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Minimal Legendrian submanifolds of $\mathbb{S}^9$ with nonnegative sectional curvature

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Abstract: In this paper, we established a complete classification of 4-dimensional compact minimal Legendrian submanifolds with nonnegative sectional curvature in the 9-dimensional unit sphere.

Key words: Sasakian structure, Legendrian submanifold, minimal submanifold, C-parallel, Calabi torus

1. Introduction

Let M^m be an m-dimensional submanifold isometrically immersed in the unit sphere $\mathbb{S}^{2n+1}$, which has canonical Sasakian structure (φ, ξ, η, g) (see Section 2 for more details). We say that M^m is *C-totally real (or integral)* if the contact form η restricted to M^m vanishes, i.e. $\eta(X) = 0$ for any $X \in TM^m$. In particular, if $m = n$, we call it a *Legendrian submanifold*. The study of these submanifolds is an important geometry topic that has been widely carried out, see, e.g., among many others, [1–5, 8, 11, 12, 18].

Now, we are interested in the problem of how to classify n-dimensional compact minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$ with nonnegative sectional curvature. For the case $n = 2$, Yamaguchi, Kon and Miyahara [19] proved that M^2 is $\mathbb{S}^2$ (M^2 is totally geodesic and $K \equiv 1$) or T^2 (M^2 is flat and $K \equiv 0$). In [8], Dillen and Vrancken settled this problem for $n = 3$ by giving the following classification:

Theorem 1.1 ([8]) *Let $x : M^3 \to \mathbb{S}^7$ be a C-totally real, minimal immersion of a 3-dimensional compact Riemannian manifold M. If the sectional curvatures K of M satisfying $K \geq 0$, then it holds the following:*

(1) M is simply connected and x is congruent to $i : S^3 \to \mathbb{S}^7$ (i.e. M is totally geodesic in $\mathbb{S}^7$), or

(2) M is a covering of T^3 with covering map π and x is congruent to $j \circ \pi : M \to \mathbb{S}^7$, or

(3) M is a covering of $\mathbb{S}^1(\sqrt{3}) \times \mathbb{S}^2(\sqrt{3}/2)$ with covering map π and x is congruent to $k \circ \pi : M \to \mathbb{S}^7$, where the map i, j and k are defined in section 5 of [8].

In this paper, we gave a complete classification of 4-dimensional compact minimal Legendrian submanifolds with nonnegative sectional curvature in $\mathbb{S}^9$, which extends the above theorem. To state our result, we first introduce several canonical examples.

Example 1.1 The totally geodesic Legendrian sphere in $\mathbb{S}^9$ (cf. [6]).

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Let L be a 5-dimensional linear subspace of $\mathbb{C}^5$ passing through the origin and such that JL is orthogonal to L. Then $\mathbb{S}^4 = \mathbb{S}^9 \cap L =: f^{(1)}(\mathbb{S}^4)$, is a 4-dimensional totally geodesic compact minimal Legendrian submanifold of $\mathbb{S}^9$.

Example 1.2 The flat torus in $\mathbb{S}^9$ (cf. [6, 8]).

Let $\mathbb{S}^1$ be a circle of radius 1 and $T^4 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Then, with the usual parameterization $u = (u_1, u_2, u_3, u_4)$ of T^4, an immersion $f^{(2)} : T^4 \to \mathbb{S}^9$ is defined by

$$f^{(2)}(u) = \tfrac{1}{\sqrt{5}}(e^{iu_1}, e^{iu_2}, e^{iu_3}, e^{iu_4}, e^{-i(u_1+u_2+u_3+u_4)}) \in \mathbb{C}^5 \simeq \mathbb{R}^{10}$$

is a flat compact minimal Legendrian submanifold of $\mathbb{S}^9$ (see [6] for detailed computation).

Example 1.3 The Calabi torus in $\mathbb{S}^9$ (cf. [11, 13]).

Let $\psi : \mathbb{S}^3 \hookrightarrow \mathbb{R}^4 : p \mapsto (y_1, y_2, y_3, y_4)$ be the inclusion mapping, and $\gamma : \mathbb{R} \hookrightarrow \mathbb{S}^3$ be the standard embedding with a parametrization

$$\gamma(t) = \big(\tfrac{2}{\sqrt{5}}e^{-\frac{it}{2}}, \tfrac{1}{\sqrt{5}}e^{i2t}\big) = (\gamma_1(t), \gamma_2(t)) \in \mathbb{C}^2.$$

Putting $f^{(3)} : \mathbb{R} \times \mathbb{S}^3 \to \mathbb{S}^9$ such that $f^{(3)}(t,p) = (\gamma_1(t)\psi(p), \gamma_2(t)) \in \mathbb{C}^5 \simeq \mathbb{R}^{10}$. For the sake of calculation, we assume that $f^{(3)}(t,p) = (x_1, x_2, \cdots, x_{10})$, then

$$\begin{cases} (x_{2i-1},\ x_{2i}) = \frac{2}{\sqrt{5}}\big(y_i \cos\frac{t}{2},\ -y_i \sin\frac{t}{2}\big), \quad 1 \le i \le 4, \\ \quad (x_9,\ x_{10}) = \frac{1}{\sqrt{5}}\big(\cos 2t,\ \sin 2t\big), \end{cases}$$

where $(y_1, y_2, y_3, y_4) = (\sin\theta_1,\ \cos\theta_1 \sin\theta_2,\ \cos\theta_1\cos\theta_2\sin\theta_3,\ \cos\theta_1\cos\theta_2\cos\theta_3)$.

Choose an orthonormal frame $\{e_i\}_{i=1}^4$ for $f^{(3)}(\mathbb{R} \times \mathbb{S}^3)$ with respect to g, where g is the induced metric of $\mathbb{S}^9 \to \mathbb{R}^{10}$, such that

$$e_1 = f^{(3)}_t, \quad e_2 = \tfrac{\sqrt{5}}{2} f^{(3)}_{\theta_1}, \quad e_3 = \tfrac{\sqrt{5}}{2\cos\theta_1} f^{(3)}_{\theta_2}, \quad e_4 = \tfrac{\sqrt{5}}{2\cos\theta_1\cos\theta_2} f^{(3)}_{\theta_3}.$$

It is easy to verify that $\eta(e_i) = 0$ for $i = 1, 2, 3, 4$, so $f^{(3)}$ is a Legendrian submanifold. Let D be the standard Euclidean flat connection, by using $g(D_{e_i}e_j, \varphi e_k) = g(h(e_i, e_j), \varphi e_k)$, $1 \le i, j, k \le 4$, we can derive the second fundamental form h satisfies

$$h(e_1, e_1) = \tfrac{3}{2}\varphi e_1, \quad h(e_1, e_j) = -\tfrac{1}{2}\varphi e_j, \quad h(e_i, e_j) = -\tfrac{1}{2}\delta_{ij}\varphi e_1, \quad i, j = 2, 3, 4. \tag{1.1}$$

Then, $f^{(3)}$ is a minimal submanifold with nonnegative sectional curvature. Furthermore, we see that $f^{(3)}(t, y_1, y_2, y_3, y_4) = f^{(3)}(\tilde{t}, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ if and only if $t = \tilde{t} \pmod{4\pi}$ and $y_i = \tilde{y}_i$, $1 \le i \le 4$, thus $f^{(3)}(\mathbb{R} \times \mathbb{S}^3)$ is isometric with $\mathbb{S}^1(2) \times \mathbb{S}^3(2/\sqrt{5})$.

Therefore, we obtain an embedding $\tilde{f}^{(3)}$ from $\mathbb{S}^1(2) \times \mathbb{S}^3(2/\sqrt{5})$ into $\mathbb{S}^9$, which is a compact minimal Legendrian submanifold with nonnegative sectional curvature.

Example 1.4 Let $\psi : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{S}^7 : (t_2, p) \mapsto \big(\frac{\sqrt{3}}{2}e^{-\frac{it_2}{\sqrt{3}}}\tau(p), \frac{1}{2}e^{i\sqrt{3}t_2}\big)$ be a Calabi torus defined in [11], here $\tau : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is the inclusion mapping. Then, following the method of Li-Wang [15], we can define

$f^{(4)}:\mathbb{R}^2\times\mathbb{S}^2\to\mathbb{S}^9$ by

$$
\begin{aligned}
f^{(4)}(t_1,t_2,p)=&(\tfrac{2}{\sqrt{5}}e^{-\frac{it_1}{5}}\psi(t_2,p),\ \tfrac{1}{\sqrt{5}}e^{\frac{i4t_1}{5}})\\
=&(\tfrac{\sqrt{3}}{\sqrt{5}}e^{-i(\frac{t_1}{5}+\frac{t_2}{\sqrt{3}})}\tau(p),\ \tfrac{1}{\sqrt{5}}e^{-i(\frac{t_1}{5}-\sqrt{3}t_2)},\ \tfrac{1}{\sqrt{5}}e^{\frac{i4t_1}{5}}).
\end{aligned}
$$

Similar to Example 1.3, putting $f^{(4)}(t_1,t_2,p)=(x_1,x_2,\cdots,x_{10})$, then we have

$$
\begin{cases}
(x_{2i-1},x_{2i})=\frac{\sqrt{3}}{\sqrt{5}}y_i\big(\cos(\frac{t_1}{5}+\frac{t_2}{\sqrt{3}}),\ -\sin(\frac{t_1}{5}+\frac{t_2}{\sqrt{3}})\big), \quad i=1,2,3,\\
(x_7,x_8)=\frac{1}{\sqrt{5}}\big(\cos(\frac{t_1}{5}-\sqrt{3}t_2),\ -\sin(\frac{t_1}{5}-\sqrt{3}t_2)\big),\\
(x_9,x_{10})=\frac{1}{\sqrt{5}}\big(\cos\frac{4t_1}{5},\ \sin\frac{4t_1}{5}\big),
\end{cases}
$$

where $(y_1,y_2,y_3)=(\sin\theta_1,\ \cos\theta_1\sin\theta_2,\ \cos\theta_1\cos\theta_2)=\tau(p)$.

Choose an orthonormal frame $\{e_i\}_{i=1}^4$ for $f^{(4)}(\mathbb{R}^2\times\mathbb{S}^2)$:

$$
e_1=\tfrac{5}{2}f^{(4)}_{t_1},\quad e_2=\tfrac{\sqrt{5}}{2}f^{(4)}_{t_2},\quad e_3=\sqrt{\tfrac{5}{3}}f^{(4)}_{\theta_1},\quad e_4=\sqrt{\tfrac{5}{3\cos^2\theta_1}}f^{(4)}_{\theta_2}.
$$

By computation we can derive that the second fundamental form h satisfies

$$
\begin{cases}
h(e_1,e_j)=-\frac{1}{2}\varphi e_j+2\delta_{1j}\varphi e_1, \quad 1\le j\le 4,\\
h(e_2,e_2)=-\frac{1}{2}\varphi e_1+\sqrt{\frac{5}{3}}\varphi e_2, \quad h(e_3,e_4)=0,\\
h(e_k,e_k)=-\frac{1}{2}\varphi e_1-\sqrt{\frac{5}{12}}\varphi e_2, \quad h(e_2,e_k)=-\sqrt{\frac{5}{12}}\varphi e_k, \quad k=3,4.
\end{cases}
\tag{1.2}
$$

Then, $f^{(4)}:\mathbb{R}^2\times\mathbb{S}^2\to\mathbb{S}^9$ is a minimal submanifold with nonnegative sectional curvatures. Furthermore, we see that $f^{(4)}(t_1,t_2,y_1,y_2,y_3)=f^{(4)}(\tilde{t}_1,\tilde{t}_2,\tilde{y}_1,\tilde{y}_2,\tilde{y}_3)$ if and only if $t_1=\tilde{t_1}\pmod{5\pi}$, $t_2=\tilde{t_2}\pmod{\sqrt{3}\pi}$ and $y_i=\tilde{y}_i$, $i=1,2,3$.

Therefore, we obain an embedding $\tilde{f}^{(4)}:T^2\times\mathbb{S}^2(\sqrt{3/5})\to\mathbb{S}^9$, which is a compact minimal Legendrian submanifold with nonnegative sectional curvature.

Having the preceding preparations, now we can state our main theorem as follows:

Theorem 1.2 *Let $x:M^4\to\mathbb{S}^9$ be a compact minimal Legendrian submanifold of the 9-dimensional unit sphere. If the sectional curvature $K\ge 0$, then it holds the following:*

(1) *$x(M^4)$ is totally geodesic and is given by $f^{(1)}$ in Example 1.1; or*

(2) *$x(M^4)$ is flat and is given by $f^{(2)}$ in Example 1.2; or*

(3) *$x(M^4)$ is congruent to $\mathbb{S}^1(2)\times\mathbb{S}^3(2/\sqrt{5})$ and is given by $\tilde{f}^{(3)}$ in Example 1.3; or*

(4) *$x(M^4)$ is congruent to $T^2\times\mathbb{S}^2(\sqrt{3/5})$ and is given by $\tilde{f}^{(4)}$ in Example 1.4.*

Remark 1.3 *We note that the Legendrian submanifolds given in (1) and (2) of Theorem 1.2 have constant sectional curvature, which have been described in [6]. While the last two examples have nonconstant sectional curvature, they are constructed by Calabi product of a 3-dimensional Legendrian submanifold in $\mathbb{S}^7$ and a point (refer to [15]).*

2. Preliminaries

In this section, we first review some basic formulas about Sasakian manifold $\mathbb{S}^{2n+1}$ and its Legendrian submanifolds (see [2, 12] for details), then we give an important property for compact, minimal, Legendrian submanifolds.

2.1. The Sasakian structure on the (2n+1)-dimensional unit sphere

As a Sasakian manifold, the unit sphere $\mathbb{S}^{2n+1}$ has constant φ-sectional curvature 1 and canonical Sasakian structure (φ, ξ, η, g): g is the induced metric; $\xi = JN$, J is the natural complex structure of $\mathbb{C}^{n+1}$ and N is the unit normal vector field of the inclusion $\mathbb{S}^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$; let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian metric on $\mathbb{C}^{n+1}$. Then for any tangent vector fields X, Y of $\mathbb{S}^{2n+1}$, it holds that:

$$\begin{cases} \varphi(X) = JX - \langle JX, N\rangle N, \quad \varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \\ \eta(X) = g(X,\xi), \quad \eta(\varphi X) = 0, \quad d\eta(X,Y) = g(X, \varphi Y), \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \operatorname{rank}(\varphi) = 2n, \end{cases} \tag{2.1}$$

$$\bar{\nabla}_X \xi = -\varphi X, \quad (\bar{\nabla}_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.2}$$

where $\bar{\nabla}$ is the Levi-Civita connection with respect to the metric g.

The curvature tensor $\bar{R}(X,Y)Z := \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z$ of $\mathbb{S}^{2n+1}$ has the expression: $\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y$.

2.2. Legendrian submanifolds of $\mathbb{S}^{2n+1}$

Let M^n be an n-dimensional submanifold isometrically immersed in the unit sphere $\mathbb{S}^{2n+1}$. Denote also by g the metric of M^n, and ∇ the Levi-Civita connection of (M^n, g). The Gauss and Weingarten formulae of $M^n \hookrightarrow \mathbb{S}^{2n+1}$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.3}$$

where $X, Y \in TM^n$ are tangent vector fields, $V \in T^{\perp}M^n$ is a normal vector field, h is the second fundamental form of M^n, A_V is the shape operator associated to V, and $\nabla^{\perp}$ is the normal connection of the normal bundle $T^{\perp}M^n$. From (2.3), we can obtain

$$g(h(X,Y), V) = g(A_V X, Y). \tag{2.4}$$

Assume that M^n is a *Legendrian* submanifold of $\mathbb{S}^{2n+1}$, i.e. the contact form η satisfies $\eta(X) = g(X,\xi) = 0$ for all $X \in TM^n$. Then ξ is a normal vector field of M^n, and by $d\eta(Y,X) = g(Y, \varphi X) = 0$, we see that $\varphi X \in T^{\perp}M^n$. From (2.3), we can also get

$$A_{\varphi Y} X = -\varphi h(X,Y), \quad \nabla^{\perp}_X \varphi Y = \varphi \nabla_X Y + g(X,Y)\xi, \quad \forall X, Y \in TM^n. \tag{2.5}$$

In the sequel, we will make the following convention on range of indices:

$$i^* = i + n, \quad \alpha^* = \alpha + n; \quad 1 \le i,j,k,l,m,p,s \le n; \quad 1 \le \alpha, \beta \le n+1.$$

Now, we choose a local *Legendre frame* $\{e_i, e_{i^*}, e_{2n+1}\}_{i=1}^n$ in $\mathbb{S}^{2n+1}$ along M^n, such that $\{e_i\}_{i=1}^n$ is an orthonormal frame field of TM^n, and $\{e_{i^*} = \varphi e_i, e_{2n+1} = \xi\}_{i=1}^n$ is the orthonormal normal vector fields of $M^n \hookrightarrow \mathbb{S}^{2n+1}$. Denote by $\{\omega^i\}$ the dual frame of $\{e_i\}$. Let $\{\omega_i^j\}$ and $\{\omega_{\alpha^*}^{\beta^*}\}$ denote the connection 1-forms of TM^n and $T^\perp M^n$, respectively:

$$\nabla e_i = \sum_{j=1}^n \omega_i^j e_j, \quad \nabla^\perp e_{\alpha^*} = \sum_{\beta=1}^{n+1} \omega_{\alpha^*}^{\beta^*} e_{\beta^*},$$

where $\omega_i^j + \omega_i^j = 0$ and $\omega_{\alpha^*}^{\beta^*} + \omega_{\beta^*}^{\alpha^*} = 0$. By (2.5), we have $\omega_i^j = \omega_{i^*}^{j^*}$ and $\omega_{i^*}^{2n+1} = \omega^i$.

Put $h_{ij}^{k^*} = g(h(e_i, e_j), \varphi e_k)$. It is easily seen that

$$h_{ij}^{k^*} = h_{ik}^{j^*} = h_{jk}^{i^*}, \quad \forall \ i, j, k. \tag{2.6}$$

From (2.2), (2.4) and the Gauss formula, we get

$$g(A_\xi e_i, e_j) = g(h(e_i, e_j), \xi) = g(\bar{\nabla}_{e_i} e_j, \xi) = -g(e_j, \bar{\nabla}_{e_i}\xi) = g(e_j, \varphi e_i) = 0.$$

$$h_{ij}^{(n+1)^*} = h_{ij}^{2n+1} := g(h(e_i, e_j), e_{2n+1}) = g(A_\xi e_i, e_j) = 0, \quad \forall \ i, j. \tag{2.7}$$

Let $R_{ijkl} := g\big(R(e_i, e_j)e_l, e_k\big)$ and $R_{ij\alpha^*\beta^*} := g\big(R(e_i, e_j)e_{\beta^*}, e_{\alpha^*}\big)$ be the components of the curvature tensors of ∇ and $\nabla^\perp$ with respect to the Legendre frame, respectively. Then the equations of Gauss, Ricci and Codazzi are given by

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_{m=1}^n (h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*}), \tag{2.8}$$

$$R_{ijk^*l^*} = \sum_{m=1}^n (h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*}), \quad R_{ijk^*(2n+1)} = 0, \tag{2.9}$$

$$h_{ij,k}^{\alpha^*} = h_{ik,j}^{\alpha^*}, \tag{2.10}$$

where $h_{ij,k}^{\alpha^*}$ is the component of the covariant differentiation of h, defined by

$$\sum_{k=1}^n h_{ij,k}^{\alpha^*}\omega^k := dh_{ij}^{\alpha^*} - \sum_{k=1}^n h_{kj}^{\alpha^*}\omega_i^k - \sum_{k=1}^n h_{ik}^{\alpha^*}\omega_j^k + \sum_{\beta=1}^{n+1} h_{ij}^{\beta^*}\omega_{\beta^*}^{\alpha^*}, \tag{2.11}$$

or, equivalently,

$$(\bar{\nabla} h)(e_k, e_i, e_j) := \nabla_{e_k}^\perp(h(e_i, e_j)) - h(\nabla_{e_k} e_i, e_j) - h(e_i, \nabla_{e_k} e_j) = \sum_{\alpha=1}^{n+1} h_{ij,k}^{\alpha^*} e_{\alpha^*}. \tag{2.12}$$

For a compact minimal Legendrian submanifold of $\mathbb{S}^{2n+1}$, we have the following result:

Lemma 2.1 (cf. [8, 17]) *If M^n is an n-dimensional compact, minimal, Legendrian submanifold of $\mathbb{S}^{2n+1}$ and if all the sectional curvatures K of M^n satisfy $K \geq 0$, then*

(i) $(\bar{\nabla}h)(u,v,w) = g\big((\bar{\nabla}h)(u,v,w), \xi(p)\big)\xi(p)$,

(ii) $R(v, A_{\varphi v}v, A_{\varphi v}v, v) = 0$, *for all* $p \in M^n$ *and* $u, v, w \in T_pM^n$.

Next, we can naturally define a modified covariant differentiation $\bar{\nabla}^{\xi}h$ by

$$(\bar{\nabla}^{\xi}h)(e_k, e_i, e_j) := (\bar{\nabla}h)(e_k, e_i, e_j) - g\big(h(e_i, e_j), \varphi e_k\big)\xi := \sum_{k=1}^{n} \tilde{h}^{k^*}_{ij,l} e_{k^*}. \tag{2.13}$$

Then, from (2.12) and Lemma 2.1, we know that a compact minimal Legendrian submanifold with nonnegative sectional curvature of $\mathbb{S}^{2n+1}$ satisfies $\bar{\nabla}^{\xi}h = 0$, now the second fundamental form is called C-*parallel*.

Moreover, the second covariant derivative $\tilde{h}^{k^*}_{ij,lp}$ is defined by

$$\sum_{p=1}^{n} \tilde{h}^{k^*}_{ij,lp}\theta_p := dh^{k^*}_{ij,l} + \sum_{p=1}^{n} h^{k^*}_{pj,l}\theta_{pi} + \sum_{p=1}^{n} h^{k^*}_{ip,l}\theta_{pj} + \sum_{p=1}^{n} h^{k^*}_{ij,p}\theta_{pl} + \sum_{p=1}^{n} h^{p^*}_{ij,l}\theta_{p^*k^*}, \tag{2.14}$$

and the components satisfy the following Ricci identity:

$$\tilde{h}^{k^*}_{ij,lp} - \tilde{h}^{k^*}_{ij,pl} = \sum_{m=1}^{n} h^{k^*}_{mj} R_{milp} + \sum_{m=1}^{n} h^{k^*}_{im} R_{mjlp} + \sum_{m=1}^{n} h^{m^*}_{ij} R_{mklp}. \tag{2.15}$$

3. Proof of Theorem 1.2

Let M^4 be a 4-dimensional compact minimal Legendrian submanifold of $\mathbb{S}^9$ with nonnegative sectional curvature. For any fixed point $p \in M^4$, we will consider a construction of typical orthonormal basis with respect to the metric g, which was introduced by Ejiri [9] and has been widely applied, see, e.g., [7, 12, 14, 16].

On $U_pM^4 = \{u \in T_pM^4 | g(u,u) = 1\}$, we define a function $f(u) = g(h(u,u), \varphi u)$. Since U_pM^4 is compact, there is a unit vector $e_1 \in U_pM^4$ at which the function $f(u)$ attains an absolute maximum, denoted by λ_1 and $\lambda_1 \geq 0$. Moreover, we have the following result.

Lemma 3.1 (cf. [10, 12]) *Let $x : M^4 \to \mathbb{S}^9$ be a compact minimal Legendrian submanifold of $\mathbb{S}^9$. If the sectional curvature K of M^4 satisfies $K \geq 0$, then for any fixed point p, there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_pM^4 such that the following hold:*

(i) $h(e_1, e_i) = \lambda_i \varphi e_i, i = 1,2,3,4$, *where* λ_1 *is the maximum of* f,

(ii) $\lambda_1 \geq 2\lambda_i$ *for* $i \geq 2$. *Moreover, if* $\lambda_1 = 2\lambda_j$ *for some* $j \geq 2$, *then* $f(e_j) = 0$,

(iii) $(2\lambda_l - \lambda_1)(1 + \lambda_1\lambda_l - \lambda_l^2) = 0, \quad l \geq 2$.

Lemma 3.2 *Let $x : M^4 \to \mathbb{S}^9$ be a compact minimal Legendrian submanifold with nonnegative sectional curvature. If it is not totally geodesic, then there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that Lemma 3.1 and the following hold:*

$$\lambda_{m+1} = \cdots = \lambda_4 = -\tfrac{m+1}{2(4-m)}\lambda_1, \quad \lambda_1 = \tfrac{2(4-m)}{\sqrt{(m+1)(9-m)}}, \quad m = 1, 2. \tag{3.1}$$

Proof Since x is minimal and $\lambda_1 > 0$, we get $\sum_{i=1}^{4}\lambda_i = 0$ and at least one of $\{\lambda_2, \lambda_3, \lambda_4\}$ is not equal to $\frac{1}{2}\lambda_1$. If there exist $l, t \geq 2$ satisfy $(2\lambda_l - \lambda_1)(2\lambda_t - \lambda_1) \neq 0$, then from (iii) of Lemma 3.1, we have $1+\lambda_1\lambda_l - \lambda_l^2 = 0 = 1+\lambda_1\lambda_t - \lambda_t^2$, thus $(\lambda_l - \lambda_t)[\lambda_1 - (\lambda_l + \lambda_t)] = 0$, together with (ii) of Lemma 3.1, we obtain that $\lambda_l = \lambda_t$.

Assume that there are $(4-m)$ $(1 \leq m \leq 3)$ elements of $\{\lambda_2, \lambda_3, \lambda_4\}$ satisfying $\lambda_i \neq \frac{1}{2}\lambda_1$, we can rearranging the order of $\{e_2, e_3, e_4\}$ such that

$$\lambda_{m+1} = \cdots = \lambda_4 \neq \tfrac{1}{2}\lambda_1, \quad 1+\lambda_1\lambda_4 - \lambda_4^2 = 0,$$

which together with $\sum_{i=1}^{4}\lambda_i = 0$ gives that $\lambda_4 = -\frac{m+1}{2(4-m)}\lambda_1$ and $\lambda_1^2 = \frac{4(4-m)^2}{(m+1)(9-m)}$.

Choose $\varepsilon = \pm 1$ such that $\varepsilon f(e_4) = \kappa \geq 0$. Since $\lambda_1 = f(e_1) = \max_{v\in U_pM^4} f(v)$, then for any $\delta \in (0,1)$ and $u = -\delta e_1 + \sqrt{1-\delta^2}\varepsilon e_4$, we have

$$f(u) = g\big(h(u,u), \varphi u\big) = -\delta^3\lambda_1 - 3\delta(1-\delta^2)\lambda_4 + (\sqrt{1-\delta^2})^3\kappa \leq \lambda_1.$$

Thus $-3\delta(1-\delta^2)\lambda_4 \leq (1+\delta^3)\lambda_1$, which equivalents to $m \leq \frac{11\delta^2 - 11\delta + 8}{-\delta^2+\delta+2} := C(\delta)$. It is not difficult to find that the function $C(\delta)$ attains the minimum value $\frac{7}{3}$ at $\delta = \frac{1}{2}$. Hence $m \leq \frac{7}{3}$ and m can only take $1, 2$.

We have completed the proof of Lemma 3.2. □

Applying Lemma 3.2, we see that there are two cases if M^4 is not totally geodesic:

Case I: $\lambda_1 = \frac{3}{2}$, $\lambda_2 = \lambda_3 = \lambda_4 = -\frac{1}{3}\lambda_1 = -\frac{1}{2}$;

Case II: $\lambda_1 = \frac{4}{\sqrt{21}}$, $\lambda_2 = \frac{1}{2}\lambda_1 = \frac{2}{\sqrt{21}}$, $\lambda_3 = \lambda_4 = -\frac{3}{4}\lambda_1 = -\frac{3}{\sqrt{21}}$.

We first deal with **Case I**.

Follow the method of [9] (or [10]), for fixed point p, we can choose $\{e_2, e_3, e_4\}$ satisfying

$$f(e_2) = \max_{v\in U_pM^4\cap\{e_1\}^\perp} f(v) := a, \qquad f(e_3) = \max_{v\in U_pM^4\cap\{e_1,\ e_2\}^\perp} f(v) := d. \tag{3.2}$$

Then $g(h(e_2,e_2), \varphi e_3) = g(h(e_2,e_2), \varphi e_4) = 0$, $g(h(e_3,e_3), \varphi e_4) = 0$.

We may assume that $g\big(h(e_2,e_3), \varphi e_4\big) = c \geq 0$ by changing the sign of e_4. Now by using (3.2), Lemma 3.1 and the minimality of M^4, we obtain the following expressions for the second fundamental form h:

$$\begin{cases} h(e_1,e_1) = \frac{3}{2}\varphi e_1, \quad h(e_1,e_j) = -\frac{1}{2}\varphi e_j, \quad j = 2,3,4, \\ h(e_2,e_2) = -\frac{1}{2}\varphi e_1 + a\varphi e_2, \quad h(e_2,e_3) = b\varphi e_3 + c\varphi e_4, \\ h(e_3,e_3) = -\frac{1}{2}\varphi e_1 + b\varphi e_2 + d\varphi e_3, \quad h(e_2,e_4) = c\varphi e_3 - (a+b)\varphi e_4, \\ h(e_4,e_4) = -\frac{1}{2}\varphi e_1 - (a+b)\varphi e_2 - d\varphi e_3, \quad h(e_3,e_4) = c\varphi e_2 - d\varphi e_4, \end{cases} \tag{3.3}$$

where $a, b, c, d \in \mathbb{R}$ and $\frac{3}{2} \geq a \geq d \geq 0$, $c \geq 0$.

We note that $h_{ij}^{k^*} = g(h(e_i,e_j), \varphi e_k)$, and $h_{ij}^{k^*}$ is totally symmetric for indices i, j, k. By (3.3) and Gauss equation (2.8), we get

$$\begin{cases} R_{2323} = \frac{5}{4} + ab - b^2 - c^2, \quad R_{2324} = 2ac, \\ R_{2424} = \frac{5}{4} - 2a^2 - b^2 - c^2 - 3ab, \quad R_{2334} = -2dc, \\ R_{3434} = \frac{5}{4} - 2d^2 - b^2 - c^2 - ab, \quad R_{2434} = -d(a+2b). \end{cases} \tag{3.4}$$

From Lemma 2.1, we know that $\bar{\nabla}^{\xi} h = 0$. Hence, by (2.15), for any $l, p \in \{1,2,3,4\}$, we have

$$\sum_{m=1}^{4} h_{mj}^{k^*} R_{milp} + \sum_{m=1}^{4} h_{im}^{k^*} R_{mjlp} + \sum_{m=1}^{4} h_{ij}^{m^*} R_{mklp} = 0. \tag{3.5}$$

Taking $(k,i,j) = (1,1,2), (1,1,3), (1,1,4)$ in (3.5), respectively, and combining with (3.3), we can get

$$R_{12lp} = R_{13lp} = R_{14lp} = 0, \quad l, p \in \{1,2,3,4\}. \tag{3.6}$$

Taking $(k,i,j) = (3,3,3), (2,2,3), (2,2,4), (4,4,4)$ in (3.5), respectively, for any $l, p \in \{1,2,3,4\}$, we have the following equations:

$$3bR_{23lp} = 0, \tag{3.7}$$

$$(a-2b)R_{23lp} - 2cR_{24lp} = 0, \tag{3.8}$$

$$2cR_{23lp} - (3a+2b)R_{24lp} = 0, \tag{3.9}$$

$$dR_{34lp} + (a+b)R_{24lp} = 0. \tag{3.10}$$

From (3.7), (3.4) we obtain that

$$\tfrac{1}{2} bR_{2324} = abc = 0, \qquad -\tfrac{1}{2} bR_{2334} = bcd = 0. \tag{3.11}$$

So we can divide the discussions into the following three subcases:

I-(i): $a = 0$; **I-(ii)**: $a \neq 0,\ b = 0$; **I-(iii)**: $a \neq 0,\ b \neq 0,\ c = 0$.

I-(i). Let $u_1 = \frac{1}{\sqrt{2}}(e_2 + e_3)$, $u_2 = \frac{1}{\sqrt{3}}(e_2 + e_3 + e_4)$, then by (3.2), we have $f(e_2) = f(e_3) = f(e_4) = f(u_1) = f(u_2) = 0$, which implies $d = b = c = 0$.

I-(ii). As $a \neq 0, b = 0$, from (3.4) and (3.8) we can get

$$aR_{2323} - 2cR_{2423} = a(\tfrac{5}{4} - c^2) - 2c(2ac) = 0,$$

which gives that $c^2 = \frac{1}{4}$. Similarly, by (3.4), (3.9) and (3.10), we get

$$2cR_{2323} - 3aR_{2423} = 2c(1 - 3a^2) = 0,$$

$$dR_{3423} + aR_{2423} = d(-2dc) + a(2ac) = 0.$$

From which we obtain that $a^2 = d^2 = \frac{1}{3}$. Since $a \geq d \geq 0$, $c \geq 0$, so $a = d = \frac{1}{\sqrt{3}}$, $c = \frac{1}{2}$. Then for $u = -\frac{1}{\sqrt{5}}(e_2 + e_3 - \sqrt{3}e_4)$, we have $g(h(u,u), \varphi u) = \sqrt{\frac{5}{3}} > \frac{1}{\sqrt{3}} = a$, which contradicts to the definition of a in (3.2), hence case I-(ii) does not occur.

I-(iii). From $b \neq 0$, $c = 0$ and (3.7), (3.9), we get the following equations:

$$R_{2323} = \tfrac{5}{4} + ab - b^2 = 0, \tag{3.12}$$

$$(3a+2b)(\tfrac{5}{4} - 2a^2 - b^2 - 3ab) = 0. \tag{3.13}$$

If $3a+2b=0$, then by (3.12), we can obtain that $b^2=\frac{3}{4}$ and $ab=-\frac{1}{2}$, thus $a^2=\frac{4}{9}b^2=\frac{1}{3}$. Substituting this and $c=0$ into (3.4), (3.10), we can get that

$$R_{2424}=\tfrac{4}{3}, \quad R_{3424}=-\tfrac{4}{3}bd, \quad dR_{3424}+(a+b)R_{2424}=-\tfrac{4}{3}b(d^2-\tfrac{1}{3})=0.$$

It follows that $d^2=\frac{1}{3}$. Since $a\geq d\geq 0$, we have $a=d=\frac{1}{\sqrt{3}}$, $b=-\frac{\sqrt{3}}{2}$. Then for $u=\frac{1}{\sqrt{2}}(-e_2+e_3)$, we have $f(u)=\frac{3}{4}\sqrt{\frac{3}{2}}>a$. This is a contradiction.

If $3a+2b\neq 0$, then (3.13) implies that $\frac{5}{4}-2a^2-b^2-3ab=0$. Using this and (3.12), we can deduce that $b^2=\frac{5}{12}$, $a^2=\frac{5}{3}$. Substituting this into (3.10) gives that

$$dR_{3434}+(a+b)R_{2434}=d(\tfrac{5}{3}-2d^2)=0, \tag{3.14}$$

this and $d\geq 0$ gives that $d=0$ or $d=\sqrt{\frac{5}{6}}$.

To sum up, we obtain the following three posibilities for **Case I**:

(1) $a=b=c=d=0$;

(2) $a=\sqrt{\frac{5}{3}},\ b=-\sqrt{\frac{5}{12}},\ c=0,\ d=0$;

(3) $a=\sqrt{\frac{5}{3}},\ b=-\sqrt{\frac{5}{12}},\ c=0,\ d=\sqrt{\frac{5}{6}}$.

Next, we will show that **Case II** does not occur.

In this case, as $\lambda_1=2\lambda_2$, by Lemma 3.1, we see that $f(e_2)=0$. Choose $\{e_3,\ e_4\}$ such that $g(h(e_2,e_2),\varphi e_4)=0$ and $g(h(e_2,e_2),\varphi e_3)=a\geq 0$. Then the second fundamental form h of $M^4\hookrightarrow\mathbb{S}^9$ can be expressed as follows:

$$\begin{cases} h(e_1,e_1)=\frac{4}{\sqrt{21}}\varphi e_1, \quad h(e_1,e_2)=\frac{2}{\sqrt{21}}\varphi e_2, \quad h(e_1,e_k)=-\frac{3}{\sqrt{21}}\varphi e_k,\ k=3,4,\\ h(e_2,e_2)=\frac{2}{\sqrt{21}}\varphi e_1+a\varphi e_3, \qquad h(e_2,e_3)=a\varphi e_2+b\varphi e_3+c\varphi e_4,\\ h(e_3,e_3)=-\frac{3}{\sqrt{21}}\varphi e_1+b\varphi e_2+d\varphi e_3+f\varphi e_4, \quad h(e_2,e_4)=c\varphi e_3-b\varphi e_4,\\ h(e_4,e_4)=-\frac{3}{\sqrt{21}}\varphi e_1-b\varphi e_2-(a+d)\varphi e_3-f\varphi e_4,\\ h(e_3,e_4)=c\varphi e_2+f\varphi e_3-(a+d)\varphi e_4, \end{cases} \tag{3.15}$$

where $a,b,c,d,f\in\mathbb{R}$.

Similar to Case I, using (3.15) and Gauss equation (2.8), we easily have

$$\begin{cases} R_{1212}=1+\lambda_1\lambda_2-\lambda_2^2=\frac{25}{21}, \qquad R_{1224}=R_{1234}=0,\\ R_{2323}=\frac{15}{21}+(ad-a^2-c^2-b^2), \quad R_{1223}=\frac{5}{\sqrt{21}}a. \end{cases} \tag{3.16}$$

Taking $(k,i,j)=(2,2,2),(2,2,3),(2,2,4),(2,3,4)$ in (3.5), respectively, and using (3.15), for any $l,p\in\{1,2,3,4\}$, we can get

$$\tfrac{2}{\sqrt{21}}R_{12lp}+aR_{32lp}=0, \tag{3.17}$$

$$bR_{32lp}+cR_{42lp}=0, \tag{3.18}$$

$$aR_{34lp} + 2cR_{32lp} - 2bR_{42lp} = 0, \tag{3.19}$$

$$(2a + d)R_{24lp} - fR_{23lp} = 0. \tag{3.20}$$

Taking $(l, p) = (1, 2)$ in (3.17)-(3.20), we obatin $a^2 = \frac{10}{21}$ and $b = c = f = 0$, then $a = \sqrt{\frac{10}{21}}$. From (3.17), we also have

$$\frac{2}{\sqrt{21}} R_{1223} + aR_{3223} = \frac{10}{21}a - a(\frac{15}{21} + ad - a^2) = 0, \tag{3.21}$$

which implies that $d = \sqrt{\frac{5}{42}}$. Let $u = -\cos\alpha e_3 - \sin\alpha e_4$, $\alpha \in (0, \frac{\pi}{2})$ such that $\tan\alpha = \sqrt{\frac{7}{3}}$, then $f(u) = \frac{3\sqrt{3}}{\sqrt{21}} > \frac{4}{\sqrt{21}} = \lambda_1$, which gives the desired contradiction. Hence, **Case II** does not occur.

Therefore, we immediately obtain the following lemma:

Lemma 3.3 *Let M^4 be a 4-dimensional compact minimal Legendrian submanifold of $\mathbb{S}^9$. If the sectional curvature $K \geq 0$ for each point p of M^4, then there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_pM^4 such that the second fundamental form h of M^4 can be expressed as one of the following:*

(i) $h(e_i, e_j) = 0, \quad 1 \leq i, j \leq 4$;

(ii) h is expressed by (3.3), with $a = \sqrt{\frac{5}{3}},\ b = -\sqrt{\frac{5}{12}},\ c = 0,\ d = \sqrt{\frac{5}{6}}$;

(iii) h is expressed by (3.3), with $a = b = c = d = 0$;

(iv) h is expressed by (3.3), with $a = \sqrt{\frac{5}{3}},\ b = -\sqrt{\frac{5}{12}},\ c = 0,\ d = 0$.

Let M^4 be given as in Lemma 3.3, then we have the following result:

Lemma 3.4 *Let $p \in M^4$, we have*

(1) *if (i) of Lemma 3.3 holds, then $K(p) \equiv 1$;*

(2) *if (ii) of Lemma 3.3 holds, then $K(p) \equiv 0$;*

(3) *if (iii) of Lemma 3.3 holds, then $0 \leq K(p) \leq \frac{5}{4}$, where $K(p) = 0$ for every plane through e_1, and $K(p) = \frac{5}{4}$ for any plane perpendicular to e_1;*

(4) *if (iv) of Lemma 3.3 holds, then $0 \leq K(p) \leq \frac{5}{3}$, where $K(p) = 0$ for every plane through e_1 or e_2, and $K(p) = \frac{5}{3}$ only for the plane determined by e_3 and e_4.*

Proof (1) If (i) of Lemma 3.3 holds, $h \equiv 0$, by Gauss equation (2.8), we obtain that $K(p) \equiv 1$.

(2) If (ii) of Lemma 3.3 holds, by (3.4), (3.6) we have $R_{ijij} = 0$ for $\forall\ i, j$, then $K(p) = 0$.

(3) If (iii) of Lemma 3.3 holds, by (3.4), (3.6), we immediately obtain that

$$R_{1212} = R_{1313} = R_{1414} = 0, \quad R_{2323} = R_{2424} = R_{3434} = \frac{5}{4}.$$

Choose an orthonormal basis $\{X, Y, Z, W\}$ of T_pM^4:

$$\begin{cases} X = \sin\theta e_1 - \cos\theta\sin\alpha e_2 + \cos\theta\cos\alpha\sin\beta e_3 + \cos\theta\cos\alpha\cos\beta e_4, \\ Y = \cos\theta e_1 + \sin\theta\sin\alpha e_2 - \sin\theta\cos\alpha\sin\beta e_3 - \sin\theta\cos\alpha\cos\beta e_4, \\ Z = \cos\alpha e_2 + \sin\alpha\sin\beta e_3 + \sin\alpha\cos\beta e_4, \quad W = \cos\beta e_3 - \sin\beta e_4, \end{cases} \tag{3.22}$$

where $\theta, \alpha, \beta \in \mathbb{R}$. Then straightforward computation shows that

$$g(R(X,Y)Y,X) = 0, \quad g(R(X,Z)Z,X) = g(R(X,W)W,X) = \tfrac{5}{4}\cos^2\theta,$$
$$g(R(Z,W)W,Z) = \tfrac{5}{4}, \quad g(R(Y,Z)Z,Y) = g(R(Y,W)W,Y) = \tfrac{5}{4}\sin^2\theta.$$

Hence $0 \le K(p) \le \frac{5}{4}$, where $K(p) = 0$ for any plane through e_1, $K(p) = \frac{5}{4}$ for any plane perpendicular to e_1.

(4) In this case, we also use (3.4) and (3.6) to get

$$R_{1212} = R_{1313} = R_{1414} = R_{2323} = R_{2424} = 0, \quad R_{3434} = \tfrac{5}{3}.$$

Then for the basis stated in (3.22), we have $g(R(X,Y)Y,X) = g(R(X,Z)Z,X) = 0$,

$$g(R(X,W)W,X) = \tfrac{5}{3}\cos^2\theta\cos^2\alpha, \quad g(R(Y,Z)Z,Y) = 0,$$
$$g(R(Y,W)W,Y) = \tfrac{5}{3}\sin^2\theta\cos^2\alpha, \quad g(R(Z,W)W,Z) = \tfrac{5}{3}\sin^2\alpha.$$

Thus the assertion of (4) immediately follows from the above arguments.

In summing up, we have completed the proof of Lemma 3.4. □

Since the second fundamental form h of $M^4 \hookrightarrow \mathbb{S}^9$ is C-parallel, we can extend the basis $\{e_i\}_{i=1}^4$ by parallel translation along geodesics through p to a normal neighborhood around p, so as to obtain a local orthonormal frame $\{E_i\}_{i=1}^4$, such that h has the same expression in any point as in p. This is stated in the following lemma, which can be proved similarly as Proposition 4.2 of [8].

Lemma 3.5 *Let M^4 be a 4-dimensional compact minimal Legendrian submanifold of $\mathbb{S}^9$ with K not constant and satisfying $K \ge 0$. Then there exists globally a unique tangent vector field E_1, and locally tangent vector fields $\{E_2, E_3, E_4\}$, such that*

(1) *$\{E_1, E_2, E_3, E_4\}$ is a local orthonormal frame,*

(2) *for any $p \in M^4$, f attains its maximum value at $E_1(p)$,*

(3) *for any $p \in M^4$, $\{E_1(p), E_2(p), E_3(p), E_4(p)\}$ satisfies (iii) or (iv) of Lemma 3.3.*

In order to prove Theorem 1.2, we need also the following uniqueness theorem for Legendrian submanifolds.

Lemma 3.6 (cf. [6]) *Let f and $\bar{f}: M^n \to \mathbb{S}^{2n+1}$ be two n-dimensional Legendrian isometric immersions of a connected Riemannian manifold M^n into the unit sphere with second fundamental forms h and $\bar{h}$, respectively. If*

$$g(f_*X, f_*Y) = g(\bar{f}_*X, \bar{f}_*Y), \quad g(h(X,Y), \varphi f_*Z) = g(\bar{h}(X,Y), \varphi \bar{f}_*Z),$$

for all vector fields X, Y, Z tangent to M^n, then there exists an isometry τ of $\mathbb{S}^{2n+1}$ such that $f = \tau \circ \bar{f}$.

Completion of the proof of Theorem 1.2

If the sectional curvature K is constant, by Lemma 3.3 and 3.4, we know that $K \equiv 1$ or $K \equiv 0$ and M^4 is totally geodesic or it is flat. According to the main result of [6], we conclude that, up to an isometry, $M^4 \hookrightarrow \mathbb{S}^9$ must be given by the immersion $f^{(1)}$, $f^{(2)}$, as described in Example 1.1, Example 1.2, repectively.

If K is not constant, by comparing Lemma 3.3 and (1.1), (1.2), we can apply Lemma 3.4, 3.5 and 3.6 to conclude that, up to an isometry, $M^4 \hookrightarrow \mathbb{S}^9$ must be given by the immersion $\tilde{f}^{(3)}$, $\tilde{f}^{(4)}$, as described in Example 1.3, Example 1.4, repectively.

We have finished the proof of Theorem 1.2.

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