Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2556 – 2571
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doi:10.55730/1300-0098.3288

Existence of fixed points in conical shells of a Banach space for sum of two operators and application in ODEs

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Received: 09.04.2022 • **Accepted/Published Online:** 31.05.2022 • **Final Version:** 05.09.2022

Abstract: In this work a new functional expansion-compression fixed point theorem of Leggett–Williams type is developed for a class of mappings of the form $T+F$, where $(I-T)$ is Lipschitz invertible map and F is a k-set contraction. The arguments are based upon recent fixed point index theory in cones of Banach spaces for this class of mappings. As application, our approach is applied to prove the existence of nontrivial nonnegative solutions for three-point boundary value problem.

Key words: Fixed point index, cone, sum of operators, Green's function, positive solution

1. Introduction

Throughout this paper, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$. Let χ and ψ be nonnegative continuous functionals on $\mathcal{P}$. For positive real numbers a and b, we define the sets:

$$\mathcal{P}(\chi, b) = \{x \in \mathcal{P} : \chi(x) \leq b\},$$

and

$$\mathcal{P}(\chi, \psi, a, b) = \{x \in \mathcal{P} : a \leq \chi(x) \text{ and } \psi(x) \leq b\}.$$

Krasnosel'skii type expansion-compression fixed point theorems give us fixed points localized in a conical shell of the form $\{x \in \mathcal{P} : a \leq \|x\| \leq b\}$, where $a, b \in (0, \infty)$, while with the Leggett–Williams theorems type, fixed points are localized in a conical shell of the form $\mathcal{P}(\chi, \psi, a, b)$.

In [2, Theorem 4.1], Anderson et al. have developed a functional expansion-compression fixed point theorem of Leggett–Williams type. They have discussed the existence of at least one solution in $\mathcal{P}(\beta, \alpha, r, R)$ or in $\mathcal{P}(\alpha, \beta, r, R)$ for the nonlinear operational equation $Ax = x$, where A is a completely continuous nonlinear map acting in $\mathcal{P}$, α is a nonnegative continuous concave functional on $\mathcal{P}$ and β is a nonnegative continuous convex functional on $\mathcal{P}$. Noting that, in [2], the authors provided more general results than those obtained in [1, 3, 14–17] for completely continuous mappings.

Recently, in 2019 a new direction of research in the theory of fixed point in ordered Banach spaces for the sum of two operators is opened by Djebali and Mebarki [8]. Then, several fixed point theorems, including Krasnosel'skii type and Leggett–Williams types theorems in cones, have been established (see [5–7, 10–12]).

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2010 *AMS Mathematics Subject Classification:* 47H10, 47H08, 34B18

These theorems have been applied to obtain existence results for nonnegative solutions of various types of boundary and/or initial value problems (see [9, 10, 12, 13]).
In this paper, we use the fixed point index theory developed in [8] and [12] to generalize the main result of [2, Theorem 4.1] for the sum $T+F$ where $(I-T)$ is Lipschitz invertible mapping with constant $\gamma>0$ and F is a k-set contraction with $k\gamma<1$.

The paper is organized as follows. In Section 2, we give some preliminary results which will be used for the proof of our main results. In Section 3, we present our main contribution. As application, in Section 4, we establish the existence of nontrivial nonnegative solutions for a nonlinear second order three-point boundary value problem. The article ends with a conclusion.

2. Auxiliary results

Let E be a real Banach space.

Definition 2.1 *A closed, convex set $\mathcal{P}$ in E is said to be cone if*

1. $\beta x\in\mathcal{P}$ *for any* $\beta\geq 0$ *and for any* $x\in\mathcal{P}$,
2. $x,-x\in\mathcal{P}$ *implies* $x=0$.

Definition 2.2 *A mapping $K:E\to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.3 *Let Ω_E be the class of all bounded sets of E. The Kuratowski measure of noncompactness $\alpha:\Omega_E\to[0,\infty)$ is defined by*

$$\alpha(Y)=\inf\left\{\delta>0: Y=\cup_{j=1}^{m}Y_j \quad \textit{and} \quad diam(Y_j)\leq\delta, \quad j\in\{1,\dots,m\}\right\},$$

where $diam(Y_j)=\sup\{\|x-y\|_X : x,y\in Y_j\}$ is the diameter of Y_j, $j\in\{1,\dots,m\}$.

For the main properties of measure of noncompactness we refer the reader to [4].

Definition 2.4 *A mapping $K:E\to E$ is said to be k-set contraction if it is continuous, bounded and there exists a constant $k\geq 0$ such that*

$$\alpha(K(Y))\leq k\alpha(Y),$$

for any bounded set $Y\subset E$.

Obviously, if $K:E\to E$ is a completely continuous mapping, then K is 0-set contraction.

Let $\mathcal{P}$ be a cone in X, $\Omega\subset\mathcal{P}$ and U is a bounded open subset of $\mathcal{P}$. Assume that $T:\Omega\to E$ is such that $(I-T)$ is Lipschitz invertible with constant $\gamma>0$, $F:\overline{U}\to E$ is a k-set contraction mapping with $0\leq k<\gamma^{-1}$. Suppose that

$$F(\overline{U})\subset(I-T)(\Omega), \tag{2.1}$$

and

$$x \neq Tx + Fx, \text{ for all } x \in \partial U \bigcap \Omega. \tag{2.2}$$

Then $x \neq (I-T)^{-1}Fx$, for all $x \in \partial U$ and the mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict $k\gamma$-set contraction. Indeed, $(I-T)^{-1}F$ is continuous and bounded; and for any bounded set B in U, we have

$$\alpha(((I-T)^{-1}F)(B)) \leq \gamma\,\alpha(F(B)) \leq k\gamma\,\alpha(B).$$

The fixed point index $i\,((I-T)^{-1}F, U, \mathcal{P})$ is well defined. Thus we put,

$$i_*\,(T+F, U\bigcap\Omega, \mathcal{P}) = \begin{cases} i\,((I-T)^{-1}F, U, \mathcal{P}), & \text{if } U\bigcap\Omega \neq \emptyset \\ 0, & \text{if } U\bigcap\Omega = \emptyset. \end{cases} \tag{2.3}$$

The proof of our theoretical result invokes the following main properties of the fixed point index i_*.

(i) *(Normalization)* If $Fx = y_0$, for all $x \in \overline{U}$, where $(I-T)^{-1}y_0 \in U \cap \Omega$, then

$$i_*\,(T+F, U\cap\Omega, \mathcal{P}) = 1.$$

(ii) *(Additivity)* For any pair of disjoint open subsets $U_1, U_2 \subset U$ such that $T+F$ has no fixed point on $(\overline{U}\setminus(U_1 \cup U_2)) \cap \Omega$, we have

$$i_*(T+F, U\cap\Omega, \mathcal{P}) = i_*(T+F, U_1\cap\Omega, \mathcal{P}) + i_*(T+F, U_2\cap\Omega, \mathcal{P}).$$

(iii) *(Homotopy invariance)* The fixed point index $i_*(T+H(.,t), U\cap\Omega, \mathcal{P})$ does not depend on the parameter $t \in [0,1]$, where

- **(a)** $H : [0,1]\times\overline{U} \to E$ is continuous and $H(t,x)$ is uniformly continuous in t with respect to $x \in \overline{U}$,
- **(b)** $H([0,1]\times\overline{U}) \subset (I-T)(\Omega)$,
- **(c)** $H(t,.) : \overline{U} \to E$ is a ℓ-set contraction with $0 \leq \ell < \gamma^{-1}$ for all $t \in [0,1]$,
- **(d)** $Tx + H(t,x) \neq x$ for all $t \in [0,1]$ and $x \in \partial U \cap \Omega$.

(iv) *(Solvability)* If $i_*(T+F, U\cap\Omega, \mathcal{P}) \neq 0$, then $T+F$ has a fixed point in $U\cap\Omega$.

For more details about the definition of the index i_* and its properties see [8, 12].

3. Main results

Our main result is as follows.

Theorem 3.1 *Let α be a nonnegative continuous concave functional on $\mathcal{P}$ and β be a nonnegative continuous convex functional on $\mathcal{P}$. Let $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible mapping with constant $\gamma > 0$, $F : \mathcal{P} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$.*
Assume that there exist four nonnegative numbers a, b, c, d and $z_0 \in \mathcal{P}$ such that $\beta((I-T)^{-1}0) < b$, $\alpha((I-T)^{-1}z_0) > c$ and

$$Fx + Tx \in \mathcal{P},\ Tx \in \mathcal{P}, \quad \textit{for all } x \in \partial\mathcal{P}(\beta, b) \cup \partial\mathcal{P}(\alpha, c),$$

$$\lambda F(\mathcal{P}(\beta,b)) \subset (I-T)(\Omega),\ \textit{for all}\ \lambda \in [0,1], \tag{3.1}$$

$$\lambda F(\mathcal{P}(\alpha,c)) + (1-\lambda)z_0 \subset (I-T)(\Omega),\ \textit{for all}\ \lambda \in [0,1]. \tag{3.2}$$

Suppose that:

($\mathcal{A}1$) *if* $x \in \mathcal{P}$ *with* $\beta(x) = b$, *then* $\alpha(Tx) \geq a$;

($\mathcal{A}2$) *if* $x \in \mathcal{P}$ *with* $\beta(x) = b$ *and* $[\alpha(x) \geq a$ *or* $\alpha(Tx+Fx) < a]$, *then* $\beta(Tx+Fx) < b$ *and* $\beta(Tx) \leq b$;

($\mathcal{A}3$) *if* $x \in \mathcal{P}$ *with* $\alpha(x) = c$, *then* $\beta(Tx+z_0) \leq d$;

($\mathcal{A}4$) *if* $x \in \mathcal{P}$ *with* $\alpha(x) = c$ *and* $[\beta(x) \leq d$ *or* $\beta(Tx+Fx) > d]$, *then* $\alpha(Tx+Fx) > c$ *and* $\alpha(Tx+z_0) \geq c$.

Then,

1. *(Expansive form)* $T+F$ *has a fixed point* x^* *in* $\mathcal{P}(\beta,\alpha,b,c)\cap\Omega$ *if*

   ($\mathcal{H}1$) $a < c$, $b < d$, $\{x \in \mathcal{P} : b < \beta(x)$ *and* $\alpha(x) < c\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\beta,b) \subset \mathcal{P}(\alpha,c)$, $\mathcal{P}(\beta,b)\cap\Omega \neq \emptyset$ *and* $\mathcal{P}(\alpha,c)$ *is bounded.*

2. *(Compressive form)* $T+F$ *has a fixed point* x^* *in* $\mathcal{P}(\alpha,\beta,c,b)\cap\Omega$ *if*

   ($\mathcal{H}2$) $c < a$, $d < b$, $\{x \in \mathcal{P} : c < \alpha(x)$ *and* $\beta(x) < b\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\alpha,c) \subset \mathcal{P}(\beta,b)$, $\mathcal{P}(\alpha,c)\cap\Omega \neq \emptyset$, *and* $\mathcal{P}(\beta,b)$ *is bounded.*

Proof We will prove the expansion form. The proof of the compression form is similar.

We list

$$U = \{x \in \mathcal{P} : \beta(x) < b\},$$

$$V = \{x \in \mathcal{P} : \alpha(x) < c\}.$$

Then, the interior of $V - U$ is given by

$$W = (V-U)^o = \{x \in \mathcal{P} :\ b < \beta(x) \text{ and } \alpha(x) < c\}.$$

Thus U, V and W are bounded, not empty and open subsets of $\mathcal{P}$. To prove the existence of a fixed point for the sum $T+F$ in $\mathcal{P}(\beta,\alpha,b,c)\cap\Omega$, it is enough for us to show that $i_*(T+F, W\cap\Omega, \mathcal{P}) \neq 0$ since W is the interior of $\mathcal{P}(\beta,\alpha,b,c)$.

Claim 1. $Tx+Fx \neq x$ for all $x \in \partial U \cap \Omega$.
Let $x_0 \in \partial U \cap \Omega$, then $\beta(x_0) = b$. Suppose that $x_0 = Tx_0 + Fx_0$, then $\beta(Tx_0+Fx_0) = b$. By the condition ($\mathcal{A}2$), if $\alpha(x_0) \geq a$, then $\beta(Tx_0+Fx_0) < b$, and if $\alpha(x_0) < a$, thus $\alpha(Tx_0+Fx_0) < a$, then $\beta(Tx_0+Fx_0) < b$. This is a contradiction. Thus we have $Tx+Fx \neq x$ for all $x \in \partial U \cap \Omega$.

Claim 2. $Tx+Fx \neq x$ for all $x \in \partial V \cap \Omega$.
Let $x_1 \in \partial V \cap \Omega$, then $\alpha(x_1) = c$. Suppose that $x_1 = Tx_1 + Fx_1$, then $\alpha(Tx_1+Fx_1) = c$. By the condition ($\mathcal{A}4$), if $\beta(x_1) \leq d$, then $\alpha(Tx_1+Fx_1) > c$, and if $\beta(x_1) > d$, thus $\beta(Tx_1+Fx_1) > d$, then $\alpha(Tx_1+Fx_1) > c$.

This is a contradiction. Thus we have $Tx + Fx \neq x$ for all $x \in \partial V \cap \Omega$.

Claim 3. Let $H_1 : [0,1] \times \overline{U} \to E$ be defined by

$$H_1(t,x) = tFx.$$

Clearly H_1 is continuous and uniformly continuous in t with respect to $x \in \overline{U}$, and from (3.2) we easily see that $H_1([0,1] \times \overline{U}) \subset (I-T)(\Omega)$. Moreover $H_1(t,.) : \overline{U} \to E$ is a k-set contraction for all $t \in [0,1]$ and $Tx + H_1(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U \cap \Omega$. Otherwise, there would exist $(t_2, x_2) \in [0,1] \times \partial U \cap \Omega$ such that $Tx_2 + H_1(t_2, x_2) = x_2$. Since $x_2 \in \partial U$, $\beta(x_2) = b$. Either $\alpha(Tx_2 + Fx_2) < a$ or $\alpha(Tx_2 + Fx_2) \geq a$.

Case (1): If $\alpha(Tx_2 + Fx_2) < a$, the convexity of β and the condition ($\mathcal{A}2$) lead

$$\begin{aligned} b = \beta(x_2) &= \beta\left(Tx_2 + H_1(t_2, x_2)\right) \\ &= \beta\left((1-t_2)Tx_2 + t_2(Tx_2 + Fx_2)\right) \\ &\leq (1-t_2)\beta(Tx_2) + t_2\beta(Tx_2 + Fx_2) \\ &< b, \end{aligned}$$

which is a contradiction.

Case (2): If $\alpha(Tx_2 + Fx_2) \geq a$, from the concavity of α and the condition ($\mathcal{A}1$), we obtain $\alpha(x_2) \geq a$. Indeed,

$$\begin{aligned} \alpha(x_2) &= \alpha\left(Tx_2 + H_1(t_2, x_2)\right) \\ &\geq (1-t_2)\alpha(Tx_2) + t_2\alpha(Tx_2 + Fx_2) \\ &\geq a, \end{aligned}$$

and thus by the condition ($\mathcal{A}2$), we have $\beta(Tx_2 + Fx_2) < b$ and $\beta(Tx_2) < b$, which is the same contradiction we arrived at in the previous case.

Being $(I-T)^{-1}0 \in U \cap \Omega$, the homotopy invariance property (iii) and the normality property (i) of the index i_* lead

$$i_*(T+F, U \cap \Omega, \mathcal{P}) = i_*(T+0, U \cap \Omega, \mathcal{P}) = 1.$$

Claim 4. Let $H_2 : [0,1] \times \overline{V} \to E$ be defined by

$$H_2(t,x) = t\,Fx + (1-t)z_0.$$

Clearly H_2 is continuous, and uniformly continuous in t with respect to $x \in \overline{V}$, and from (3.2) we easily see that $(H_2([0,1] \times \overline{V})) \subset (I-T)(\Omega)$. Moreover $H_2(t,.) : \overline{V} \to E$ is a k-set contraction for all $t \in [0,1]$ and $Tx + H_2(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial V \cap \Omega$. Otherwise, there would exist $(t_3, x_3) \in [0,1] \times \partial V \cap \Omega$ such that $Tx_3 + H_2(t_3, x_3) = x_3$. Since $x_3 \in \partial V$ we have that $\alpha(x_3) = c$. Either $\beta(Tx_3 + Fx_3) \leq d$ or $\beta(Tx_3 + Fx_3) > d$.

Case (1): If $\beta(Tx_3 + Fx_3) > d$. the concavity of α and the condition ($\mathcal{A}4$) lead

$$\begin{aligned} c = \alpha(x_3) &= \alpha(Tx_3 + H_2(t_3, x_3)) \\ &= \alpha(t_3(Tx_3 + Fx_3) + (1-t_3)(Tx_3 + z_0)) \\ &\geq t_3\alpha(Tx_3 + Fx_3) + (1-t_3)\alpha(Tx_3 + z_0) \\ &> c. \end{aligned}$$

This is a contradiction.

Case (2): If $\beta(Tx_3 + Fx_3) \leq d$, from the convexity of β and the condition $(\mathcal{A}3)$, we obtain $\beta(x_3) \leq d$. Indeed,

$$\begin{aligned} \beta(x_3) &= \beta(Tx_3 + H_2(t_3, x_3)) \\ &\leq t_3\beta(Tx_3 + Fx_3) + (1 - t_3)\beta(Tx_3 + z_0) \\ &\leq d, \end{aligned}$$

and thus by the condition $(\mathcal{A}4)$, we have $\alpha(Tx_3 + Fx_3) > c$, this is the same contradiction that we found in the previous case.
Hence, the homotopy invariance property (iii) of the fixed index i_* yields

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + z_0, V \cap \Omega, \mathcal{P}),$$

and by the solvability property (iv) of the index i_* (since $(I - T)^{-1}z_0 \notin V$ the index cannot be nonzero) we have

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + z_0, V \cap \Omega, \mathcal{P}) = 0.$$

Since U and W are disjoint open subsets of V and $T + F$ has no fixed points in $\overline{V} - (U \cup W)$ (by Claims 1 and 2), from the additivity property (ii) of the index i_*, we deduce

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + F, U \cap \Omega, \mathcal{P}) + i_*(T + F, W \cap \Omega, \mathcal{P}).$$

Consequently, we get

$$i(T + F, W \cap \Omega, \mathcal{P}) = -1,$$

and thus by the solvability property (iv) of the fixed point index i_*, the sum $T + F$ has a fixed point $x^* \in W \cap \Omega \subset \mathcal{P}(\beta, \alpha, b, c) \cap \Omega$. □

Now we add restrictions on the operator $T + F$ of Theorem 3.1 and we combine the expansive form and the compressive form to establish a multiplicity result.

Theorem 3.2 *Let α be a nonnegative continuous concave functional on $\mathcal{P}$ and β, γ be nonnegative continuous convex functionals on $\mathcal{P}$ for all $x \in \mathcal{P}$. Let $T : \Omega \subset \mathcal{P} \to E$ be such that $(I - T)$ is Lipschitz invertible mapping with constant $\gamma > 0$, $F : \mathcal{P} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$.*
Assume that there exist six nonnegative numbers $a < c < r$, $b < d < R$ and $z_0 \in \mathcal{P}$ such that

$$\beta((I - T)^{-1}0) < b, \ \gamma((I - T)^{-1}0) < R, \ \ \alpha((I - T)^{-1}z_0) > c,$$

$$Fx + Tx \in \mathcal{P}, \ Tx \in \mathcal{P}, \quad \text{for all } x \in \partial\mathcal{P}(\beta, b) \cup \partial\mathcal{P}(\alpha, c) \cup \partial\mathcal{P}(\gamma, R),$$

$$\lambda F(\mathcal{P}(\gamma, R)) \subset (I - T)(\Omega), \ \text{for all } \lambda \in [0, 1],$$

$$\lambda F(\mathcal{P}(\alpha, c)) + (1 - \lambda)z_0 \subset (I - T)(\Omega), \ \text{for all } \lambda \in [0, 1].$$

In addition to the assumptions $(\mathcal{A}1) - (\mathcal{A}4)$ of Theorem 3.1, we suppose that the following conditions hold:

$(\mathcal{B}1)$ *if $x \in \mathcal{P}$ with $\gamma(x) = R$, then $\alpha(Tx) \geq r$;*

$(\mathcal{B}2)$ *if $x \in \mathcal{P}$ with $\gamma(x) = R$ and $[\alpha(x) \geq r$ or $\alpha(Tx + Fx) < r]$, then $\gamma(Tx + Fx) < R$ and $\gamma(Tx) \leq R$.*

If the two following conditions hold,

($\mathbb{H}1$) $\{x \in \mathcal{P} : b < \beta(x)$ *and* $\alpha(x) < c\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$, $\mathcal{P}(\beta, b) \cap \Omega \neq \emptyset$ *and* $\mathcal{P}(\alpha, c)$ *is bounded,*

($\mathbb{H}2$) $\{x \in \mathcal{P} : c < \alpha(x)$ *and* $\gamma(x) < R\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\alpha, c) \subset \mathcal{P}(\gamma, R)$, $\mathcal{P}(\alpha, c) \cap \Omega \neq \emptyset$, *and* $\mathcal{P}(\gamma, R)$ *is bounded,*

then, $T + F$ *has at least two nontrivial fixed points* $x_1, x_2 \in \mathcal{P}$ *such that*

$$x_1 \in \mathcal{P}(\beta, \alpha, b, c) \cap \Omega \quad \textit{and} \quad x_2 \in \mathcal{P}(\alpha, \gamma, c, R) \cap \Omega.$$

Proof We list

$$U = \{x \in \mathcal{P} : \beta(x) < b\},$$

$$V = \{x \in \mathcal{P} : \alpha(x) < c\},$$

$$Y = \{x \in \mathcal{P} : \gamma(x) < R\}.$$

Then, the interior of $V - U$ is given by

$$W = (V - U)^o = \{x \in \mathcal{P} : \ b < \beta(x) \text{ and } \alpha(x) < c\},$$

and the interior of $Y - V$ is given by

$$Z = (Y - V)^o = \{x \in \mathcal{P} : \ c < \alpha(x) \text{ and } \gamma(x) < R\}.$$

Thus U, V, Y and W, Z are bounded, not empty and open subsets of $\mathcal{P}$. To prove the existence of two fixed point for the sum $T + F$ in $\mathcal{P}(\beta, \alpha, b, c) \cap \Omega$ and $\mathcal{P}(\alpha, \gamma, c, R) \cap \Omega$ it is enough for us to show that $i_*(T + F, W \cap \Omega, \mathcal{P}) \neq 0$ and $i_*(T + F, Z \cap \Omega, \mathcal{P}) \neq 0$ since W is the interior of $\mathcal{P}(\beta, \alpha, b, c)$ and Z is the interior of $\mathcal{P}(\alpha, \gamma, c, R)$.
The use of the fixed point index here is similar to the proof of Theorem 3.1. □

4. Applications of our approach

In the sequel, we will investigate the three-point boundary value problem:

$$\begin{aligned} &y'' + f(t, y) = 0, \quad t \in (0, 1), \\ &y(0) = ky(\eta), \quad y(1) = 0, \end{aligned} \tag{4.1}$$

where $\eta \in (0, 1)$, $k > 0$ with $k(1 - \eta) < 1$ and $f \in \mathcal{C}([0, 1] \times [0, \infty))$. Set $B = \frac{1+k\eta}{1-k(1-\eta)}$ and suppose that

($\mathcal{C}$1) $\tilde{A} < f(t, y) \leq a_1(t) + a_2(t)|y|^p$ for $t \in [0, 1]$ and $y \in [0, \infty)$, $a_1, a_2 \in \mathcal{C}([0, 1])$, $0 \leq a_1, a_2 \leq A$ on $[0, 1]$, for some positive constants $A, \tilde{A}$ and p.

($\mathcal{C}$2) $\epsilon \in (0, 1)$, and there exist $a, b, c, d, z_0, \rho > 0$ such that

$$\max(d, \tfrac{2z_0}{\epsilon}, \tfrac{1}{\Lambda}(c - z_0)) < b \leq \rho; \quad 3z_0 > a; \quad z_0 \leq c < \min(a, 3\, z_0, \tfrac{\eta}{3}\left(1 - \tfrac{\eta}{2}\right)\tilde{A} + (1 - \tfrac{1}{\epsilon})z_0);$$

$$\frac{\epsilon AB(1+b^p)+3z_0}{\epsilon} \leq \rho; \quad (1 - \epsilon)\frac{c}{\Lambda} + 3z_0 \leq d, \quad \text{where } \Lambda = \frac{\min\left(\epsilon \frac{\eta^2}{18}\left(1 - \frac{\eta}{2}\right)\tilde{A}, z_0\right)}{\epsilon\, \rho},$$

and

$$AB(1+b^p) < b. \tag{4.2}$$

Remark 4.1 *1. We end this section by an illustrative example, in which we give the constants $\epsilon, a, b, c, d, \rho, z_0$ and the function f that satisfy ($\mathcal{C}$1)-($\mathcal{C}$2). After setting the constants $A, \tilde{A}$ and p, we choose the constants $\epsilon, a, b, d, z_0, c$ and ρ.*

2. Discussion of Hypothesis (4.2):

(a) *If $p = 1$, the inequality (4.2) may be rewritten as $(\frac{1}{AB} - 1)\, b > 1$. A necessary condition for (4.2) to hold is that $A < \frac{1}{B}$.*

(b) *If $p \neq 1$, the inequality (4.2) can be written as $Kb - b^p > 1$ with $K = \frac{1}{AB}$.*
Consider the continuous function $\Phi(x) = Kx - x^p$ on $[0, \infty)$, then

$$\Phi'(x) = 0 \Leftrightarrow x = x_0 = \sqrt[p-1]{\frac{K}{p}}.$$

(i) When $p < 1$, the function Φ verifies $\Phi(0) = 0$ and $\lim\limits_{x\to+\infty} \Phi(x) = +\infty$. Moreover, Φ is decreasing on $[0, x_0)$ and increasing on (x_0, ∞) and assumes $\frac{K}{p}\sqrt[p-1]{\frac{K}{p}}(p-1)$ as a minimum at the point x_0. Hence for every real number $r > 0$, there exists a constant $b > 0$ with $\Phi(b) > r$. In particular $\Phi(b) > 1$.

(ii) When $p > 1$, the function Φ verifies $\Phi(0) = 0$ and $\lim\limits_{x\to+\infty} \Phi(x) = -\infty$. Moreover, Φ is increasing on $[0, x_0)$ and decreasing on (x_0, ∞) and assumes $\frac{K}{p}\sqrt[p-1]{\frac{K}{p}}\,(p-1)$ as a maximum at $x = x_0$. Hence the inequality $\Phi(b) > 1$ has a solution $b > 0$ if and only if $\frac{K}{p}\sqrt[p-1]{\frac{K}{p}}\,(p-1) > 1$.

4.1. Existence of at least one nonnegative solution

Our first existence result is as follows.

Theorem 4.2 *Suppose $(\mathcal{C}1)$ and $(\mathcal{C}2)$. Then the problem (4.1) has at least one nontrivial nonnegative solution $y \in \mathcal{C}^2([0,1])$ such that $c < \min\limits_{t\in[\frac{\eta}{3},\frac{\eta}{2}]} y(t) + z_0$ and $\max\limits_{t\in[0,1]} |y(t)| < b$.*

Proof To prove our main result, we will use Theorem 3.1.

Set

$$H(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

In [18] it is proved that the solution of the problem (4.1) can be expressed in the following form

$$y(t) = \int_0^1 G(t,s)\, f(s, y(s))ds, \quad t \in [0,1],$$

where

$$G(t,s) = H(t,s) + \frac{k(1-t)}{1-k(1-\eta)} H(\eta, s), \quad t, s \in [0,1].$$

Note that $0 \leq H(t,s) \leq 1$, $t, s \in [0,1]$. Hence,

$$\begin{aligned} 0 \leq G(t,s) \leq 1 + \frac{k}{1-k(1-\eta)} &= \frac{1-k+k\eta+k}{1-k(1-\eta)} \\ &= \frac{1+k\eta}{1-k(1-\eta)} = B, \quad t, s \in [0,1]. \end{aligned}$$

Moreover, for $t, s \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]$, we have

$$H(t,s) \geq \frac{\eta}{3}\left(1 - \frac{\eta}{2}\right)$$

and

$$G(t,s) \geq H(t,s) \geq \frac{\eta}{3}\left(1 - \frac{\eta}{2}\right).$$

Next,

$$H_t(t,s) = \begin{cases} -s, & 0 \leq s \leq t \leq 1, \\ 1-s, & 0 \leq t \leq s \leq 1. \end{cases}$$

Hence, $|H_t(t,s)| \leq 1$, $t, s \in [0,1]$, and

$$\begin{aligned} |G_t(t,s)| &= \left| H_t(t,s) - \frac{k}{1-k(1-\eta)} H(\eta, s) \right| \\ &\leq |H_t(t,s)| + \frac{k}{1-k(1-\eta)} H(\eta, s) \\ &\leq 1 + \frac{k}{1-k(1-\eta)} = \frac{1+k\eta}{1-k(1-\eta)} = B, \quad t, s \in [0,1]. \end{aligned}$$

Let $E = \mathcal{C}([0,1])$ be endowed with the maximum norm

$$\|y\| = \max_{t \in [0,1]} |y(t)|.$$

For $y \in \mathcal{P}$, let us define

$$\alpha(y) = \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} y(t) + z_0, \quad \beta(y) = \max_{t \in [0,1]} |y(t)|.$$

Obvious that, since $\frac{2z_0}{\epsilon} < b \leq \rho$, we get $\Lambda < 1$.

Define

$$\begin{aligned} \mathcal{P} &= \left\{ y \in E : \ y(t) \geq 0, \ t \in [0,1], \ \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} y(t) \geq \Lambda \|y\| \right\}, \\ \Omega &= \{ y \in \mathcal{P} : \|y\| \leq \rho \}. \end{aligned}$$

For $y \in \mathcal{P}$, define the operators

$$\begin{aligned} Ty(t) &= (1-\epsilon)y(t) + 2z_0, \\ Fy(t) &= \epsilon \int_0^1 G(t,s)f(s,y(s))ds - 2z_0, \quad t \in [0,1]. \end{aligned}$$

Note that if $y \in \mathcal{P}$ is a fixed point of the operator $T+F$, then it is a solution to the problem (4.1). Next, if $y \in \mathcal{P}$ and $\|y\| \leq b$, we have

$$\begin{aligned} |Ty(t)| &\leq (1-\epsilon)y(t) + 2z_0 \\ &\leq (1-\epsilon)b + 2z_0 \\ &< b, \quad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} |Ty(t) + Fy(t)| &= \left|(1-\epsilon)y(t) + \epsilon \int_0^1 G(t,s)f(s,y(s))ds\right| \\ &\leq (1-\epsilon)y(t) + \epsilon \int_0^1 G(t,s)\left(a_1(s) + a_2(s)|y(s)|^p\right)ds \\ &\leq (1-\epsilon)b + \epsilon\left(A + A\|y\|^p\right)\int_0^1 G(t,s)ds \\ &\leq (1-\epsilon)b + \epsilon AB(1+b^p) \\ &< b, \quad t \in [0,1]. \end{aligned}$$

Therefore, if $y \in \mathcal{P}$ and $\|y\| \leq b$, we have

$$\|Ty\| < b, \tag{4.3}$$

and

$$\|Ty + Fy\| < b. \tag{4.4}$$

1. For $y, z \in \mathcal{P}$, we have

$$|(I-T)y(t) - (I-T)z(t)| = \epsilon|y(t) - z(t)|, \quad t \in [0,1].$$

Hence,

$$\|(I-T)y - (I-T)z\| = \epsilon\|y - z\|.$$

Thus, $I - T : \mathcal{P} \to E$ is Lipschitz invertible operator with constant $\gamma = \frac{1}{\epsilon}$.

2. Let $y \in \mathcal{P}$. Then

$$\begin{aligned} |Fy(t))| &\leq \epsilon\left|\int_0^1 G(t,s)f(s,y(s))ds\right| + 2z_0 \\ &\leq \epsilon AB(1 + \|y\|^p) + 2z_0, \quad t \in [0,1], \end{aligned}$$

whereupon

$$\|Fy\| \leq \epsilon AB(1 + \|y\|^p) + 2z_0 < \infty.$$

Moreover,

$$\begin{aligned}\left|\frac{d}{dt}Fy(t)\right| &= \left|\epsilon\int_0^1 G_t(t,s)f(s,y(s))ds\right| \\ &\leq AB\,\epsilon(1+\|y\|^p)<\infty, \quad t\in[0,1].\end{aligned}$$

Consequently, by Arzelà–Ascoli compactness criteria $F:\mathcal{P}\to E$ is completely continuous. Then $F:\mathcal{P}\to E$ is a 0-set contraction.

3. For $y\in E$, we have

$$(I-T)^{-1}y=\frac{y+2z_0}{\epsilon}\cdot$$

Hence,

$$\alpha\left((I-T)^{-1}z_0\right)=\alpha\left(\frac{3z_0}{\epsilon}\right)=\frac{3z_0}{\epsilon}+z_0\geq c,$$

and

$$\beta\left((I-T)^{-1}0\right)=\beta\left(\frac{2z_0}{\epsilon}\right)=\frac{2z_0}{\epsilon}<b.$$

Suppose that $y\in\mathcal{P}$ with $\beta(y)=b$. Then

$$\alpha(Ty)=\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]}Ty(t)+z_0\geq 3z_0>a.$$

Consequently, $(\mathcal{A}1)$ holds.

4. Let $y\in\mathcal{P}$ with $\beta(y)=b$ and $[\alpha(y)\geq a$ or $\alpha(Ty+Fy)<a]$. Then, using (4.3) and (4.4), we obtain

$$\beta(Ty)<b \quad \text{and} \quad \beta(Ty+Fy)<b.$$

Consequently, $(\mathcal{A}2)$ holds.

5. Let $y\in\mathcal{P}$ with $\alpha(y)=c$, we get

$$\|y\|\leq\frac{1}{\Lambda}\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]}y(t)\leq\frac{1}{\Lambda}\,\alpha(y)=\frac{c}{\Lambda}.$$

Hence,

$$\beta(Ty+z_0)\leq(1-\epsilon)\frac{c}{\Lambda}+3z_0\leq d.$$

Consequently, $(\mathcal{A}3)$ holds.

6. Suppose that $y \in \mathcal{P}$ with $\alpha(y) = c$. Then

$$\begin{aligned}
\alpha(Ty + Fy) &= \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} (Ty(t) + Fy(t)) + z_0 \\
&\geq \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} \left((1-\epsilon)y(t) + \epsilon \int_0^1 G(t,s) f(s, y(s)) ds \right) \\
&\geq (1-\epsilon) \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} y(t) + \epsilon \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} \int_{\frac{\eta}{3}}^{\frac{\eta}{2}} G(t,s) f(s, y(s)) ds \\
&\geq (1-\epsilon)(c - z_0) + \epsilon \frac{\eta}{3} \left(1 - \frac{\eta}{2}\right) \tilde{A} \\
&> c.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\alpha(Ty + z_0) &= \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} (Ty(t) + z_0) + z_0 \\
&= \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} Ty(t) + 2z_0 \\
&= (1-\epsilon) \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} y(t) + 4z_0 \\
&\geq 4z_0 > c.
\end{aligned}$$

Consequently, $(\mathcal{A}4)$ holds.

7. Let $b_1 = 2\,z_0$. Then

$$\alpha(b_1) = 3\,z_0 > c$$

and

$$\beta(b_1) = 2\,z_0 < b.$$

Therefore

$$\{y \in \mathcal{P} : c < \alpha(y) \quad \text{and} \quad \beta(y) < b\} \cap \Omega \neq \emptyset.$$

8. Let $y \in \mathcal{P}(\alpha, c)$. Then $y \in \mathcal{P}$ and $\alpha(y) \leq c$. Hence,

$$\|y\| \leq \frac{1}{\Lambda} \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} y(t) \leq \frac{1}{\Lambda}(c - z_0) \leq b.$$

Thus, $y \in \mathcal{P}(\beta, b)$ so $\mathcal{P}(\alpha, c) \subset \mathcal{P}(\beta, b)$ and $\mathcal{P}(\beta, b)$ is bounded. Since $0 \in \mathcal{P}(\alpha, c)$, we get $\mathcal{P}(\alpha, c) \cap \Omega \neq \emptyset$.

9. Let $\lambda \in [0,1]$ be fixed and $u \in \mathcal{P}(\beta, b)$ be arbitrary chosen. Take

$$v(t) = \frac{2(1-\lambda)z_0 + \lambda\epsilon \int_0^1 G(t,s) f(s, u(s)) ds}{\epsilon}, \quad t \in [0,1].$$

We have $v(t) \geq 0$, $t \in [0,1]$, and

$$v(t) \leq \frac{\epsilon AB(1 + b^p) + 2z_0}{\epsilon} \leq \rho, \quad t \in [0,1].$$

Moreover,

$$\begin{aligned}
\min_{t\in[\frac{\eta}{3},\frac{\eta}{2}]} v(t) &\geq \frac{\lambda\epsilon \int_{\frac{\eta}{3}}^{\frac{\eta}{2}} \min_{t\in[\frac{\eta}{3},\frac{\eta}{2}]} G(t,s)f(s,u(s))ds + 2(1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\lambda\epsilon\left(\frac{\eta}{2}-\frac{\eta}{3}\right)\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)\widetilde{A} + (1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\min\left(\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right)}{\epsilon} \\
&= \frac{\min\left(\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right)}{\epsilon\,\rho}\,\rho \\
&\geq \Lambda\|v\|.
\end{aligned}$$

Therefore $v \in \Omega$. Also,

$$\begin{aligned}
\lambda F u(t) &= \epsilon\lambda \int_0^1 G(t,s)f(s,u(s))ds - \lambda 2z_0 \\
&= \epsilon\,\frac{\epsilon\int_0^1 G(t,s)f(s,u(s))ds + 2(1-\lambda)z_0}{\epsilon} - 2z_0 \\
&= \epsilon\, v(t) - 2z_0 \\
&= (I-T)v(t), \quad t\in[0,1].
\end{aligned}$$

Therefore

$$\lambda F(\mathcal{P}(\beta,b)) \subset (I-T)(\Omega).$$

10. Let $\lambda\in[0,1]$ be fixed and $\tilde{u}\in\mathcal{P}(\alpha,c)$ be arbitrarily chosen. So

$$\|\tilde{u}\| \leq \frac{1}{\Lambda}\min_{t\in[\frac{\eta}{3},\frac{\eta}{2}]}\tilde{u}(t) \leq \frac{1}{\Lambda}(c-z_0) \leq b.$$

Set

$$w(t) = \frac{\lambda\epsilon\int_0^1 G(t,s)f(s,\tilde{u}(s))ds + 3(1-\lambda)z_0}{\epsilon}, \quad t\in[0,1].$$

We have that $w(t)\geq 0$, $t\in[0,1]$, and

$$w(t) \leq \frac{\epsilon AB(1+b^p)+3z_0}{\epsilon} \leq \rho, \quad t\in[0,1],$$

so

$$\|w\| \leq \frac{\epsilon AB(1+b^p)+3z_0}{\epsilon} \leq \rho.$$

Moreover,

$$\begin{aligned}
\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} w(t) &\geq \frac{\lambda\epsilon\int_{\frac{\eta}{3}}^{\frac{\eta}{2}}\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} G(t,s)f(s,\tilde{u}(s))ds + 3(1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\lambda\epsilon\left(\frac{\eta}{2}-\frac{\eta}{3}\right)\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)\widetilde{A} + (1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\min\left(\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right)}{\epsilon} \\
&= \frac{\min\left(\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right)}{\epsilon\,\rho}\rho \\
&\geq \Lambda\|w\|.
\end{aligned}$$

Thus, $w\in\Omega$. Next,

$$\begin{aligned}
\lambda F\tilde{u}(t) + (1-\lambda)z_0 &= -2\lambda z_0 + \lambda\epsilon\int_0^1 G(t,s)f(s,\tilde{u}(s))ds + z_0 - \lambda z_0 \\
&= \lambda\epsilon\int_0^1 G(t,s)f(s,\tilde{u}(s))ds + (1-3\lambda)z_0 \\
&= \epsilon\frac{\lambda\epsilon\int_0^1 G(t,s)f(s,\tilde{u}(s))ds + 3(1-\lambda)z_0}{\epsilon} - 2z_0 \\
&= \epsilon w(t) - 2z_0 \\
&= (I-T)w(t), \quad t\in[0,1].
\end{aligned}$$

Therefore

$$\lambda F(\mathcal{P}(\alpha,a)) + (1-\lambda)z_0 \subset (I-T)(\Omega).$$

By Theorem 3.1, it follows that the problem (4.1) has at least one solution $y\in\Omega$ such that

$$\beta(y) < b \quad \text{and} \quad \alpha(y) > c.$$

□

4.1.1. An example

Consider the boundary value problem:

$$\begin{aligned}
y'' + \frac{y^2}{(200+t^2)(1+y)} + \frac{1}{500}(1+t) &= 0, \quad t\in(0,1), \\
y(0) = y\left(\tfrac{1}{2}\right), \quad y(1) &= 0.
\end{aligned} \tag{4.5}$$

Here

$$f(t,y) = \frac{y^2}{(200+t^2)(1+y)} + \frac{1}{500}(1+t), \quad t\in[0,1], \quad y\in[0,\infty), \quad k=1, \quad \eta=\frac{1}{2}.$$

We have, $f\in\mathcal{C}([0,1]\times\mathbb{R}^+)$ and $0<\frac{1}{500}\leq f(t,y)\leq a_1(t)+a_2(t)\,|y|^2$ for $t\in[0,1]$ and $y\in[0,\infty)$, where $p=2$, $a_1(t)=\frac{1}{500}(1+t)$, $a_2(t)=\frac{1}{200+t^2}$, $0\leq a_1,a_2\leq\frac{1}{200}$ on $[0,1]$. So, the condition $(\mathcal{C}1)$ holds.

Take the constants

$$\epsilon = \tfrac{1}{2}, \quad B = 3, \quad A = \tfrac{1}{200}, \quad \widetilde{A} = \tfrac{1}{500}, \quad b = \tfrac{41}{50}, \quad d = \tfrac{4}{5},\ \rho = \tfrac{4}{3}$$

$$c = z_0 = 2 \times 10^{-6},\ a = \tfrac{5}{2} \times 10^{-6},\ \Lambda = \frac{\min((\frac{1}{2}\frac{1}{72}\frac{3}{4})\times\frac{1}{500},\, 10^{-6})}{\frac{2}{3}} = \tfrac{3}{2} \times 10^{-6} < 1.$$

We have

$$z_0 \leq c < \min\left(a,\ (1-\epsilon)(c - z_0) + \epsilon\frac{\eta}{3}\left(1 - \frac{\eta}{2}\right)\tilde{A}, 3z_0\right) = \frac{5}{2} \times 10^{-6},$$

$$\frac{2z_0}{\epsilon} = 4\, z_0 = 8 \times 10^{-6} < b,$$

$$\frac{1}{\Lambda}(c - z_0) = 0 \leq b,$$

$$(1-\epsilon)\frac{c}{\Lambda} + 3z_0 = \frac{2}{3} + 6 \times 10^{-6} < d,$$

$$\frac{\epsilon AB(1+b^p) + 3z_0}{\epsilon} = \frac{3}{500}\left(1 + \left(\frac{41}{50}\right)^2\right) + 12 \times 10^{-6} \leq \frac{4}{5} = \rho,$$

$$AB(1+b^p) = \frac{3}{200}\left(1 + \left(\frac{41}{50}\right)^2\right) = \frac{25}{1000} < b.$$

Thus, $(\mathcal{C}2)$ holds. By Theorem 4.2, it follows that the problem (4.5) has at least one nonnegative solution.

5. Conclusion

In this paper, the functional expansion-compression fixed point theorem of Leggett–Williams type developed in [2] is extended to the class of mappings of the form $T + F$, where $(I - T)$ is Lipschitz invertible map and F is a k-set contraction. As application of some obtained theoretical results, we give new results on the existence of nonnegative solutions for a second order differential equation subjected to three-point boundary value problem.

Acknowledgment

The authors acknowledge the support of "Direction Générale de la Recherche Scientifique et du Développement Technologique (DGRSDT)", MESRS, Algeria.

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