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On the BMO spaces associated with the Laplace-Bessel differential operator

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Abstract: In this paper, the characteristic properties of the space of functions of bounded mean oscillation called the B-BMO associated with the Laplace-Bessel differential operator are obtained. The John-Nirenberg type inequality on the B-BMO space and a relation between the B-Poisson integral and the B-BMO functions are proved.

Key words: Laplace-Bessel differential operator, Generalized translation operator, BMO spaces, B-BMO spaces, Weighted Lebesque spaces

1. Introduction

The Laplace-Bessel differential operator Δ_B which is an important technical tool in Fourier-Bessel harmonic analysis is defined by

$$\Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \left(\frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n} \frac{\partial^2}{\partial x_n} \right), \quad (\nu > 0,\ x_n > 0).$$

This operator is a hybrid differential operator which is obtained by applying the Laplace differential operator in the first $n-1$ variable and the Bessel differential operator in the last variable.

The relevant Fourier-Bessel harmonic analysis associated with the Bessel differential operator B_t (or Laplace-Bessel differential operator Δ_B) has been a research area for many mathematicians such as Delsarte, Levitan, Kipriyanov, Klyuchantsev, Lyakhov, Stempak, Gadjiev, Aliev, Guliev, Bayrakci, Sezer, Hasanov and many others [1–5, 8, 11, 14–16, 19–22, 24–26].

The classical BMO space of functions of bounded mean oscillation, was first introduced by John and Nirenberg [18] in 1961. The space BMO shares similar properties with the space L_∞, and it often serves as a substitute for it. For instance, classical singular integrals do not map L_∞ to L_∞ but L_∞ to BMO. Many interpolations between L_p and BMO work very well between L_p and L_∞. This space has been studied by several authors, e.g., John, Nirenberg, Fefferman, Stein, Garnett, Carleson, Chang, Sadosky, Jones, Meyers, Janson, and others [6, 7, 9, 10, 12, 17, 18, 23].

The space of functions of bounded mean oscillation associated with the Laplace-Bessel differential operator Δ_B, called the B-BMO space was defined by Guliyev [14]. In recent years, the B-BMO space has been used by many mathematicians such as Guliyev, Abasova, Aliyeva, Shirinova, Hasanov, Ayazoglu, and Bayrakci to obtain the boundedness of some integral operators in suitable function spaces [1, 14–16].

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In this article, we obtain that the characteristic properties provided by the classical BMO space are also valid in the B-BMO space. We also state the John-Nirenberg-type inequality in B-BMO spaces and prove the relationship between the B-Poisson integrals and the B-BMO functions.

The paper is organized as follows. Section 2 contains basic definitions and results. The characteristic properties of the B-BMO space (Theorem 3.3, Theorem 3.5), the relationship between the B-Poisson integral and the B-BMO function (Theorem 3.6, Theorem 3.7) and finally the John-Nirenberg type inequality and application (Theorem 3.8, Theorem 3.9) are given in Section 3.

2. Definitions, notations, and preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Also let for $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, $|x|=\left(\sum\limits_{k=1}^{n}x_k^2\right)^{\frac{1}{2}}$. Denote $\mathbb{R}_+^n=\{x\in\mathbb{R}^n : x=(x_1,\cdots,x_{n-1},x_n),\ x_n>0\}$. The Lebesgue measure of a measurable set $E\subset\mathbb{R}_+^n$ is denoted by $|E|_\nu=\int\limits_E x_n^{2\nu}dx$, $dx=dx_1dx_2\cdots dx_n$ and $\nu>0$ is a fixed parameter.

Suppose that $E(x,r)=\{y\in\mathbb{R}_+^n : |x-y|<r\}$ denotes the "ball" of radius $r>0$ centered at $x\in\mathbb{R}_+^n$. It is known that $|E(0,r)|_\nu=r^{n+2\nu}\omega(n,\nu)$ where $\omega(n,\nu)=|E(0,1)|_\nu$.

Denote by T^y the generalized translation operator, acting according to the law

$$T^y f(x)=\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)}\int\limits_0^\pi f\left(x'-y';\sqrt{(x_n^2-2x_ny_n\cos\theta+y_n^2)}\right)\sin^{2\nu-1}\theta d\theta,$$

where $x=(x',x_n),y=(y',y_n),x',y'\in\mathbb{R}^{n-1}$ and

$$\left(\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)}\right)^{-1}=\int\limits_0^\pi\sin^{2\nu-1}\theta d\theta.$$

Note that the generalized translation operator T^y is closely related with the Laplace-Bessel differential operator Δ_B (see [8, 21, 22] for details).

Let $L_{p,\nu}\left(\mathbb{R}_+^n\right),1\le p<\infty$ be the space of all measurable functions on $\mathbb{R}_+^n$ with the norm

$$\|f\|_{p,\nu}=\left(\int\limits_{\mathbb{R}_+^n}|f(x)|^p x_n^{2\nu}dx\right)^{\frac{1}{p}}<\infty.$$

In the case of $p=\infty$, the space $L_\infty\left(\mathbb{R}_+^n\right)$ is equipped with the norm $\|f\|_\infty=ess\sup\limits_{x\in\mathbb{R}_+^n}|f(x)|$. Also, we denote by $L_1^{loc}\left(\mathbb{R}_+^n\right)$ the set of locally integrable functions on $\mathbb{R}_+^n$.

Lemma 2.1 *[16] For all $x\in\mathbb{R}_+^n$ the following equality is valid:*

$$\int\limits_{E(0,r)}T^zf(x)\,z_n^{2\nu}dz=\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)}\int\limits_{E(\widetilde{x},r)}f\left(y',\sqrt{y_n^2+y_{n+1}^2}\right)d\widetilde{y}$$

where $E(\widetilde{x},r) = E((x,0),r) = \{\widetilde{y} = (y, y_{n+1}) \in \mathbb{R}^n \times (0,\infty) : \ |\widetilde{x} - \widetilde{y}| < r\}, \quad y' \in \mathbb{R}^{n-1}, \ dy = dy_1...dy_{n-1}dy_n.$

Proof Let $\widetilde{x} = (x,0) = (x_1,...,x_n,0), \ \widetilde{y} = (y_1,...,y_n,y_{n+1})$. We set $\widetilde{w} = \widetilde{x} - \widetilde{y} = (x_1 - y_1, ..., x_n - y_n, -y_{n+1})$. Thus we have

$$\int\limits_{E(\widetilde{x},r)} f\left(y', \sqrt{y_n^2 + y_{n+1}^2}\right) y_{n+1}^{2\nu-1} dy_{n+1} dy = \int\limits_{E(0,r)} f\left(x' - w', \sqrt{(x_n - w_n)^2 + w_{n+1}^2}\right) w_{n+1}^{2\nu-1} dw_{n+1} dw.$$

Now, let us apply the substitutions for $(z_1,...,z_n) \in \mathbb{R}_+^n$: $\quad w' = z', \quad w_n = z_n \cos\theta, \quad w_{n+1} = z_n \sin\theta,$
$0 \le \theta < \pi, \quad z_n \ge 0$. Finally, we get

$$\frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)} \int\limits_{E(0,r)} \int\limits_0^\pi f\left(x' - z', \sqrt{(x_n - z_n\cos\theta)^2 + z_n^2 \sin^2\theta}\right) z_n^{2\nu-1} \sin^{2\nu-1}\theta \ z_n d\theta dz = \int\limits_{E(0,r)} T^z f(x) \, z_n^{2\nu} dz.$$

□

3. Main definitions and results

Definition 3.1 *(Guliev [14]) The B-BMO space, generated by the generalized translation operator is defined as the space of locally integrable functions f with the norm (see Remark below):*

$$\|f\|_{B\text{-}BMO} = \sup_{\substack{r>0 \\ x\in\mathbb{R}_+^n}} \frac{1}{|E(0,r)|_\nu} \int\limits_{E(0,r)} \left|T^y f(x) - f_{E(0,r)}(x)\right| y_n^{2\nu} dy < \infty$$

where

$$f_{E(0,r)}(x) = \frac{1}{|E(0,r)|_\nu} \int\limits_{E(0,r)} T^y f(x) \, y_n^{2\nu} dy.$$

It is a simple fact that $B\text{-}BMO$ is a linear space, that is, if $f, g \in B\text{-}BMO$ and $\alpha \in \mathbb{C}$ then, $f + g$ and αf are also in $B\text{-}BMO$ and

$$\begin{aligned} \|f+g\|_{B\text{-}BMO} &\le \|f\|_{B\text{-}BMO} + \|g\|_{B\text{-}BMO}, \\ \|\alpha f\|_{B\text{-}BMO} &= |\alpha| \, \|f\|_{B\text{-}BMO}. \end{aligned}$$

Remark 3.2 $\|\cdot\|_{B\text{-}BMO}$ *is not a norm. The problem is that if* $\|f\|_{B\text{-}BMO} = 0$, *this does not imply that* $f = 0$ *but that* f *is a constant. Moreover, every constant function* c *satisfies* $\|c\|_{B\text{-}BMO} = 0$. *Although* $\|\cdot\|_{B\text{-}BMO}$ *is only a seminorm, it can be taken as a norm when there is no possibility of confusion.*

Now, let us begin with a list of basic properties of the space $B\text{-}BMO$.

Theorem 3.3 *Let* $f \in L_1^{loc}\left(\mathbb{R}_+^n\right)$. *The following properties of the space* $B\text{-}BMO$ *are valid:*

a) $L_\infty\left(\mathbb{R}_+^n\right) \subsetneqq B\text{-}BMO$ *and* $\|f\|_{B\text{-}BMO} \le 2\|f\|_\infty$.

b) Suppose that there exists an $A > 0$ *such that for all* $x \in \mathbb{R}_+^n$ *and all* $r > 0$ *there exists a constant* $c_{x,r}$ *such that*

$$\sup_{\substack{r>0 \\ x\in\mathbb{R}_+^n}} \frac{1}{|E(0,r)|_\nu} \int\limits_{E(0,r)} \left|T^y f(x) - c_{x,r}\right| y_n^{2\nu} dy \le A. \tag{3.1}$$

Then, $f \in B\text{-}BMO$ *and* $\|f\|_{B\text{-}BMO} \leq 2A$.

c) If $f \in B\text{-}BMO$ *and* $\lambda > 0$ *then, the function* $\delta^{\lambda} f$ *defined by* $\delta^{\lambda} f(x) = f(\lambda x)$ *is also in* $B\text{-}BMO$ *and*

$$\left\|\delta^{\lambda} f\right\|_{B\text{-}BMO} = \|f\|_{B\text{-}BMO}. \tag{3.2}$$

Proof a) Let $f \in L_{\infty}\left(\mathbb{R}_{+}^{n}\right)$. Since

$$\begin{aligned} |T^{y} f(x)| &\leq \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)} \int_{0}^{\pi} \left| f\left(x'-y'; \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\right)\right| \sin^{2\nu-1}\alpha d\alpha \\ &\leq ess\sup_{x\in\mathbb{R}_{+}^{n}} |f(x)| \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)} \int_{0}^{\pi} \sin^{2\nu-1}\alpha d\alpha = \|f\|_{\infty} \end{aligned}$$

we have

$$\left|f_{E(0,r)}(x)\right| \leq \frac{1}{|E(0,r)|_{\nu}} \int_{E(0,r)} |T^{y} f(x)|\, y_n^{2\nu} dy \leq \|f\|_{\infty}.$$

Hence,

$$\begin{aligned} &\frac{1}{|E(0,r)|_{\nu}} \int_{E(0,r)} \left|T^{y} f(x) - f_{E(0,r)}(x)\right| y_n^{2\nu} dy \leq \\ &\leq \frac{1}{|E(0,r)|_{\nu}} \int_{E(0,r)} |T^{y} f(x)|\, y_n^{2\nu} dy + \frac{1}{|E(0,r)|_{\nu}} \int_{E(0,r)} \left|f_{E(0,r)}(x)\right| y_n^{2\nu} dy \\ &\leq \|f\|_{\infty} + \|f\|_{\infty} = 2\|f\|_{\infty}. \end{aligned}$$

Thus, $\|f\|_{B\text{-}BMO} \leq 2\|f\|_{\infty}$ is obtained by taking the supremum over all $x \in \mathbb{R}_{+}^{n}$, $r > 0$.

b) Firstly, since

$$\left|T^{y} f(x) - f_{E(0,r)}(x)\right| \leq |T^{y} f(x) - c_{x,r}| + \left|c_{x,r} - f_{E(0,r)}(x)\right|$$

and

$$\begin{aligned} \left|c_{x,r} - f_{E(0,r)}(x)\right| &= \left|c_{x,r} - \frac{1}{|E(0,r)|_{\nu}} \int_{E(0,r)} T^{y} f(x)\, y_n^{2\nu} dy\right| \\ &\leq \frac{1}{|E(0,r)|_{\nu}} \int_{E(0,r)} |T^{y} f(x) - c_{x,r}|\, y_n^{2\nu} dy \end{aligned}$$

we have

$$\frac{1}{|E(0,r)|_{\nu}} \int_{E(0,r)} \left|T^{y} f(x) - f_{E(0,r)}(x)\right| y_n^{2\nu} dy \leq \frac{2}{|E(0,r)|_{\nu}} \int_{E(0,r)} |T^{y} f(x) - c_{x,r}|\, y_n^{2\nu} dy.$$

Finally, by taking supremum over all $x \in \mathbb{R}_+^n$, $r > 0$ we have $f \in B\text{-}BMO$ and $\|f\|_{B\text{-}BMO} \leq 2A$.

c) For the property (3.2), let us show $\left(\delta^\lambda f\right)_{E(0,r)}(x) = f_{E(0,\lambda r)}(\lambda x)$. For this, we have

$$\begin{aligned} T^y\left(\delta^\lambda f\right)(x) &= \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi \left(\delta^\lambda f\right)\left(x'-y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\right) \sin^{2\nu-1}\alpha d\alpha \\ &= \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi f\left(\lambda x' - \lambda y', \sqrt{\left(\lambda x_n\right)^2 - 2\left(\lambda x_n\right)\left(\lambda y_n\right)\cos\alpha + \left(\lambda y_n\right)^2}\right) \sin^{2\nu-1}\alpha d\alpha \\ &= T^{\lambda y} f(\lambda x). \end{aligned} \tag{3.3}$$

Thus we get

$$\begin{aligned} \left(\delta^\lambda f\right)_{E(0,r)}(x) &= \frac{1}{|E(0,r)|_\nu} \int_{E(0,r)} T^y\left(\delta^\lambda f\right)(x)\, y_n^{2\nu} dy \\ &= \frac{1}{|E(0,r)|_\nu} \int_{E(0,r)} T^{\lambda y} f(\lambda x)\, y_n^{2\nu} dy \\ &= \cdots (z = \lambda y,\ dz = \lambda^n dy,\ z_n^{2\nu} = \lambda^{2\nu} y_n^{2\nu}) \cdots \\ &= \frac{1}{|E(0,r)|_\nu} \frac{1}{\lambda^{n+2\nu}} \int_{\lambda E(0,r)} T^z f(\lambda x)\, z_n^{2\nu} dz \\ &= (f)_{\lambda E(0,r)}(\lambda x) = (f)_{E(0,\lambda r)}(\lambda x). \end{aligned} \tag{3.4}$$

Hence by taking into account (3.3) and (3.4), we obtain $\left\|\delta^\lambda f\right\|_{B\text{-}BMO} = \|f\|_{B\text{-}BMO}$. □

Remark 3.4 *We indicate that $L_\infty\left(\mathbb{R}_+^n\right)$ is a proper subspace of B-BMO. As in case of the classical $BMO-$ space, we claim that the function $f(x) = \log|x|$ is in B-BMO but not in $L_\infty\left(\mathbb{R}_+^n\right)$, (cf.([25], p.141).*

Our next goal is to determine the connection between the B-BMO function and $B-$Poisson integral and present John-Nirenberg-type inequality for B-BMO functions. These results have promising applications for the classical BMO-functions, such as determining the characterization of BMO-space and the connection between BMO-functions and the Carleson measure (see [7, 9, 10, 13, 25] for details).

Theorem 3.5 *Let f be in B-BMO. Then,*

a)

$$\left|f_{E(0,r)}(x) - f_{E(0,s)}(x)\right| \leq \left(\frac{s}{r}\right)^{n+2\nu} \|f\|_{B\text{-}BMO}, \quad r \leq s. \tag{3.5}$$

b)

$$\left|f_{E(0,r)}(x) - f_{E(0,2^m r)}(x)\right| \leq 2^{n+2\nu} m \|f\|_{B\text{-}BMO}, \ m \in \mathbb{N}. \tag{3.6}$$

Proof a) For $r \leq s$, we have

$$\begin{aligned}
\left|f_{E(0,r)}(x) - f_{E(0,s)}(x)\right| &\leq \frac{1}{|E(0,r)|_\nu} \int\limits_{E(0,r)} |T^y f(x) - f_{E(0,s)}(x)| y_n^{2\nu} dy \\
&\leq \frac{|E(0,s)|_\nu}{|E(0,r)|_\nu} \frac{1}{|E(0,s)|_\nu} \int\limits_{E(0,s)} \left(T^y f(x) - f_{E(0,s)}(x)\right) y_n^{2\nu} dy \\
&\leq \left(\frac{s}{r}\right)^{n+2\nu} \|f\|_{B\text{-}BMO}.
\end{aligned}$$

b) Firstly let $m = 1$. By using (3.5) we get

$$\left|f_{E(0,r)}(x) - f_{E(0,2r)}(x)\right| \leq \left(\frac{2r}{r}\right)^{n+2\nu} \|f\|_{B\text{-}BMO} = 2^{n+2\nu} \|f\|_{B\text{-}BMO}.$$

Finally using this inequality, we obtain

$$\begin{aligned}
\left|f_{E(0,r)}(x) - f_{E(0,2^m r)}(x)\right| &\leq \left|f_{E(0,r)}(x) - f_{E(0,2r)}(x)\right| + \cdots + \left|f_{E(0,2^{m-1}r)}(x) - f_{E(0,2^m r)}(x)\right| \\
&\leq 2^{n+2\nu} m \|f\|_{B\text{-}BMO}, \quad m \in \mathbb{N}.
\end{aligned}$$

□

Theorem 3.6 *There exists a constant* $c(n,\nu) > 0$ *such that for all* $f \in B\text{-}BMO$ *we have*

$$\sup_{\substack{x \in \mathbb{R}_+^n \\ t>0}} \int\limits_{\mathbb{R}_+^n} |T^y f(x) - V_t f(x)| P_\nu(y,t) y_n^{2\nu} dy \leq c(n,\nu) \|f\|_{B\text{-}BMO}. \tag{3.7}$$

Here $P_\nu(y,t)$ *denotes the B-Poisson kernel introduced in [2] and*

$$(V_t f)(x) = \int\limits_{\mathbb{R}_+^n} T^y f(x) P_\nu(y,t) y_n^{2\nu} dy$$

the B-Poisson integral of f.

Proof We have

$$\begin{aligned}
&|T^y f(x) - V_t f(x)| P_\nu(y,t) y_n^{2\nu} dy \leq \\
&\leq \int\limits_{\mathbb{R}_+^n} \left|T^y f(x) - f_{E(0,t)}(x)\right| P_\nu(y,t) y_n^{2\nu} dy + \int\limits_{\mathbb{R}_+^n} \left|V_t f(x) - f_{E(0,t)}(x)\right| P_\nu(y,t) y_n^{2\nu} dy \\
&= I_1 + I_2.
\end{aligned} \tag{3.8}$$

Now, let us calculate the estimates I_1 and I_2. Since

$$P_\nu(y,t) = d_\nu(n) \frac{t}{(t^2 + |y|^2)^{(n+2\nu+1)/2}}$$

and

$$d_\nu(n) = \frac{(2\pi)^{1-n}\, 2^{1-\nu+n/2}\Gamma\left(n+2\nu+1/2\right)}{\sqrt{\pi}\Gamma^2\left(n+1/2\right)}, \qquad \text{(see [2] for details)}$$

we obtain

$$\begin{aligned}
I_1 &= \int\limits_{\mathbb{R}_+^n} \left|T^y f(x) - f_{E(0,t)}(x)\right| P_\nu(y,t)\, y_n^{2\nu} dy = d_\nu(n) \int\limits_{\mathbb{R}_+^n} \frac{t\left|T^y f(x) - f_{E(0,t)}(x)\right|}{\left(t^2+|y|^2\right)^{\frac{n+2\nu+1}{2}}} y_n^{2\nu} dy \\
&\leq \int\limits_{E(0,t)} \frac{t\left|T^y f(x) - f_{E(0,t)}(x)\right|}{\left(t^2+|y|^2\right)^{\frac{n+2\nu+1}{2}}} y_n^{2\nu} dy + \\
&+ \sum_{k=0}^{\infty} \int\limits_{E(0,2^{k+1}t)\setminus E(0,2^k t)} \frac{t\left(\left|T^y f(x) - f_{E(0,2^{k+1}t)}(x)\right| + \left|f_{E(0,2^{k+1}t)} - f_{E(0,t)}(x)\right|\right)}{\left(t^2+|y|^2\right)^{(n+2\nu+1)/2}} y_n^{2\nu} dy \\
&\leq \frac{1}{t^{n+2\nu}} \int\limits_{E(0,t)} \left|T^y f(x) - f_{E(0,t)}(x)\right| y_n^{2\nu} dy + \sum_{k=0}^{\infty} 2^{-k(n+2\nu+1)} \times
\end{aligned}$$

$$\left(\frac{1}{t^{n+2\nu}} \int\limits_{2^{k+1}E(0,t)} \left|T^y f(x) - f_{2^{k+1}E(0,t)}(x)\right| y_n^{2\nu} dy + \frac{1}{t^{n+2\nu}} \left|f_{2^{k+1}E(0,t)}(x) - f_{E(0,t)(x)}\right| 2^{(k+1)(n+2\nu)} \omega(n,\nu)\, t^{n+2\nu} \right)$$

$$\begin{aligned}
&\leq \omega(n,\nu)\left\|f\right\|_{B\text{-}BMO} + \omega(n,v)\left\|f\right\|_{B\text{-}BMO} \left(2^{n+2\nu} \sum_{k=0}^{\infty} \frac{1}{2^k} + 2^{2(n+2\nu)} \sum_{k=0}^{\infty} \frac{k+1}{2^k} \right) \\
&\leq c(n,\nu)\left\|f\right\|_{B\text{-}BMO}. \qquad (3.9)
\end{aligned}$$

Also, by taking into account

$$\int\limits_{\mathbb{R}_+^n} P_\nu(y,t)\, y_n^{2\nu} dy = 1, \quad \text{(see [2])}$$

we obtain

$$\begin{aligned}
I_2 &= \int\limits_{\mathbb{R}_+^n} \left|V_t f(x) - f_{E(0,t)}(x)\right| P_\nu(y,t)\, y_n^{2\nu} dy \\
&= \left|V_t f(x) - f_{E(0,t)}(x)\right| \\
&\leq \int\limits_{\mathbb{R}_+^n} \left|T^y f(x) - f_{E(0,t)}(x)\right| P_\nu(y,t)\, y_n^{2\nu} dy \\
&\leq c(n,\nu)\left\|f\right\|_{B\text{-}BMO}
\end{aligned}$$

which combined with the inequality in (3.9). Therefore, this estimate combined with (3.8) yields the desired result and finishes the proof.

The following theorem gives a sufficient condition for the inverse inequality in (3.7). □

Theorem 3.7 *There is a constant* $\widetilde{c}(n,\nu)$ *such that for all* $f\in L_1^{loc}\left(\mathbb{R}_+^n\right)$ *for which*

$$\int\limits_{\mathbb{R}_+^n}\frac{\left|T^yf\left(x\right)\right|}{\left(1+\left|y\right|^2\right)^{n+2\nu+1}}y_n^{2\nu}dy<\infty$$

we have

$$\sup_{\substack{x\in\mathbb{R}_+^n\\ t>0}}\int\limits_{\mathbb{R}_+^n}\left|T^yf\left(x\right)-V_tf\left(x\right)\right|P_\nu\left(y,t\right)y_n^{2\nu}dy\geq\widetilde{c}\left(n,\nu\right)\left\|f\right\|_{B\text{-}BMO}.$$

Proof Let

$$A=\sup_{\substack{x\in\mathbb{R}_+^n\\ t>0}}\int\limits_{\mathbb{R}_+^n}\left|T^yf\left(x\right)-V_tf\left(x\right)\right|P_\nu\left(y,t\right)y_n^{2\nu}dy.$$

For $y\in E(0,t)$, $|y|<t$ we have $\left(t^2+\left|y\right|^2\right)^{(n+2\nu+1)/2}\leq c_1\left(n,\nu\right)t^{n+2\nu+1}$ and $P_\nu\left(y,t\right)\geq c_2\left(n,\nu\right)t^{-(n+2\nu)}$ which gives

$$A\geq\frac{c_2\left(n,\nu\right)}{t^{n+2\nu}}\int\limits_{\mathbb{R}_+^n}\left|T^yf\left(x\right)-V_tf\left(x\right)\right|y_n^{2\nu}dy.$$

Taking into account (3.1) implies that

$$\left\|f\right\|_{B\text{-}BMO}\leq\frac{2A}{c_2\left(n,\nu\right)}.$$

This concludes the theorem. □

Theorem 3.8 *For all* $f\in B\text{-}BMO$, *all* $t>0$, $x\in\mathbb{R}_+^n$ *and all* $\alpha>0$ *there exist* $c_1=c_1\left(n,\nu\right)$, $c_2=c_2\left(n,\nu\right)>0$ *such that*

$$\left|\left\{x\in E(0,t):\quad\left|T^yf(x)-f_{E(0,t)}(x)\right|>\alpha\right\}\right|_\nu\leq c_1e^{-\alpha c_2/\|f\|_{B\text{-}BMO}}.\tag{3.10}$$

Proof For the sake of simplicity, we assume that $\left\|f\right\|_{B\text{-}BMO}=1$. We apply the Calderon-Zygmund decomposition to the function $T^yf-f_{E(0,t)}$ inside the $E(0,t)$. The rest part of the proof is similar to the proof of the well-known John-Nirenberg inequality for the classical BMO functions, (see [18], (cf. [13], p.124) for details). □

By using Theorem 3.8, we obtain the following important $L_{p,\nu}$ characterization of $B\text{-}BMO$.

Theorem 3.9 *Let* f *be in* $B\text{-}BMO$. *Then, for* $1\leq p<\infty$ *there exists* $c=c\left(n,\nu,p\right)$, $d=d\left(n,\nu,p\right)>0$ *such that*

$$c\left\|f\right\|_{B\text{-}BMO}\leq\sup_{\substack{x\in\mathbb{R}_+^n\\ t>0}}\left(\frac{1}{\left|E(0,t)\right|_\nu}\int\limits_{E(0,t)}\left|T^yf\left(x\right)-f_{E(0,t)}\left(x\right)\right|^py_n^{2\nu}dy\right)^{1/p}\leq d\left\|f\right\|_{B\text{-}BMO}.\tag{3.11}$$

Proof By using (3.10) and the Gamma function $\Gamma(z)=\int\limits_0^\infty t^{z-1}e^{-t}dt$ we have

$$\begin{aligned}
&\frac{1}{|E(0,t)|_\nu}\int\limits_{E(0,t)}\left|T^y f(x)-f_{E(0,t)}(x)\right|^p y_n^{2\nu}dy=\\
=&\ \frac{1}{|E(0,t)|_\nu}\int\limits_0^\infty p\alpha^{p-1}\left|\{x\in E(0,t):\left|T^y f(x)-f_{E(0,t)}(x)\right|^p>\alpha\}\right|_\nu d\alpha\\
\le&\ \frac{pc_1}{|E(0,t)|_\nu}\int\limits_0^\infty \alpha^{p-1}e^{-\alpha c_2/\|f\|_{B\text{-}BMO}}d\alpha\\
=&\ \cdots(\beta=\frac{\alpha c_2}{\|f\|_{B\text{-}BMO}},\ d\beta=\frac{c_2}{\|f\|_{B\text{-}BMO}}d\alpha)\cdots\\
=&\ c_3\|f\|^{p-1}_{B\text{-}BMO}\|f\|_{B\text{-}BMO}\int\limits_0^\infty \beta^{p-1}e^{-\beta}d\beta\\
=&\ d^p\|f\|^p_{B\text{-}BMO},\qquad d=d(n,v,p)>0. \qquad (3.12)
\end{aligned}$$

Hence,

$$\sup_{\substack{x\in\mathbb{R}^n_+\\ t>0}}\left(\frac{1}{|E(0,t)|_\nu}\int\limits_{E(0,t)}\left|T^y f(x)-f_{E(0,t)}(x)\right|^p y_n^{2\nu}dy\right)^{\frac{1}{p}}\le d\|f\|_{B\text{-}BMO}.$$

Also by taking into account the Hölder inequality $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<\infty$ we obtain

$$\begin{aligned}
\int\limits_{E(0,t)}|T^y f(x)-f_{E_t}(x)|y_n^{2\nu}dy &\le \left(\int\limits_{E(0,t)}|T^y f(x)-f_{E_t}(x)|^p y_n^{2\nu}dy\right)^{\frac{1}{p}}\left(\int\limits_{E(0,t)}y_n^{2\nu}dy\right)^{\frac{1}{q}}\\
&=\left(\int\limits_{E_t}|T^y f(x)-f_{E_t}(x)|^p y_n^{2\nu}dy\right)^{\frac{1}{p}}|E(0,t)|_\nu^{\frac{1}{q}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{|E(0,t)|_\nu}\int\limits_{E(0,t)}\left|T^y f(x)-f_{E(0,t)}(x)\right|y_n^{2\nu}dy &\le \frac{|E(0,t)|_\nu^{\frac{1}{q}}}{|E(0,t)|_\nu}\left(\int\limits_{E(0,t)}\left|T^y f(x)-f_{E(0,t)}(x)\right|^p y_n^{2\nu}dy\right)^{\frac{1}{p}}\\
&=\left(\frac{1}{|E(0,t)|_\nu}\int\limits_{E(0,t)}\left|T^y f(x)-f_{E(0,t)}(x)\right|^p y_n^{2\nu}dy\right)^{\frac{1}{p}}
\end{aligned}$$

and

$$\|f\|_{B\text{-}BMO} \leq \sup_{\substack{x\in\mathbb{R}_+^n \\ t>0}} \left(\frac{1}{|E(0,t)|_\nu} \int\limits_{E(0,t)} \left| T^y f(x) - f_{E(0,t)}(x) \right|^p y_n^{2\nu} dy \right)^{\frac{1}{p}}, \ 1 \leq p < \infty. \tag{3.13}$$

Finally, combining (3.12) and (3.13) yields that (3.11). □

Corollary 3.10 *For all* $1 \leq p < \infty$ *we have*

$$\|f\|_{B\text{-}BMO} \approx \sup_{\substack{x\in\mathbb{R}_+^n \\ t>0}} \left(\frac{1}{|E(0,t)|_\nu} \int\limits_{E(0,t)} \left| T^y f(x) - f_{E(0,t)}(x) \right|^p y_n^{2\nu} dy \right)^{\frac{1}{p}}.$$

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