the order to the orders given by Figure 3, Figure 4 or Figure 5, we still get ordered semigroups. The hypersemigroup $(S, \circ)$ that corresponds to Figure 3 is given by Table 7, the hypersemigroup $(S, \circ)$ that corresponds to Figure 4 is given by Table 8, the hypersemigroup $(S, \circ)$ that corresponds to Figure 5 is given by Table 9. Each of these hypersemigroups is regular, intra-regular, right (left) regular, right (left) quasi-regular, and semisimple and the hypersemigroup $\big(S \cup \{e\}, \bar{\circ}\big)$ is, respectively, regular, intra-regular, right (left) regular, right (left) quasi-regular, and semisimple as well.

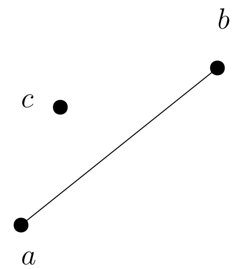

Figure 3. The order of Example 3.6 regarding to Figure 3.

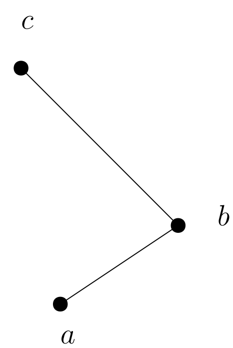

Figure 4. The order of Example 3.6 regarding to Figure 4.

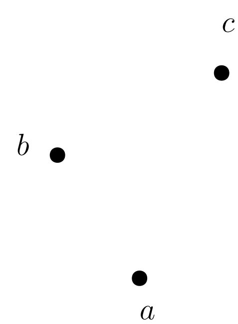

Figure 5. The order of Example 3.6 regarding to Figure 5.

Table 7. The hyperoperation $\circ$ of Example 3.6 regarding to Figure 3.

$\circ$	a	b	c
a	$\{a\}$	$\{a,b\}$	$\{c\}$
b	$\{a\}$	$\{a,b\}$	$\{c\}$
c	$\{a\}$	$\{a,b\}$	$\{c\}$

Table 8. The hyperoperation $\circ$ of Example 3.6 regarding the Figure 4.

$\circ$	a	b	c
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$
b	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$
c	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$

Table 9. The hyperoperation $\circ$ of Example 3.6 regarding to Figure 5.

$\circ$	a	b	c
a	$\{a\}$	$\{b\}$	$\{c\}$
b	$\{a\}$	$\{b\}$	$\{c\}$
c	$\{a\}$	$\{b\}$	$\{c\}$

Example 3.7 We consider the ordered semigroup $S = \{a, b, c, d, e, f, g, h\}$ given by Table 10 and Figure 6. Using the Light's associativity test and its extending form for ordered semigroups (see [6]), we can immediately see that this is really an ordered semigroup. From this semigroup, using the methodology described in [10] (see also [1]) we get the hypersemigroup with the hyperoperation $\circ$ given by Table 11. We get an element t not included in S and, according to Theorem 2.2, the set $S \cup \{t\}$ is an hypersemigroup given by Table 12 and having t as the identity. The hypersemigroup $(S, \circ)$ is regular, intra-regular, right (left) regular, right (left) quasi-regular and semisimple and, by Theorems 3.1, 3.2, 3.3, 3.4 and 3.5, the hypersemigroup $S \cup \{t\}$ is regular, intra-regular, right (left) regular, right (left) quasi-regular and semisimple as well.

Table 10. The multiplication table of the semigroup of Example 3.7.

$\cdot$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	a	b	c	d	e	f	g	h
c	a	b	c	d	e	f	g	h
d	a	b	c	d	e	f	g	h
e	a	b	c	d	e	f	g	h
f	a	b	c	d	e	f	g	h
g	a	b	c	d	e	f	g	h
h	a	b	c	d	e	f	g	h

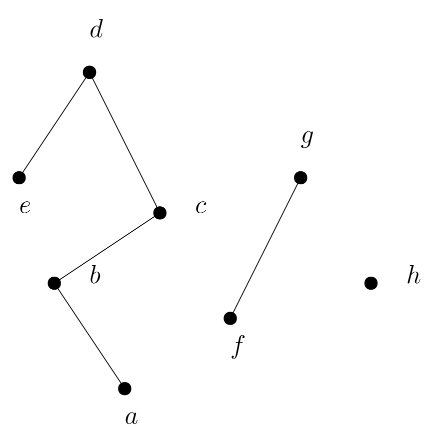

Figure 6. The order of Example 3.7.

Table 11. The hyperoperation $\circ$ of Example 3.7.

$\circ$	a	b	c	d	e	f	g	h
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$
b	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$
c	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$
d	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$
e	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$
f	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$
g	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$
h	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$

Table 12. The hyperoperation $\overline{\circ}$ of Example 3.7.

$\overline{\circ}$	a	b	c	d	e	f	g	h	t
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$	$\{b\}$
c	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$	$\{c\}$
d	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$	$\{d\}$
e	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$	$\{e\}$
f	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$	$\{f\}$
g	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$	$\{g\}$
h	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d,e\}$	$\{e\}$	$\{f\}$	$\{f,g\}$	$\{h\}$	$\{h\}$
t	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$	$\{g\}$	$\{h\}$	$\{t\}$

4. The ideals, the subidempotent bi-ideals or quasi-ideals of $(S,\circ)$ are ideals, bi-ideals or quasi-ideals of $\big(S\cup\{e\},\overline{\circ}\big)$

Proposition 4.1 *If A is an ideal of $(S,\circ)$, then it is an ideal of $\big(S\cup\{e\},\overline{\circ}\big)$.*

Proof Let $A*S\subseteq A$. Then $A\,\overline{*}\,\big(S\cup\{e\}\big)\subseteq A$. Indeed: Let $x\in A\,\overline{*}\,\big(S\cup\{e\}\big)$. Then $x\in a\,\overline{\circ}\,y$ for some $a\in A$, $y\in S\cup\{e\}$. If $y\in S$, then $a\,\overline{\circ}\,y=a\circ y=\{a\}*\{y\}\subseteq A*S\subseteq A$ and so $x\in A$. If $y=e$, then $a\,\overline{\circ}\,y=a\,\overline{\circ}\,e=\{a\}\subseteq A$ and again $x\in A$. Similarly, if A is a left ideal of S, then it is a left ideal of $S\cup\{e\}$. $\square$

Property (E) of Theorem 2.2 follows as an application.

We apply Proposition 4.1 to Examples 4.2 and 4.3.

Example 4.2 We consider the hypersemigroup $S=\{a,b,c,d\}$ of Example 2.14 in [12] with the hyperoperation defined by Table 13.

The set $\{a,c,d\}$ is an ideal of $(S,\circ)$ and, according to Proposition 4.1, the set $\{a,c,d\}$ is an ideal of the hypersemigroup $\big(S\cup\{e\},\overline{\circ}\big)$ as well its hyperoperation given by Table 14.

Independently, we can check that $\{a,c,d\}$ is an ideal of $S\cup\{e\}$ as follows:

Table 13. The hyperoperation $\circ$ of Example 4.2.

$\circ$	a	b	c	d
a	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$
b	$\{a,d\}$	$\{b\}$	$\{a,d\}$	$\{a,d\}$
c	$\{a,d\}$	$\{a,d\}$	$\{c\}$	$\{a,d\}$
d	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$

Table 14. The hyperoperation $\bar{\circ}$ of Example 4.2.

$\bar{\circ}$	a	b	c	d	e
a	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$
b	$\{a,d\}$	$\{b\}$	$\{a,d\}$	$\{a,d\}$	$\{b\}$
c	$\{a,d\}$	$\{a,d\}$	$\{c\}$	$\{a,d\}$	$\{c\}$
d	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$	$\{d\}$
e	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$

$$\begin{aligned}\{a,c,d\} \bar{*} \big(S \cup \{e\}\big) &= \{a,c,d\} \bar{*} \{a,b,c,d,e\} = (a \,\bar{\circ}\, a) \cup (a \,\bar{\circ}\, b) \cup (a \,\bar{\circ}\, c) \cup (a \,\bar{\circ}\, d) \cup (a \,\bar{\circ}\, e) \cup (c \,\bar{\circ}\, a) \cup \\ &\quad (c \,\bar{\circ}\, b) \cup (c \,\bar{\circ}\, c) \cup (c \,\bar{\circ}\, d) \cup (c \,\bar{\circ}\, e) \cup (d \,\bar{\circ}\, a) \cup (d \,\bar{\circ}\, b) \cup (d \,\bar{\circ}\, c) \cup (d \,\bar{\circ}\, d) \cup (d \,\bar{\circ}\, e) \\ &= \{a,c,d\} \subseteq \{a,c,d\}.\end{aligned}$$

$$\begin{aligned}\big(S \cup \{e\}\big) \bar{*} \{a,c,d\} &= \{a,b,c,d,e\} \bar{*} \{a,c,d\} = (a \,\bar{\circ}\, a) \cup (b \,\bar{\circ}\, a) \cup (c \,\bar{\circ}\, a) \cup (d \,\bar{\circ}\, a) \cup (e \,\bar{\circ}\, a) \cup (a \,\bar{\circ}\, c) \cup \\ &\quad (b \,\bar{\circ}\, c) \cup (c \,\bar{\circ}\, c) \cup (d \,\bar{\circ}\, c) \cup (e \,\bar{\circ}\, c) \cup (a \,\bar{\circ}\, d) \cup (b \,\bar{\circ}\, d) \cup (c \,\bar{\circ}\, d) \cup (d \,\bar{\circ}\, d) \cup (e \,\bar{\circ}\, d) \\ &= \{a,c,d\} \subseteq \{a,c,d\}.\end{aligned}$$

Example 4.3 We consider the hypersemigroup $S = \{a,b,c,d,e\}$ of Example 14 in [10] given by Table 15 and the hyperoperation $\bar{\circ}$ of the hypersemigrop $\big(S \cup \{t\}\big)$ given by Table 16. The set $\{a\}$ is an ideal of $(S, \circ)$ and, according to Proposition 4.1, it is an ideal of $\big(S \cup \{t\}, \bar{\circ}\big)$ as well. Independently,

$\{a\} \bar{*} \big(S \cup \{t\}\big) = \{a\} \bar{*} \{a,b,c,d,e,t\} = \{a\} \subseteq \{a\}$ i.e. $\{a\}$ is a right ideal of $\big(S \cup \{t\}\big)$ and
$\big(S \cup \{t\}\big) \bar{*} \{a\} = \{a,b,c,d,e,t\} \bar{*} \{a\} = \{a\} \subseteq \{a\}$ i.e. $\{a\}$ is a left ideal of $\big(S \cup \{t\}\big)$.

Table 15. The hyperoperation $\circ$ of Example 4.3.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,d\}$	$\{a\}$
c	$\{a\}$	$\{a,e\}$	$\{a,c\}$	$\{a,c\}$	$\{a,e\}$
d	$\{a\}$	$\{a,b\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b\}$
e	$\{a\}$	$\{a,e\}$	$\{a\}$	$\{a,c\}$	$\{a\}$

Table 16. The hyperoperation $\bar{\circ}$ of Example 4.3.

$\bar{\circ}$	a	b	c	d	e	t
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,d\}$	$\{a\}$	$\{b\}$
c	$\{a\}$	$\{a,e\}$	$\{a,c\}$	$\{a,c\}$	$\{a,e\}$	$\{c\}$
d	$\{a\}$	$\{a,b\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b\}$	$\{d\}$
e	$\{a\}$	$\{a,e\}$	$\{a\}$	$\{a,c\}$	$\{a\}$	$\{e\}$
t	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{t\}$

The following question arises: If A is a bi-ideal or quasi-ideal of $(S,\circ)$, then is A a bi-ideal or quasi-ideal of of $\big(S\cup\{e\},\bar{\circ}\big)$?

Regarding the bi-ideals, we have the following proposition.

Proposition 4.4 *If A is subidempotent bi-ideal of $(S,\circ)$, then it is a bi-ideal of $\big(S\cup\{e\},\bar{\circ}\big)$.*

Proof Let $A*A\subseteq A$ and $A*S*A\subseteq A$. Then $A\,\bar{*}\,\big(S\cup\{e\}\big)\,\bar{*}\,A\subseteq A$. Indeed: Let $x\in A\,\bar{*}\,\big(S\cup\{e\}\big)\,\bar{*}\,A$. Then

$$x\in u\,\bar{\circ}\,b \text{ for some } u\in A\,\bar{*}\,\big(S\cup\{e\}\big),\ b\in A$$

and

$$u\in c\,\bar{\circ}\,y \text{ for some } c\in A,\ y\in S\cup\{e\}.$$

If $y=e$ then, since $c\in S$, we have $u\in c\,\bar{\circ}\,y=c\,\bar{\circ}\,e=\{c\}$, and $x\in u\,\bar{\circ}\,b=c\,\bar{\circ}\,b$. Since $c,b\in S$, we have $c\,\bar{\circ}\,b=c\circ b$, then $x\in c\circ b=\{c\}*\{b\}\subseteq A*A\subseteq A$ and so $x\in A$. If $y\in S$ then, since $c\in S$, we have $u\in c\,\bar{\circ}\,y=c\circ y\subseteq S$. Since $u,b\in S$, we have $x\in u\,\bar{\circ}\,b=u\circ b$. Then

$$x\in u\circ b=\{u\}*\{b\}\subseteq(c\circ y)*\{b\}=\{c\}*\{y\}*\{b\}\subseteq A*S*A\subseteq A$$

and again $x\in A$. □

Remark 4.5 Every subidempotent subset A of $(S,\circ)$ is a subidempotent subset of $\big(S\cup\{e\},\bar{\circ}\big)$ as well. Indeed: if $A*A\subseteq A$ and $x\in A\,\bar{*}\,A$, then $x\in a\,\bar{\circ}\,b$ for some $a,b\in A$. Since $a,b\in S$, we have $a\,\bar{\circ}\,b=a\circ b$, then $x\in a\circ b=\{a\}*\{b\}\subseteq A*A\subseteq A$ and so $x\in A$. As a result, *if A is a subidempotent bi-ideal of $(S,\circ)$, then it is a subidempotent bi-ideal of $\big(S\cup\{e\},\bar{\circ}\big)$.*

Proposition 4.6 *If Q is a quasi-ideal of $(S,\circ)$, then it is a quasi-ideal of $\big(S\cup\{e\},\bar{\circ}\big)$ as well.*

Proof Let $(Q*S)\cap(S*Q)\subseteq Q$. Then $\Big(Q\,\bar{*}\,\big(S\cup\{e\}\big)\Big)\cap\Big(\big(S\cup\{e\}\big)\,\bar{*}\,Q\Big)\subseteq Q$. Indeed:

Let $t\in\Big(Q\,\bar{*}\,\big(S\cup\{e\}\big)\Big)\cap\Big(\big(S\cup\{e\}\big)\,\bar{*}\,Q\Big)$. Then $t\in Q\,\bar{*}\,\big(S\cup\{e\}\big)$ and $t\in\big(S\cup\{e\}\big)\,\bar{*}\,Q$. Then

$$t\in a\,\bar{\circ}\,b \text{ for some } a\in Q, b\in S\cup\{e\}$$

and

$$t \in c \,\bar{\circ}\, d \text{ for some } c \in S \cup \{e\}, d \in Q.$$

We have

$t \in a \,\bar{\circ}\, b,\ a \in Q,\ \big(b \in S \text{ or } b = e\big),$

$t \in c \,\bar{\circ}\, d,\ d \in Q,\ \big(c \in S \text{ or } c = e\big).$

We have to check the following cases:

(1) $t \in a \,\bar{\circ}\, b,\ a \in Q,\ b \in S,\ t \in c \,\bar{\circ}\, d,\ d \in Q,\ c \in S,$

(2) $t \in a \,\bar{\circ}\, b,\ a \in Q,\ b \in S,\ t \in c \,\bar{\circ}\, d,\ d \in Q,\ c = e,$

(3) $t \in a \,\bar{\circ}\, b,\ a \in Q,\ b = e,\ t \in c \,\bar{\circ}\, d,\ d \in Q,\ c \in S,$

(4) $t \in a \,\bar{\circ}\, b,\ a \in Q,\ b = e,\ t \in c \,\bar{\circ}\, d,\ d \in Q,\ c = e.$

(1) Let $t \in a \,\bar{\circ}\, b,\ a \in Q,\ b \in S,\ t \in c \,\bar{\circ}\, d,\ d \in Q,\ c \in S$. Since $a, b \in S$, we have $a \,\bar{\circ}\, b = a \circ b$. Since $c, d \in S$, we have $c \,\bar{\circ}\, d = c \circ d$. Then we have $t \in a \circ b = \{a\} * \{b\} \subseteq Q * S$ and $t \in c \circ d = \{c\} * \{d\} \subseteq S * Q$. Hence $t \in (Q * S) \cap (S * Q) \subseteq Q$ and so $t \in Q$.

(2) From (2), we have $t \in c \,\bar{\circ}\, d,\ d \in Q,\ c = e$. Since $d \in S,\ c = e$, we have $t = e \,\bar{\circ}\, d = \{d\}$. Then $t = d \in Q$ and so $t \in Q$.

(3) From (3), we have $t \in a \,\bar{\circ}\, b,\ a \in Q,\ b = e$. Since $a \in S,\ b = e$, we have $t \in a \,\bar{\circ}\, b = a \,\bar{\circ}\, e = \{a\}$. Then $t = a \in Q$ and so $t \in Q$.

(4) From (4), we have $t \in a \,\bar{\circ}\, b,\ a \in Q,\ b = e$; the same with (3). So $t \in Q$. □

Example 4.7 We consider the hypersemigroup $(S, \circ)$ of the Example 4.2 given by Table 13. The set $\{a, c, d\}$ is an ideal of $(S, \circ)$, so it is a quasi-ideal and a bi-ideal of $(S, \circ)$ as well. Independently,

$$\Big(\{a,c,d\} * \{a,b,c,d\}\Big) \cap \Big(\{a,b,c,d\} * \{a,c,d\}\Big) = \{a,c,d\} \cap \{a,c,d\} = \{a,c,d\}$$

and so $\{a, c, d\}$ is a quasi-ideal of $(S, \circ)$. We also have

$$\Big(\{a,c,d\} * \{a,b,c,d\}\Big) * \{a,c,d\} = \{a,c,d\} * \{a,c,d\} = \{a,c,d\}$$

and so $\{a, c, d\}$ is a bi-ideal of $(S, \circ)$. By Proposition 4.6, the set $\{a, c, d\}$ is a quasi-ideal of $\big(S \cup \{e\}, \bar{\circ}\big)$. The set $\{a, c, d\}$ is a subidempotent subset of $(S, \circ)$ and, according to Remark 4.5, the set $\{a, c, d\}$ is a subidempotent bi-ideal of $\big(S \cup \{e\}, \bar{\circ}\big)$ (its hyperoperation given by Table 14).

Let us finally give some information about the interior ideals. A nonempty subset A of an hypersemigroup $(S, \circ)$ is called *interior ideal* of S is $S * A * S \subseteq A$. That is, if $x \in u \circ s$ and $u \in t \circ b$ for some $s, t \in S$, $b \in A$, then $x \in A$. Regarding the interior ideals the following proposition holds.

Proposition 4.8 *If $(S, \circ)$ is an hypersemigroup and A is an ideal of $(S, \circ)$, then it is an interior ideal $\big(S \cup \{e\}, \bar{\circ}\big)$.*

Proof Since A is an ideal of $(S, \circ)$, by Proposition 4.1, A is an ideal of $\big(S \cup \{e\}, \bar{\circ}\big)$; that is $\big(S \cup \{e\}\big) \,\bar{*}\, A \subseteq A$ and $A \,\bar{*}\, \big(S \cup \{e\}\big) \subseteq A$. Then we have $\Big(\big(S \cup \{e\}\big) \,\bar{*}\, A\Big) \,\bar{*}\, \big(S \cup \{e\}\big) \subseteq A \,\bar{*}\, \big(S \cup \{e\}\big) \subseteq A$. □

Example 4.9 According to Proposition 4.8, the ideal $\{a, c, d\}$ of the Example 4.2 and the ideal $\{a\}$ of the Example 4.3 are interior ideals of $\left(S \cup \{e\}, \overline{\circ}\right)$.

Remark 4.10 For any semigroup $(S, \cdot)$ defined by a table of the form of Table 17 (like Tables 4 or 10) and any order relation $\leq$ on $(S, \cdot)$, the corresponding hypersemigroup $(S, \circ)$ defined by $a \circ b := \{t \in S \mid t \leq ab\}$ is regular, intra-regular, right (left) regular, right (left) quasi-regular and semisimple and the hypersemigroup $(S, \overline{\circ})$ defined in Theorem 2.2 is, respectively, so.

Table 17. The multiplication of the semigroup of Remark 4.10.

$\cdot$	a	b	c	d	$\cdot$	$\cdot$	$\cdot$
a	a	b	c	d	$\cdot$	$\cdot$	$\cdot$
b	a	b	c	d	$\cdot$	$\cdot$	$\cdot$
c	a	b	c	d	$\cdot$	$\cdot$	$\cdot$
d	a	b	c	d	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

5. Isomorphic hypersemigroups

Definition 5.1 *Two hypersemigroups $(S, \circ)$ and $(T, \overline{\circ})$ are called isomorphic if there exists a (1-1) mapping f of S onto T such that $f(a \circ b) \subseteq f(a) \;\overline{\circ}\; f(b)$ for every $a, b \in S$; in the sense that if $u \in a \circ b$, then $f(u) \in f(a) \,\overline{\circ}\, f(b)$.*

Theorem 5.2 *If $(S, \circ, *)$ and $(T, \overline{\circ}, \overline{*})$ are two isomorphic hypersemigroups and $(S, \circ, *)$ is regular, then so is $(T, \overline{\circ}, \overline{*})$.*

Proof Let $f : (S, \circ, *) \rightarrow (T, \overline{\circ}, \overline{*})$ be an isomorphism, $(S, \circ, *)$ be regular and $a \in T$. Then there exists $x \in T$ such that $a \in (a \;\overline{\circ}\; x) \;\overline{*}\; \{a\}$. Indeed: Since f is onto, there exists $b \in S$ such that $f(b) = a$. Since $(S, \circ)$ is regular, there exists $y \in S$ such that $b \in (b \circ y) * \{b\}$. Then $b \in u \circ b$ for some $u \in b \circ y$. Since $f(u \circ b) \subseteq f(u) \;\overline{\circ}\; f(b)$ and $b \in u \circ b$, we have $f(b) \in f(u) \;\overline{\circ}\; f(b)$. Since $f(b \circ y) \subseteq f(b) \;\overline{\circ}\; f(y)$ and $u \in b \circ y$, we have $f(u) \in f(b) \;\overline{\circ}\; f(y)$. Then we have

$$a = f(b) \in f(u) \,\overline{\circ}\, f(b) = \{f(u)\} \;\overline{*}\; \{f(b)\} \subseteq \Big(f(b) \,\overline{\circ}\, f(y)\Big) \;\overline{*}\; \{f(b)\}.$$

We put $f(y) := x$ and we have $x \in T$ and $a \in (a \,\overline{\circ}\, x) \;\overline{*}\; \{a\}$. □

In a similar way, we have the following:

Theorem 5.3 *If $(S, \circ, *)$ and $(T, \overline{\circ}, \overline{*})$ are two isomorphic hypersemigroups and $(S, \circ, *)$ is intra-regular, right (left) regular, right (left) quasi-regular or semisimple, then $(T, \overline{\circ}, \overline{*})$, respectively, is so.*

Remark 5.4 A second proof of the first part of the theorems of Section 3 can be given in the way indicated in Section 5 as well. Of course, we have to prove the Theorem 5.3 in that case. However, Section 3 shows exactly the role of the operations $\overline{\circ}$ and $*$ that according to the bibliography the two operations are the same.

The hypersemigroups $(S, \circ)$ and $(S, \overline{\circ})$ considered in Theorem 2.2 are isomorphic under the identity mapping. Indeed, for the one to one and onto mapping $i : (S, \circ) \rightarrow (S, \overline{\circ}) \mid a \rightarrow i(a) := a$, we have $i(a \circ b) \subseteq i(a) \,\overline{\circ}\, i(b)$; that is $u \in a \circ b$ implies $u \in a \,\overline{\circ}\, b$. This is clear as $a, b \in S$ implies $a \circ b = a \,\overline{\circ}\, b$. So the Theorem 2.2 can be given in the following way as well.

Theorem 5.5 *Every hypersemigroup can be embedded in an hypersemigroup having an identity element.*

Note In Remark 3.2 in [11], the set $\mathcal{P}^*(S)$ of all nonempty subsets of S is a $\vee e$-semigroup and not le-semigroup as for $A, B \in \mathcal{P}^*(S)$ the intersection $A \cap B$ can be empty. So in [11, Example 3.15], and in Conclusion part of the paper, the word le-semigroup should be replaced by $\vee e$-semigroup (this is obvious from Figure 3 of the example as well).

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