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(Co)Limit calculations in the category of 2-crossed R-modules

Elis SOYLU YILMAZ*
Department of Mathematics and Computer Science, Science and Art Faculty, Eskişehir Osmangazi University, Eskişehir, Turkey

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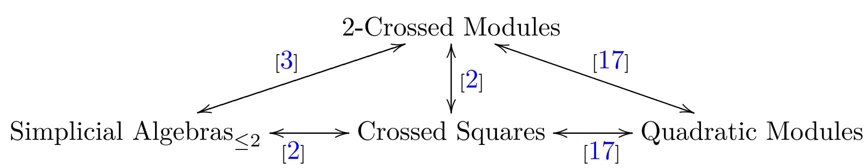
Abstract: In this work, we obtain how to construct finite limits and colimits for 2-crossed R-Modules over groups denoted with $\mathbf{X_2Mod/R}$. We give direct construction of the pullback object to show that this category has finite products over the terminal object. We also show finite coproducts and (co)completeness.

Key words: 2-crossed module, coproduct, pullback

1. Introduction

Whitehead introduced the crossed module notion for groups in [20] also known as 2-type groups. 2-crossed modules have been represented by Conduché as characterized of 3-types [11]. Baues [5] defined another version of this concept, as quadratic modules. Brown and Gilbert have given an alternative definition known as braided regular crossed modules for 3-type of groups using the automorphisms structures [6]. Brown, Sivera and Higgins [7] constructed the coproduct for crossed modules of groups. In [4] Arvasi and Ulualan have investigated the relationships between various algebraic types like crossed squares, 2-crossed modules, and simplicial groups.

Crossed squares defined in [16] and in [13] and [14] for commutative algebras. In [12] Conduché obtain a 2-crossed module from the mapping cone of a crossed square. Baues introduced quadratic modules by using simplicial groups which can be regarded as a 2-crossed module endowed with nilpotency conditions. For more details on 2-crossed modules see ([1], [21]). Some of functional relations of 2-crossed modules can be diagramatized as ([3], [2], [17]).



Freeness and coproducts are fundamental tools to construct tensor product for crossed complexes. Freeness of quadratic modules in [2] and 2-crossed modules are given in [18]. In [9] construct the coproduct in

*Correspondence: esoylu@ogu.edu.tr
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$\mathbf{X_2Mod}/(\mathbf{M} \longrightarrow \mathbf{N})$ that is they fix the precrossed module at the end. Also some colimit theorems and calculations can be seen in [8], [10], [15], and [19].

The main purposes of this paper are:

- to give the construction and existence of all finite limits in $\mathbf{X_2Mod/R}$.
- to compose the product of the $\mathbf{X_2Mod/R}$.
- to express the completeness of the $\mathbf{X_2Mod/R}$.
- to introduce the coequilaser of the $\mathbf{X_2Mod/R}$.
- to prove the cocompleteness of the category.

We give an example about these structures at the end of this paper.

2. Preliminaries

Definition 2.1 *A crossed module [20], $\partial : R \longrightarrow G$ satisfies two conditions with the action*

$$G \times R \longrightarrow R$$
$$(g, r) \mapsto^g r$$

for $g \in G$ and $r \in R$. These conditions are:

- $\varkappa_1)\partial({}^g r) = g\partial(r)g^{-1}$
- $\varkappa_2)^{\partial(r)}(r') = rr'r^{-1}$

for $r, r' \in R$ and $g \in G$. With condition $\varkappa_1$ this structure is called a precrossed module. The condition $\varkappa_2$ is known as the peiffer condition. We will denote this category with the crossed module is denoted by **XMod** *and a crossed module $\partial : R \longrightarrow G$ with triple (R, G, ∂).*

Examples:

i) Let $i : R \hookrightarrow G$ be the inclusion map and R be a normal subgroup of G, (R, G, i) is a crossed module. Also, if $\delta : S \longrightarrow G$ crossed module, then δS is a normal subgroup in G.

ii) For any group R, $\partial : R \longrightarrow Aut(R)$ is a crossed module. $Aut(R)$ corresponds to the inner automorphisms.

Definition 2.2 *A 2-crossed module [11] of groups is complex*

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

of groups with the action of N on L and M. If the mapping

$$\{-, -\} : M \times M \longrightarrow L$$

known as the peiffer lifting satisfies the following conditions,

- $\partial_2\{m_1, m_2\} = {m_2}^{\partial_1(m_1)}(m_1 m_2^{-1}(m_1)^{-1})$
- $\{\partial_2(l_1), \partial_2(l_2)\} = [l_2, l_1]$
- $\{m_1 m_2, m_3\} = \{m_2, m_3\}^{\partial_1(m_1)}\{m_1, m_2 m_3 (m_2)^{-1}\}$

  $\{m_1, m_2 m_3\} = \{m_1, m_2\}^{m_1 m_2 (m_1)^{-1}}\{m_1, m_3\}$
- $\{m_1, \partial_2(l_1)\}\{\partial_2(l_1), m_1\} = {l_1}^{\partial_1(m_1)}{l_1}^{-1}$
- ${}^{n}\{m_1, m_2\} = \{{}^{n}m_1, {}^{n}m_2\}$

  for $m_1, m_2, m_3 \in M, l_1, l_2 \in L, n \in N$.

Let $(L_1, M_1, N_1, \partial_2, \partial_1, \{-,-\}_1)$ *and* $(L_2, M_2, N_2, \delta_2, \delta_1, \{-,-\}_2)$ *be 2-crossed modules.* $f = (f_1, f_2, f_3) : (L_1, M_1, N_1) \longrightarrow (L_2, M_2, N_2)$ *is a 2-crossed module morphism if the diagram*

$$\begin{array}{ccccccc} M_1 \times M_1 & \xrightarrow{\{-,-\}_1} & L_1 & \xrightarrow{\partial_2} & M_1 & \xrightarrow{\partial_1} & N_1 \\ \downarrow{\scriptstyle f_2 \times f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ M_2 \times M_2 & \xrightarrow[\{-,-\}_2]{} & L_2 & \xrightarrow[\delta_2]{} & M_2 & \xrightarrow[\delta_1]{} & N_2 \end{array}$$

is commutative and

$$\begin{aligned} f_2({}^{n_1}m_1) &= {}^{f_3(n_1)}f_2(m_2) \\ f_1({}^{n_1}l_1) &= {}^{f_3(n_1)}f_1(l_1) \end{aligned}$$

for $n_1 \in N_1, m_1 \in M_1$ *and* $l_1 \in L_1$.

3. Finite limits in $\mathbf{X_2Mod/R}$

We give the construction and the existence of all finite limits in $\mathbf{X_2Mod/R}$. In particular for two 2-crossed R-modules we construct their product to use this to obtain finite limits in $\mathbf{X_2Mod/R}$.

Proposition 3.1 *Equaliser object exists in* $\mathbf{X_2Mod/R}$.

Proof Let $(\alpha, \beta) : (A_2, A_1, R, \partial_2, \partial_1) \to (B_2, B_1, R, \delta_2, \delta_1)$ be morphisms in $\mathbf{X_2Mod/R}$ as given with the diagram

$$\begin{array}{ccccc} A_2 & \xrightarrow{\partial_2} & A_1 & \xrightarrow{\partial_1} & R \\ {\scriptstyle \beta_2}\downarrow\downarrow{\scriptstyle \alpha_2} & & {\scriptstyle \beta_1}\downarrow\downarrow{\scriptstyle \alpha_1} & & \| \\ B_2 & \xrightarrow[\delta_2]{} & B_1 & \xrightarrow[\delta_1]{} & R \end{array}$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. Define

$$C_i = \{a_i \in A_i : \alpha(a_i) = \beta(a_i)\}$$

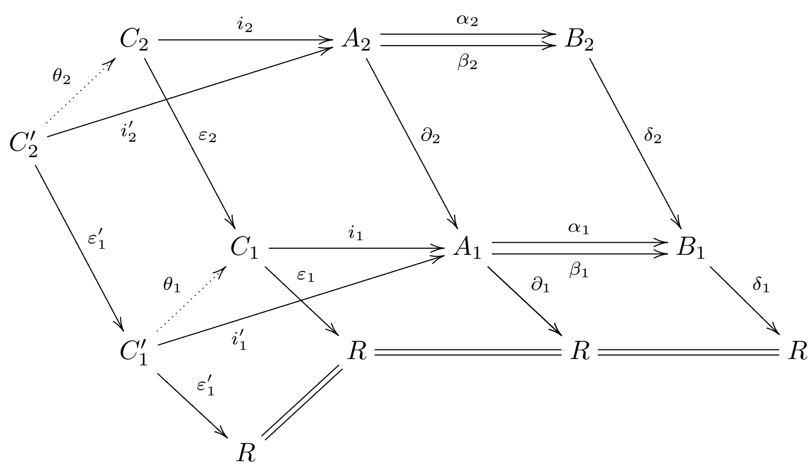
for $i = 1,2$. With induced morphisms $\varepsilon_1, \varepsilon_2$ via ∂_1 and ∂_2, $(C_2, C_1, R, \varepsilon_2, \varepsilon_1)$ becomes a sub 2-crossed R-module of $(A_2, A_1, R, \partial_2, \partial_1)$. The inclusion $i = (i_2, i_1) : (C_2, C_1, R, \varepsilon_2, \varepsilon_1) \to (A_2, A_1, R, \partial_2, \partial_1)$ is a 2-crossed R-module morphism and satisfies

$$(\alpha_k i_k)(c_k) = \alpha_k(i_k(c_k)) = \beta_k(i_k(c_k)) = (\beta_k i_k)(c_k) \quad , k = 1,2$$

for all $c_1 \in C_1$ and $c_2 \in C_2$. If $i' = (i'_2, i'_1) : (C'_2, C'_1, R, \varepsilon'_2, \varepsilon'_1) \to (A_2, A_1, R, \partial_2, \partial_1)$ is another 2-crossed R-module morphism such that

$$(\alpha_k i'_k)(c'_k) = (\beta_k i'_k)(c'_k) \quad , k = 1,2$$

for all $c'_1 \in C'_1$ and $c'_2 \in C'_2$. Then from the definition of C_1 and C_2 we get $i'_2(c'_2) \in C_2$ and $i'_1(c'_1) \in C_1$. Thus we can define $\theta = (\theta_2, \theta_1) : (C'_2, C'_1, \varepsilon') \to (C_2, C_1, \varepsilon)$ as $\theta_k(c'_k) = i'_k(c_k)$ for $k = 1,2$. Since (i'_2, i'_1) is a 2-crossed R-module morphism, (θ_2, θ_1) is the unique morphism making the diagram



commutative. Thus the morphism (i_2, i_1) is the equalizer of (α, β). □

Proposition 3.2 *Pullback object exists in* $\mathbf{X_2Mod/R}$.

Proof Let $(\alpha_2, \alpha_1) : (A_2, A_1, R, \partial_2, \partial_1) \to (B_2, B_1, R, \delta_2, \delta_1)$ and $(\beta_2, \beta_1) : (C_2, C_1, R, \varepsilon_2, \varepsilon_1) \to (B_2, B_1, R, \delta_2, \delta_1)$ be two morphisms of 2-crossed R-modules where

$$\varkappa_1 : A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} R$$

$$\varkappa_2 : B_2 \xrightarrow{\delta_2} B_1 \xrightarrow{\delta_1} R$$

$$\varkappa_3 : C_2 \xrightarrow{\varepsilon_2} C_1 \xrightarrow{\varepsilon_1} R$$

are 2-crossed R-modules with peiffer liftings $\{\_,\_\}_C : C_1 \times C_1 \to C_2$, $\{\_,\_\}_B : B_1 \times B_1 \to B_2$ and $\{\_,\_\}_A : A_1 \times A_1 \to A_2$. Define:

$$P_k = \{(a_k, c_k) : \alpha_k(a_k) = \beta_k(c_k)\} \subset A_k \times C_k, \qquad k = 1,2$$

$$p_1 : P_1 \longrightarrow B_1, \quad (a_1, c_1) \mapsto \alpha_1(a_1) = \beta_1(c_1)$$

and

$$\omega : P_2 \longrightarrow P_1, \quad (a_2, c_2) \mapsto (\partial_2(a_2), \varepsilon_2(c_2)).$$

Then we have the following diagram.

$$\begin{array}{ccc} P_1 & \xrightarrow{\delta_1 p_1} & R \\ {\scriptstyle p_1}\downarrow & \nearrow {\scriptstyle \delta_1} & \\ B_1 & & \end{array}$$

Since

$$\begin{aligned} \delta_1 p_1({}^r(a_1, c_1)) &= \delta_1 p_1(({}^r a_1, {}^r c_1)) \\ &= \delta_1(\alpha_1({}^r a_1)) \\ &= \delta_1(\alpha_1({}^r c_1)) \\ &= r\delta_1(\alpha_1(c_1))r^{-1} \\ &= r(\delta_1 p_1(a_1, c_1))r^{-1} \end{aligned}$$

for all $r \in R$ and $(a_1, c_1) \in P_1$, the map $\delta_1 p_1$ is a precrossed module. Next, we will show that

$$\varkappa' : P_2 \xrightarrow{\omega} P_1 \xrightarrow{\delta_1 p_1} R$$

is an object in $\mathbf{X_2Mod/R}$ where peiffer lifting $\{\_,\_\}_P : P_1 \times P_1 \to P_2$ is defined by $[(a_1, c_1),\ (a_1', c_1')] \mapsto (\{a_1, a_1'\}_A, \{c_1, c_1'\}_C)$ for all $(a_1, c_1),\ (a_1', c_1') \in P_1$.

1) ω and $\delta_1 p_1$ are R-equivariant. R acts on itself by conjugation.

2) For all $(a_1, c_1),\ (a_1', c_1') \in P_1$,

$$\begin{aligned} \omega\{(a_1, c_1),\ (a_1', c_1')\}_P &= \omega(\{a_1, a_1'\}_A, \{c_1, c_1'\}_C) \\ &= [\partial_2\{a_1, a_1'\}_A, \varepsilon_2\{c_1, c_1'\}_C] \\ &= \left({}^{\partial_1(a_1)}a_1' a_1 (a_1')^{-1}(a_1)^{-1}, {}^{\varepsilon_1(c_1)}c_1' c_1 (c_1')^{-1}(c_1)^{-1}\right) \\ &= \left({}^{\delta_1\alpha_1(a_1,c_1)}a_1' a_1 (a_1')^{-1}(a_1)^{-1}, {}^{\delta_1\beta_1(a_1,c_1)}c_1' c_1 (c_1')^{-1}(c_1)^{-1}\right) \\ &= \left({}^{\delta_1\alpha_1(a_1,c_1)}(a_1', c_1')(a_1, c_1)(a_1', c_1')^{-1}(a_1, c_1)^{-1}\right) \end{aligned}$$

3) For all $(a_2, c_2),\ (a_2', c_2') \in P_2$.

$$\begin{aligned} \{\omega(a_2, c_2), \omega(a_2', c_2')\}_P &= \{(\partial_2(a_2), \varepsilon_2(c_2)), (\partial_2(a_2'), \varepsilon_2(c_2'))\}_P \\ &= (\{\partial_2(a_2), \varepsilon_2(c_2)\}_A, \{\partial_2(a_2'), \varepsilon_2(c_2')\}_C) \\ &= [a_2, a_2'][c_2, c_2'] \\ &= [(a_2, c_2), (a_2', c_2')] \end{aligned}$$

4) For all $(a_2,c_2)\in P_2,\ (a_1,c_1)\in P_1$.

$$\begin{aligned}\{(a_1,c_1),\omega(a_2,c_2)\}_P\{\omega(a_2,c_2),(a_1,c_1)\}_P &= \{(a_1,c_1),(\partial_2(a_2),\varepsilon_2(c_2))\}_P\{((\partial_2(a_2),\varepsilon_2(c_2)),(a_1,c_1))\}_P\\ &=(\{a_1,\partial_2(a_2)\}_A,\{c_1,\varepsilon_2(c_2)\}_C)(\{\partial_2(a_2),a_1\}_A,\{\varepsilon_2(c_2),c_1\}_C)\\ &=(\{a_1,\partial_2(a_2)\}_A\{\partial_2(a_2),a_1\}_A,\{c_1,\varepsilon_2(c_2)\}_C\{\varepsilon_2(c_2),c_1\}_C)\\ &=({}^{\partial_1(a_1)}a_2\,(a_2)^{-1},{}^{\varepsilon_1(c_1)}c_2\,(c_2)^{-1})\\ &=({}^{\delta_1\alpha_1(a_1,c_1)}a_2\,(a_2)^{-1},{}^{\delta_1\beta_1(a_1,c_1)}c_2\,(c_2)^{-1})\\ &={}^{\delta_1\alpha_1(a_1,c_1)}(a_2,c_2)(a_2,c_2)^{-1}\end{aligned}$$

5) For $(a_0,c_0),(a_1,c_1),(a_2,c_2)\in P_1$

i)

$$\{(a_0,c_0),(a_1,c_1)(a_2,c_2)\}_P=\{(a_0,c_0),(a_1,c_1)\}_P^{(a_0,c_0)(a_1,c_1)(a_0,c_0)^{-1}}\{(a_0,c_0),(a_2,c_2)\}_P$$

ii)

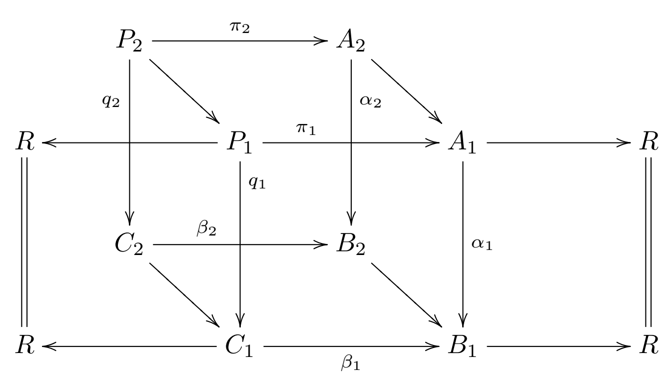
$$\{(a_0,c_0)(a_1,c_1),(a_2,c_2)\}_P={}^{\delta_1p_1}\{((a_1,c_1),(a_2,c_2)\}_P\left\{(a_0,c_0),(a_1,c_1)(a_2,c_2)(a_1,c_1)^{-1}\right\}_P$$

proof left to the reader as an exercise.

6) For $r\in R$ and $(a_0,c_0),(a_1,c_1)\in P_1$

$$\begin{aligned}{}^r\{(a_0,c_0),(a_1,c_1)\}_P &={}^r(\{a_0,a_1\}_A,\{c_0,c_1\}_C)\\ &=({}^r\{a_0,a_1\}_A,{}^r\{c_0,c_1\}_C)\\ &=(\{{}^ra_0,{}^ra_1\}_A,\{{}^rc_0,{}^rc_1\}_C)\\ &=\{({}^ra_0,{}^rc_0),({}^ra_1,{}^rc_1)\}_P\\ &=\{{}^r(a_0,c_0),{}^r(a_1,c_1)\}_P\end{aligned}$$

With the induced morphisms $\pi_{1,2}:\varkappa'\to\varkappa_1$ and projection morphisms $q_{1,2}:\varkappa'\to\varkappa_3$ we have the following commutative diagram

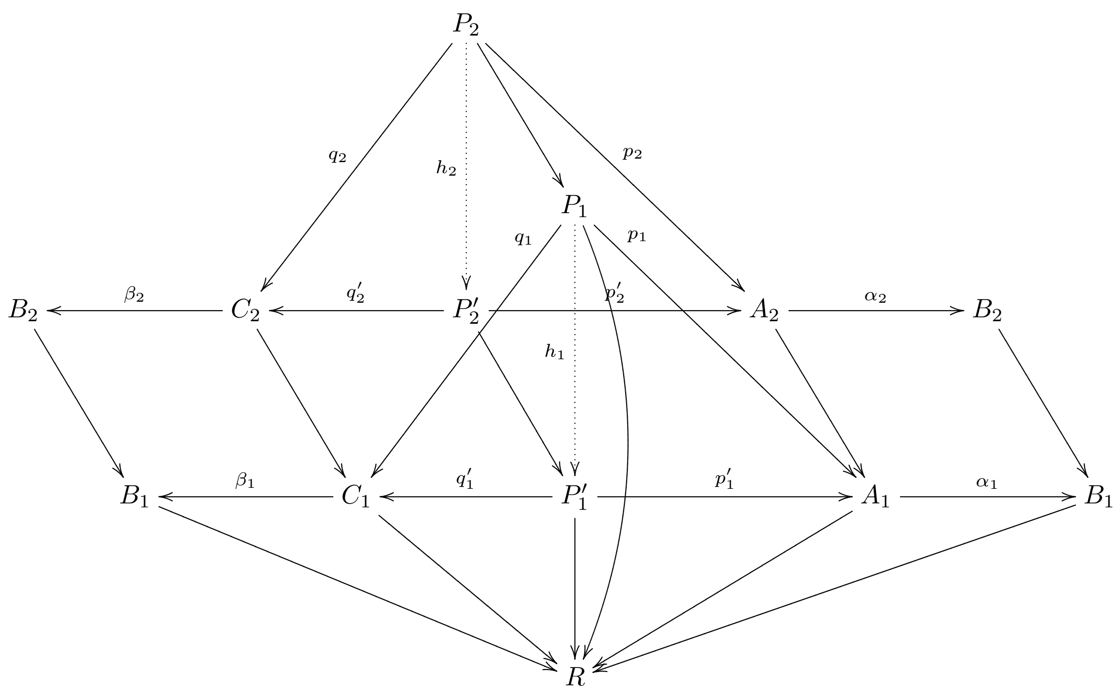


Here $\alpha\pi=\beta q$, the 2-crossed R-module morphisms commute and the morphisms π and q satisfy the universal property. Let $(\pi_1',\pi_2'):\varkappa'\to\varkappa_1$ and $(q_1',q_2'):\varkappa'\to\varkappa_3$ be any morphisms in $\mathbf{X_2Mod/R}$ such that $\alpha\pi'=\beta q'$

and

$$\varkappa'' : P_2' \to P_1' \to R$$

then there is a unique morphism $(h_1, h_2) : \varkappa'' \to \varkappa'$ given by $h_k(p_k') = (\pi_k'(p_k'), q_k'(p_k'))$ for $k = 1, 2$ and $p_1' \in P_1, p_2' \in P_2$ such that the diagram



commutes. This shows that pullback object exists in $\mathbf{X_2Mod/R}$. □

Proposition 3.3 $\mathbf{X_2Mod/R}$ *has finite products.*

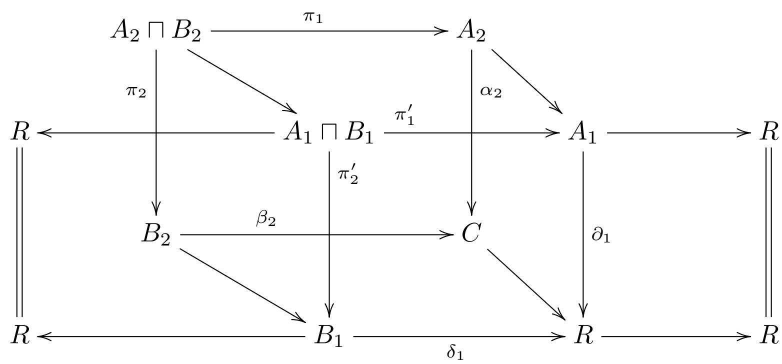
Proof For two 2-crossed R-modules say $\varkappa_1$ and $\varkappa_2$, the product $\varkappa_1 \sqcap \varkappa_2$ will be the pullback over the terminal object $\varkappa_t$ where

$$\varkappa_1 : A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} R$$

$$\varkappa_2 : B_2 \xrightarrow{\delta_2} B_1 \xrightarrow{\delta_1} R$$

$$\varkappa_t : C \xrightarrow{\sigma_2} R \xrightarrow{Id} R$$

with peiffer liftings $\{\_,\_\}_C : R \times R \to C$, $\{\_,\_\}_B : B_1 \times B_1 \to B_2$ and $\{\_,\_\}_A : A_1 \times A_1 \to A_2$ and projection maps $\pi_1 : A_2 \sqcap B_2 \longrightarrow A_2, \pi_2 : A_2 \sqcap B_2 \longrightarrow B_2, \pi_1' : A_1 \sqcap B_1 \longrightarrow A_1$ and $\pi_2' : A_1 \sqcap B_1 \longrightarrow B_1$ the diagram



is commutative. For all $a_k \in A_k, b_k \in B_k$, $k = 1,2$; $\vartheta_1 : A_1 \sqcap B_1 \to R$ and $\vartheta_2 : A_2 \sqcap B_2 \to A_1 \sqcap B_1$ is given with

$$\vartheta_1 (a_1, b_1) = \partial_1 \pi_1 (a_1, b_1) = \delta_1 \pi_2 (a_1, b_1)$$

$$\vartheta_2 (a_2, b_2) = (\partial_2 (a_2), \delta_2 (b_2))$$

with peiffer lifting $\{\_,\_\}_\sqcap : (A_1 \sqcap B_1) \times (A_1 \sqcap B_1) \to A_2 \sqcap B_2$

$$\{(a_1, b_1), (a_1', b_1')\}_\sqcap = (\{a_1, a_1'\}_A, \{b_1, b_1'\}_B)$$

we get a 2-crossed R-module $A_2 \sqcap B_2 \to A_1 \sqcap B_1 \to R$. Using induction $\mathbf{X_2Mod/R}$ has finite products.

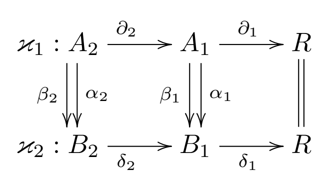
Conclusion 3.4 *Since* $\mathbf{X_2Mod/R}$ *has finite products and equalisers,* $\mathbf{X_2Mod/R}$ *has a limit for any functor from a finite category to* $\mathbf{X_2Mod/R}$. *Having all finite limits we say that* $\mathbf{X_2Mod/R}$ *is finitely complete.*

□

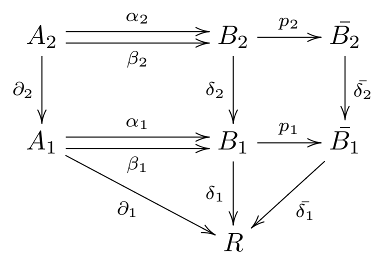
4. Finite colimits in $\mathbf{X_2Mod/R}$

Proposition 4.1 *In* $\mathbf{X_2Mod/R}$ *every pair of morphisms with common domain and codomain has a coequaliser.*

Proof Let $(\alpha, \beta) : \varkappa_1 \to \varkappa_2$ be two 2-crossed R-module morphisms as given



Let N_k be a normal subgroup of B_k generated by elements of the form $\alpha_k(a_k) - \beta_k(a_k)$, for $k = 1,2$ and for all $a_k \in A_k$. Taking $\bar{B}_2 = B_2/N_2$, $\bar{B}_1 = B_1/N_1$ and defining $\bar{\delta}_1 : \bar{B}_1 \to R$, $(b_1 + N_1) \mapsto \delta_1(b_1)$; $\bar{\delta}_2 : \bar{B}_2 \to \bar{B}_1$, $(b_2 + N_2) \mapsto \delta_2(b_2) + N_2$ with peiffer lifting $\{\_,\_\}_N : \bar{B}_1 \times \bar{B}_1 \to \bar{B}_2$, $[(b_1 + N_1), (b_1' + N_1)] \mapsto \{b_1, b_1'\}_B + N_2$, $\bar{\varkappa} = \left(\bar{B}_2, \bar{B}_1, R, \bar{\delta_2}, \bar{\delta_1}\right)$ becomes a 2-crossed R-module and the induced morphism $p{:}\varkappa_2 \to \bar{\varkappa}$ is a 2-crossed R-module morphism such that the diagram

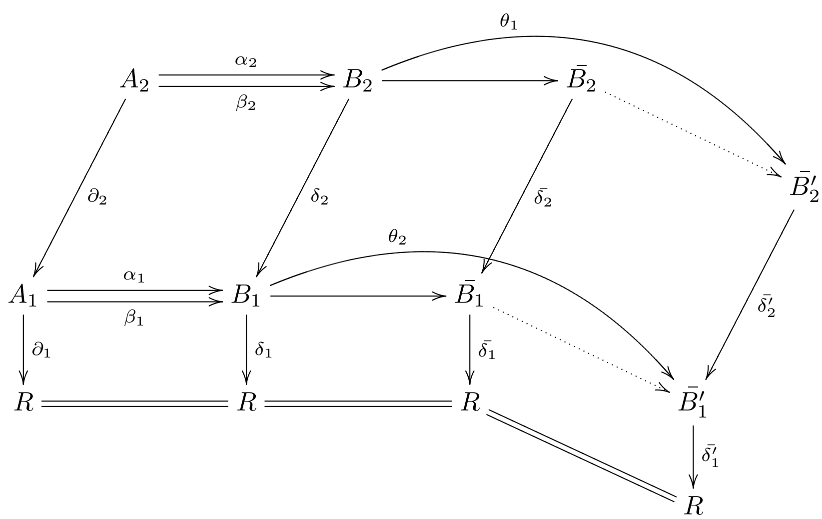


is commutative. Suppose there exists another 2-crossed module morphism $p':\varkappa_2 \to \varkappa'$ then there is a unique morphism in $\mathbf{X_2Mod/R}$ say $\varphi : \bar{\varkappa} \to \varkappa'$ in which

$$\varphi_k(b_k + N_k) = p'_k(b_k)$$

satisfying $\varphi_k p_k = p'_k$ for $k = 1, 2$ (where $\overline{\varkappa} : \overline{B_2} \xrightarrow{\overline{\delta_2}} \overline{B_1} \xrightarrow{\overline{\delta_1}} R$). Then the morphism $p = (p_1, p_2)$ is universal making the following diagram commutative.

Thus p is the coequaliser of α and β



Let

$$\varkappa_1 : A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} R$$

$$\varkappa_2 : B_2 \xrightarrow{\delta_2} B_1 \xrightarrow{\delta_1} R$$

be two 2-crossed R-modules. Since B_1 acts on A_1 via δ_1 we can form the semidirect product $B_1 \ltimes A_1$ with

$$(b_1, a_1)(b'_1, a'_1) = (b_1 \, (b'_1)^{\delta_1(b'_1)}, a'_1)$$

and

$${}^r(b_1, a_1) = ({}^r b_1, {}^r a_1)$$

for $(b_1,a_1),(b_1',a_1') \in B_1 \ltimes A_1$ and $r \in R$. We get injections $i_1 : B_1 \to B_1 \ltimes A_1$, $j_1 : A_1 \to B_1 \ltimes A_1$ and define $\sigma_1 : B_1 \ltimes A_1 \to R$ as $\sigma_1(b_1,a_1) = \delta_1(b_1) + \partial_1(a_1)$. Let N_1 be the normal subgroup of $B_1 \ltimes A_1$ generated by the elements

$$(b_1,a_1)(b_2,a_2) - \sigma_1(b_1,a_1)\cdot(b_2,a_2)$$

$$(b_1,a_1)(b_2,a_2) - (b_1,a_1)\cdot\sigma_1(b_2,a_2)$$

Thus the factor group $B_1 \ltimes A_1/N_1$ can be defined with the induced morphism $B_1 \ltimes A_1/N_1 \to R$ as $\bar{\sigma}_1[(b_1,a_1)+N_1] = \sigma_1(b_1,a_1)$. Clearly $\bar{\sigma}_1$ is a precrossed module. Furthermore B_2 acts on A_2 via $\delta_2\delta_1$ and $\delta^2 = 0$, the semidirect product of A_2 and B_2 is the direct product. With injections $i_2 : B_2 \to B_2 \ltimes A_2$, $j_1 : A_2 \to B_2 \ltimes A_2$ we define $\sigma_2 : B_2 \ltimes A_2 \to B_1 \ltimes A_1$ as $\sigma_2(b_2,a_2) = [\delta_2(b_2), \partial_2(a_2)]$. With the same method we construct $B_1 \ltimes A_1/N_1$, we can form $B_2 \ltimes A_2/N_2$ and the morphism $\bar{\sigma}_2[(b_2,a_2)+N_2] = [\delta_2(b_2),\partial_2(a_2)]+N_1$ and the action of $B_1 \ltimes A_1/N_1$ on $B_2 \ltimes A_2/N_2$ is given via σ_1. □

Proposition 4.2 $(B_2 \ltimes A_2)/N_2 \to (B_1 \ltimes A_1)/N_1 \to R$ *is a 2-crossed R-module.*

Proof First we define peiffer lifting $\{\_,\_\}_{\bar{N}} : (B_1 \ltimes A_1/N_1) \times (B_1 \ltimes A_1/N_1) \to B_2 \ltimes A_2/N_2$ as $[(b_1,a_1)+N_1,(b_1',a_1')+N_1] \mapsto (\{b_1,b_1'\}_B + N_2, \{a_1,a_1'\}_A + N_2)$ for $(b_1,a_1)+N_1,(b_1',a_1')+N_1 \in (B_1 \ltimes A_1/N_1)$.

1) $\bar{\sigma}_2$ and $\bar{\sigma}_1$ are R-equivariant. R acts on itself by conjugation.

2) For $(b_1,a_1)+N_1,(b_1',a_1')+N_1 \in (B_1 \ltimes A_1/N_1)$

$$\begin{aligned}
\bar{\sigma}_2\{(b_1,a_1)+N_1,(b_1',a_1')+N_1\}_N &= \bar{\sigma}_2(\{b_1,b_1'\}_B + N_2, \{a_1,a_1'\}_A + N_2) \\
&= \left[\bar{\delta}_2\{b_1,b_1'\}_B, \bar{\partial}_2\{a_1,a_1'\}_A\right] \\
&= ({}^{\bar{\delta}_1(b_1+N_1)}(b_1'+N_1)(b_1+N_1)(b_1'+N_1)^{-1}(b_1+N_1)^{-1}, \\
&\quad {}^{\bar{\partial}_1(a_1+N_1)}(a_1'+N_1)(a_1+N_1)(a_1'+N_1)^{-1}(a_1+N_1)^{-1}) \\
&= ({}^{\bar{\sigma}_1[(b_1,a_1)+N_1]}(b_1'+N_1)(b_1+N_1)(b_1'+N_1)^{-1}(b_1+N_1)^{-1}, \\
&\quad {}^{\bar{\sigma}_1[(b_1,a_1)+N_1]}(a_1'+N_1)(a_1+N_1)(a_1'+N_1)^{-1}(a_1+N_1)^{-1}) \\
&= {}^{\bar{\sigma}_1[(b_1,a_1)+N_1]}((b_1',a_1')+N_1)((b_1,a_1)+N_1) \\
&\quad ((b_1',a_1')+N_1)^{-1}((b_1,a_1)+N_1)^{-1}
\end{aligned}$$

3) For $(b_2,a_2)+N_2,(b_2',a_2')+N_2 \in (B_2 \ltimes A_2/N_2)$

$$\begin{aligned}
\{\bar{\sigma}_2(b_2,a_2)+N_2, \bar{\sigma}_2(b_2',a_2')+N_2\}_N &= \{(\delta_2(b_2),\partial_2(a_2))+N_1,(\delta_2(b_2'),\partial_2(a_2'))+N_1\}_P \\
&= (\{\delta_2(b_2),\delta_2(b_2')\}_B + N_2, \{\partial_2(a_2),\partial_2(a_2')\}_A + N_2) \\
&= ([b_2,b_2']+N_2,[a_2,a_2']+N_2) \\
&= [(b_2,a_2),(b_2',a_2')]+N_2
\end{aligned}$$

4) For $(b_2,a_2)+N_2 \in (B_2 \ltimes A_2/N_2), (b_1,a_1)+N_1 \in (B_1 \ltimes A_1/N_1)$,

$$\begin{aligned}
&\{(b_1,a_1)+N_1,\bar{\sigma}_2(b_2,a_2)+N_2\}_N\,\{\bar{\sigma}_2(b_2,a_2)+N_2,(b_1,a_1)+N_1\}_N\\
&=(\{b_1+N_1,\delta_2b_2+N_2\}_B\,\{\delta_2b_2+N_2,b_1+N_1\}_B\,,\{a_1+N_1,\partial_2a_2+N_2\}_A\,\{\partial_2a_2+N_2,a_1+N_1\}_A)\\
&=((b_2+N_2)^{\delta_1b_1+N_1}\,(b_2+N_2)^{-1}\,,(a_2+N_2)^{\partial_1a_1+N_1}\,(a_2+N_2)^{-1})\\
&=((b_2+N_2)^{\bar{\delta}_1(b_1,a_1)+N_1}\,(b_2+N_2)^{-1}\,,(a_2+N_2)^{\bar{\partial}_1(b_1,a_1)+N_1}\,(a_2+N_2)^{-1})\\
&=((b_2,a_2)+N_2)^{\bar{\sigma}_1(b_1,a_1)+N_1}\,((b_2,a_2)+N_2)^{-1}
\end{aligned}$$

5) For $(b_k,a_k)+N_1\in(B_1\ltimes A_1/N_1),\ k=0,1,2$

$$\begin{aligned}
i.\,\{(b_0,a_0)+N_1,(b_1,a_1)(b_2,a_2)+N_1\}_N &= \{(b_0,a_0)+N_1,(b_1,a_1)+N_1\}_N^{(b_0,a_0)(b_1,a_1)(b_0,a_0)^{-1}}\\
&\quad\ \{(b_0,a_0)+N_1,(b_2,a_2)+N_1\}_N\\
ii.\,\{(b_0,a_0)(b_1,a_1)+N_1,(b_2,a_2)+N_1\}_N &= {}^{\bar{\sigma}(b_0,a_0)+N_1}\{(b_1,a_1)+N_1,(b_2,a_2)+N_1\}_N\\
&\quad\ \left\{(b_0,a_0)+N_1,(b_1,a_1)(b_2,a_2)(b_1,a_1)^{-1}\right\}_N
\end{aligned}$$
left to the reader.

6) For $r\in R$ and $(b_0,a_0)+N_1,(b_1,a_1)+N_1\in(B_1\ltimes A_1/N_1)$

$$\begin{aligned}
{}^r\{(b_0,a_0)+N_1,(b_1,a_1)+N_1\}_N &= {}^r(\{b_0,b_1\}_B+N_2,\{a_0,a_1\}_A+N_2)\\
&= ({}^r\{b_0,b_1\}_B+N_2,{}^r\{a_0,a_1\}_A+N_2)\\
&= \{{}^r(b_0,a_0)+N_1,{}^r(b_1,a_1)+N_1\}
\end{aligned}$$

□

Theorem 4.3 *The constructed 2-crossed module*

$$\varkappa:(B_2\times A_2)/N_2\overset{\bar{\delta}_2}{\to}(B_1\times A_1)/N_1\overset{\bar{\delta}_1}{\to}R$$

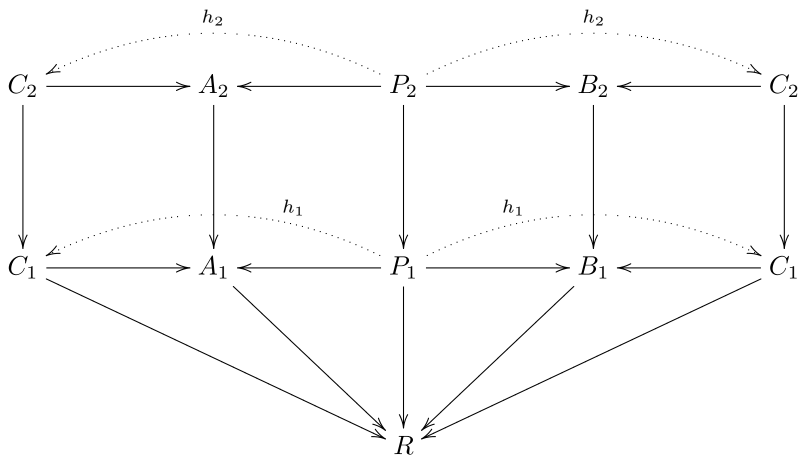
with the morphisms (i_1,j_1), (i_2,j_2) *is the coproduct of the 2-crossed modules.*

Proof Let $\varkappa_C=(C_2,C_1,R,\partial_2',\partial_1')$ be any 2-crossed R-module and $\alpha:(B_2,B_1,R,\delta_2,\delta_1)\to(C_2,C_1,R,\partial_2',\partial_1')$ and $\beta:(A_2,A_1,R,\partial_2,\partial_1)\to(C_2,C_1,R,\partial_2',\partial_1')$ be 2-crossed R-module morphisms as given by the following diagram.

$$\begin{array}{ccccc}
A_2 & \xrightarrow{\partial_2} & A_1 & \xrightarrow{\partial_1} & R\\
\downarrow{\scriptstyle\beta_2} & & \downarrow{\scriptstyle\beta_1} & & \|\\
C_2 & \xrightarrow{\partial_2'} & C_1 & \xrightarrow{\partial_1'} & R\\
\uparrow{\scriptstyle\alpha_2} & & \uparrow{\scriptstyle\alpha_1} & & \|\\
B_2 & \xrightarrow{\delta_2} & B_1 & \xrightarrow{\delta_1} & R
\end{array}$$

Then there exists a map $h:\varkappa\to\varkappa_C$ given by $h_k\left[(b_k,a_k)+N_k\right]=\alpha_k\left(b_k\right)\beta_k\left(a_k\right)$. $h=(h_1,h_2)$ is a unique

2-crossed R-module morphism making the diagram



commutative. □

The construction of coproducts in $\mathbf{X_2Mod/R}$ induces the functor

$$\circ : \mathbf{X_2Mod/R} \times \mathbf{X_2Mod/R} \to \mathbf{X_2Mod/R}$$

which is left adjoint to the diagonal functor

$$\triangle : \mathbf{X_2Mod/R} \to \mathbf{X_2Mod/R} \times \mathbf{X_2Mod/R}.$$

Conclusion 4.4 $\mathbf{X_2Mod/R}$ *is cocomplete.*

Example 4.5 *Let*

$$\varkappa_1 : B_2 \xrightarrow{\delta_2} B_1 \xrightarrow{\delta_1} R$$

$$\varkappa_2 : A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} R$$

be two 2-crossed R*-modules with* $\partial_1(A_1) \subset \delta_1(B_1)$ *and peiffer liftings* $\{\_,\_\}_A, \{\_,\_\}_B$. *Let* $\sigma_1 : \delta_1(B_1) \to B_1$ *be an* R*-equivariant section of* δ_1. *Then the morphisms*

$$\begin{aligned} i_1 &: B_1 \to B_1 \times A_1/[A_1, B_1]; & b_1 &\mapsto (b_1, 0) \\ i_2 &: B_2 \to B_2 \times B_1; & b_2 &\mapsto (b_2, 0) \\ j_1 &: A_1 \to B_1 \times A_1/[A_1, B_1]; & a_1 &\mapsto (\sigma_1(a_1), [a_1]) \\ j_2 &: A_2 \to B_2 \times A_2; & a_2 &\mapsto (0, a_2) \end{aligned}$$

give a coproduct of 2-crossed modules,

$$\varkappa_1 : B_2 \xrightarrow{\delta_2} B_1 \xrightarrow{\delta_1} R$$

with

$$\begin{aligned} \varepsilon_1 &: B_1 \times (A_1/[A_1,B_1]) \to R; & (b_1,[x]) \mapsto \delta_1(b_1) \\ \varepsilon_2 &: B_2 \times A_2 \to B_1 \times (A_1/[A_1,B_1]); & (b_2,b_1) \mapsto (\delta_2(b_2),0) \end{aligned}$$

and peiffer lifting

$$\begin{aligned} \{\_,\_\}_{cop} : (B_1 \times (A_1/[A_1,B_1])) \times (B_1 \times (A_1/[A_1,B_1])) &\to B_2 \times A_2 \\ [(b_1,[x]),(b_1',[x])] &\mapsto (\{b_1,b_1'\}_B, [\{x_1,x_1'\}_A]) \end{aligned}$$

Here $[A_1, B_1]$ *is the normal subgroup of* A_1 *generated by the elements* $a^{-1}a^b$ *for all* $a \in A_1, b \in B_1$.

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