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Ulam's type stability analysis of fractional difference equation with impulse: Gronwall inequality approach

Rabia Ilyas BUTT*, Mujeeb ur REHMAN
Department of Mathematics, School of Natural Sciences,
National University of Sciences and Technology, H-12, Islamabad, Pakistan

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Abstract: In this paper, we present a new Gronwall inequality with an impulsive effect. The existence and uniqueness of the solution is investigated through fixed point theorems. Moreover, with the help of newly developed inequality Ulam's type stability criterion is developed for impulsive fractional difference equation. At last, a model is given to help the hypothetical outcome.

Key words: Gronwall inequality, fractional difference equation, impulsive effect, existence and uniqueness, fixed point theorems; Ulam-Hyers-Rassias stability

1. Introduction

The wide-ranging applications of continuous fractional differential equations in recent years are due to their emphasis in many disciplines of science, engineering, and mathematics. Many properties of memory and hereditary are well illustrated by fractional order derivatives [16]. Fractional calculus is an excellent tool for real-life models e.g., signal processing, control engineering, fluid mechanics, and diffusion processes [24, 26, 27]. Nowadays, the study of discrete fractional calculus has become part of great interest due to its consolidation in several scientific areas. Discretized real-life models are new concepts introduced by discrete fractional calculus. Discrete fractional calculus is more applicable for studying population growth, image processing, and tumor growth [10, 19] etc. Having no truncation errors is the main feature of discrete fractional calculus while functions are defined on a discrete set.

Fractional difference equations is an emerging research topic in fractional calculus. Stability analysis is one of the most significant and well-developed topics in continuous fractional calculus but is rather rare in the discrete case. In order to discuss stability of fractional differential/difference equations, various research articles have been appeared [12–14, 20, 21, 25, 30] but still this direction is an open dilemma for researchers. In the most recent articles, Koksal [23] has investigated stability analysis of fractional differential equations with unknown parameters. Ameen et al. [7] have investigated Ulam stability for delay fractional differential equations with a generalized Caputo derivative. Khan et al. [22] discussed the existence-uniqueness of a solution and the Hyers-Ulam stability.

Researchers are using different approaches to check the stability of fractional differential as well as difference equations. Gronwall inequality is one of the most widely used approaches to check stability. Gronwall inequality

*Correspondence: rabi_butt@hotmail.com, rabia.ilyas@sns.nust.edu.pk

2010 *AMS Mathematics Subject Classification:* 39A30 is discussed by many authors in continuous as well as discrete fractional calculus [1, 6, 9, 11, 17, 32]. Wang et al. [28, 29] used Gronwall inequality to analyze Ulam's type stability of impulsive ordinary differential equations as well as a fractional differential equation. Ali et al. [5] used fixed point theorem to derive the condition for the existence of solution for impulsive fractional differential equation and discussed Ulam type stability. Wu et al. [31] considered the impulsive fractional difference equation and discussed its stability. Ding [15] discussed Ulam-Hyers stability of fractional impulsive differential equations using picard operator and Gronwall inequality. Asma et al. [8] discussed about Ulam-Hyers stability of impulsive fractional differential equation with three-point boundary conditions.

Motivated by all above-mentioned articles, in this paper, we develop a new fractional difference inequality of Gronwall type with impulsive effect which is further used to obtain stability criteria. To the best of our knowledge, this inequality has not been previously produced in the literature. We consider a fractional difference equation and investigate Ulam's type stability with impulsive effects. The problem under consideration is as follows:

$$\begin{cases} {}^c\Delta_a^\zeta u(t) = \mathscr{H}(t+\zeta, u(t+\zeta)),\ t \in \mathbb{N}_{a+1-\zeta},\ t \neq a+n_l+1-\zeta,\ 0<\zeta\leq 1, \\ u_{n_l+1} = \overline{u}_{n_l+1} + I_l(\overline{u}_{n_l+1}),\ t = a+n_l+1-\zeta,\ l \in \mathbb{N}_1, \\ u(a) = u_0, \end{cases} \tag{1.1}$$

where ${}^c\Delta_a^\zeta$ is delta Caputo fractional difference with $0<\zeta\leq 1$. Furthermore, function $\mathscr{H}$ is defined on $\mathbb{N}_a \times \mathbb{R}$ and $I_l : \mathbb{R} \to \mathbb{R}$.

This paper is organized in following manner: In Section 2, we list some basic definitions and lemmas which are necessary for proving main results. In Section 3, we provide the main results of the article including Gronwall inequality and existence-uniqueness of solution of above-mentioned problem using Banach contraction principle, Schaefer's fixed point theorem and Leray Schauder type theorem. Further in Section 4, newly developed Gronwall inequality is used to obtain stability criteria for fractional difference equation with impulse effects. To demonstrate the theoretical results, an example is also provided in Section 5.

2. Preliminaries

In the following section, we present some basic definitions and properties for the purpose of acquainting with basic fractional calculus theory. Furthermore, some necessary lemmas are also provided that are helpul in proving our main results. One can follow ([2–4, 18]) and references cited in.

Definition 2.1 *[18] For $0<\zeta$, the ζ-th order fractional sum of u defined on $\mathbb{N}_a$ is given by*

$$\Delta_a^{-\zeta} u(t) := \frac{1}{\Gamma(\zeta)} \sum_{s=a}^{t-\zeta} (t-\sigma(s))^{\underline{\zeta-1}} u(s),$$

where $\sigma(r) = r+1,\ t \in \mathbb{N}_{a+\zeta}$, and falling function $r^{\underline{\zeta}}$ is given by

$$r^{\underline{\zeta}} = \frac{\Gamma(r+1)}{\Gamma(r+1-\zeta)}.$$

Definition 2.2 *[18] Let u be defined on $\mathbb{N}_a$ and $0<\zeta$. Choose positive integer N such that $N-1<\zeta\leq N$ then the Riemann-Liouville delta fractional difference is defined as*

$$\Delta_a^\zeta u(t) := \Delta^N \Delta^{(-N-\zeta)} u(t),\ t \in \mathbb{N}_{a+N-\zeta}.$$

For $0 < \zeta \leq 1$*, we have following*

$$\Delta_a^{\zeta} u(t) := \frac{1}{\Gamma(-\zeta)} \sum_{s=a}^{t+\zeta} (t-\sigma(s))^{\underline{-\zeta-1}} u(s),\ t \in \mathbb{N}_{a+1-\zeta}.$$

Definition 2.3 *[4] For* $0 < \zeta,\ \zeta \notin \mathbb{N}$*, the Caputo difference of* u *defined on* $\mathbb{N}_a$ *is given by*

$${}^{c}\Delta_a^{\zeta} u(t) = \Delta^{-(m-\zeta)}\ \Delta^m u(t) = \frac{1}{\Gamma(m-\zeta)} \sum_{s=a}^{t-m+\zeta} (t-\sigma(s))^{\underline{m-\zeta-1}} \Delta^m u(s),$$

where $\Delta u(t) = u(t+1) - u(t)$ *and* $t \in \mathbb{N}_{a+m-\zeta},\ m = [\zeta] + 1$.
For $\zeta = m,\ {}^{c}\Delta_a^{\zeta} u(t) := \Delta^m u(t)$.

Definition 2.4 *[4] For* $0 < \zeta \leq 1$*, Discrete Leibniz integral law is defined as:*

$$\Delta_{a+1-\zeta}^{-\zeta}\ {}^{c}\Delta_a^{\zeta} u(t) = u(t) - u(a),\ t \in \mathbb{N}_{a+1}.$$

Lemma 2.5 *[31] Let* u *defined on* $\mathbb{N}_a$*, be a solution of fractional sum equation*

$$\begin{aligned} u(t) =& u(t^*) - \frac{1}{\Gamma(\zeta)} \sum_{s=a+1-\zeta}^{t^*-\zeta} (t^*-\sigma(s))^{\underline{\zeta-1}} \mathscr{H}(s+\zeta, u(s+\zeta)) \\ &+ \frac{1}{\Gamma(\zeta)} \sum_{s=a+1-\zeta}^{t-\zeta} (t-\sigma(s))^{\underline{\zeta-1}} \mathscr{H}(s+\zeta, u(s+\zeta)),\ t \in \mathbb{N}_{a+1}. \end{aligned}$$

if and only if u *is a solution of the following cauchy problem*

$${}^{c}\Delta_a^{\zeta} u(t) = \mathscr{H}(t+\zeta, u(t+\zeta)),\ t \in \mathbb{N}_{a+1-\zeta},\ 0 < \zeta \leq 1,$$

subject to

$$u(a) = u(t^*) - \frac{1}{\Gamma(\zeta)} \sum_{s=a+1-\zeta}^{t^*-\zeta} (t^*-\sigma(s))^{\underline{\zeta-1}} \mathscr{H}(s+\zeta, u(s+\zeta)).$$

Lemma 2.6 *(Banach contraction principle) Let* $\mathscr{A}$ *be a nonempty complete metric space with a contraction mapping* $\mathcal{T} : \mathscr{A} \to \mathscr{A}$*. Then* $\mathcal{T}$ *has at least one fixed point in* $\mathscr{A}$*. That is, there exists* $u \in \mathscr{A}$ *such that* $\mathcal{T}u = u$*.*

Lemma 2.7 *(Schaefer's fixed point theorem) Let* $\mathcal{E}$ *be a linear normed space and* $\psi : \mathcal{E} \to \mathcal{E}$ *be a compact operator. Suppose that the set*

$$\mathcal{S} = \{u \in \mathcal{E} \mid u = \lambda\psi u,\ 0 < \lambda < 1\},$$

is bounded. Then ψ *has a fixed point in* $\mathcal{E}$*.*

Lemma 2.8 *(Nonlinear Alternative Leray Schauder Theorem) Let* $\mathcal{S}$ *be a Banach space,* $\mathcal{D}$ *be a closed and convex subset of* $\mathcal{S}$*,* $\mathcal{V}$ *be an open subset of* $\mathcal{D}$ *and* $0 \in \mathcal{V}$*. Suppose that the operator* $\mathcal{T} : \overline{\mathcal{V}} \to \mathcal{D}$ *is a continuous and compact map. Then, either* *(i)* $\mathcal{T}$ *has a fixed point* $u^* \in \overline{\mathcal{V}}$, *or*

(ii) there is $u \in \partial\mathcal{V}$ *and* $\delta > 1$ *such that* $\delta u = \mathcal{T}(u)$.

Lemma 2.9 *[30] Assume that the discrete functions* $\mathscr{F}$ *and* $\mathscr{G}$ *are nonnegative and nondecreasing. Furthermore, there exists a positive constant* M *such that* $\mathscr{G}(t) \leq M$, *where* $t \in \mathbb{N}_a$. *If we have following sum inequality*

$$u(t) \leq \mathscr{F}(t) + \mathscr{G}(t)\Delta_{a+1-\zeta}^{-\zeta} u(t+\zeta),$$

where $t \in \mathbb{N}_{a+1}$, *then*

$$u(t) \leq \mathscr{F}(t) e_\zeta(\mathscr{G}(t), t)$$

where $e_\zeta(\mathscr{G}(t), t)$ *is discrete Mittag-Leffler function.*

Lemma 2.10 *[31] Let* u *be defined on* $\mathbb{N}_a$ *and suppose that the following inequality holds,*

$$u(t) \leq \mathscr{F}(t) + \mathscr{G}(t)\Delta_{a+1-\zeta}^{-\zeta} u(t+\zeta),\ t \in \mathbb{N}_a,\ 0 < \zeta \leq 1,$$

$$u_{n_l+1} = \overline{u}_{n_l+1} + q_l \overline{u}_{n_l+1}, -1 < q_l < 0,\ t = a + n_l + 1 - \zeta,\ l \in \mathbb{N}_1,\ u(a) = u_0.$$

then

$$u(t) \leq \mathscr{F}(t) e_\zeta^*(\mathscr{G}(t), t-a),\ t \in \{a + n_l + 1, ..., a + n_{l+1}\},\ l = 1, 2, ..., N, ...,$$

where discrete functions $\mathscr{F}$ *and* $\mathscr{G}$ *are nonnegative and nondecreasing. Furthermore, there exists a positive constant* M *such that* $\mathscr{G}(t) \leq M$, *where* $t \in \mathbb{N}_a$.

Now to investigate Ulam's type stability concepts for fractional difference equations having impulsive effects, we consider the following inequalities with a positive real number ϵ and a continuous function φ defined on $\mathbb{N}_a$.

$$\begin{cases} |{}^c\Delta_a^\zeta v(t) - \mathscr{H}(t+\zeta, v(t+\zeta))| & \leq \epsilon,\ t \in \mathbb{N}_{a+1-\zeta},\ t \neq a + n_l + 1 - \zeta. \\ |v_{n_l+1} - \overline{v}_{n_l+1} - I_l(\overline{v}_{n_l+1})| & \leq \epsilon,\ t = a + n_l + 1 - \zeta,\ l \in \mathbb{N}_1. \end{cases} \tag{2.1}$$

$$\begin{cases} |{}^c\Delta_a^\zeta v(t) - \mathscr{H}(t+\zeta, v(t+\zeta))| & \leq \varphi(t),\ t \in \mathbb{N}_{a+1-\zeta},\ t \neq a + n_l + 1 - \zeta. \\ |v_{n_l+1} - \overline{v}_{n_l+1} - I_l(\overline{v}_{n_l+1})| & \leq \varphi(t),\ t = a + n_l + 1 - \zeta,\ l \in \mathbb{N}_1. \end{cases} \tag{2.2}$$

$$\begin{cases} |{}^c\Delta_a^\zeta v(t) - \mathscr{H}(t+\zeta, v(t+\zeta))| & \leq \epsilon\varphi(t),\ t \in \mathbb{N}_{a+1-\zeta}, t \neq a + n_l + 1 - \zeta. \\ |v_{n_l+1} - \overline{v}_{n_l+1} - I_l(\overline{v}_{n_l+1})| & \leq \epsilon\varphi(t),\ t = a + n_l + 1 - \zeta,\ l \in \mathbb{N}_1. \end{cases} \tag{2.3}$$

Definition 2.11 *Let* u *and* v *defined on* $\mathbb{N}_a$ *being solutions of problem* (1.1) *and inequality* (2.1) *respectively. For each* $\epsilon > 0$ *and real number* $c > 0$, *if*

$$|v(t) - u(t)| \leq c\epsilon,$$

then problem (1.1) *is Ulam-Hyers stable.*

Definition 2.12 *Let u and v defined on $\mathbb{N}_a$ being solutions of problem (1.1) and inequality (2.1) respectively. If there exists θ defined on $\mathbb{N}_a$ such that $\theta(0)=0$, then problem (1.1) is called generalized Ulam-Hyers stable, provided*

$$|v(t)-u(t)| \leq \theta(\epsilon).$$

Definition 2.13 *Let u and v defined on $\mathbb{N}_a$ being solutions of problem (1.1) and inequality (2.3) respectively. If there exists $\eta>0$ such that for each $\epsilon>0$, problem (1.1) is Ulam-Hyers-Rassias stable with respect to φ, if following holds*

$$|v(t)-u(t)| \leq \eta\epsilon\varphi(t).$$

Definition 2.14 *Let u and v defined on $\mathbb{N}_a$ being solutions of problem (1.1) and inequality (2.2) respectively. If there exists $\eta>0$ such that*

$$|v(t)-u(t)| \leq \eta\varphi(t),$$

then problem (1.1) is called generalized Ulam-Hyers-Rassias stable with respect to φ.

3. Main results

In this section, we present our main results. First, we start by proving the existence and uniqueness of the solution to problem (1.1). Then we develop a new delta discrete fractional inequality of Gronwall type to check Ulam's type stability.

3.1. Existence and uniqueness

Theorem 3.1 *A function u defined on $\mathbb{N}_a$ is a solution of fractional difference equation with impulse condition*

$$ {}^{c}\Delta_a^{\zeta}u(t) = \mathscr{H}(t+\zeta, u(t+\zeta)),\ t\in\mathbb{N}_{a+1-\zeta},\ 0<\zeta\leq 1,\ t\neq a+n_l+1-\zeta,$$
$$u_{n_l+1} = \overline{u}_{n_l+1} + I_l(\overline{u}_{n_l+1}),\ t=a+n_l+1-\zeta,\ l\in\mathbb{N}_1,\ u(a)=u_0.$$

iff u is solution of summation equation

$$u(t)=\begin{cases} u_0+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)),\ t\in\{a+n_0+1,...,a+n_1\}, \\ \vdots \\ u_0+\sum_{i=1}^{l} I_i(\overline{u}_{n_i+1})+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)),\ t\in\{a+n_l+1,...,a+n_{l+1}\}, \\ \vdots \\ u_0+\sum_{i=1}^{N} I_i(\overline{u}_{n_i+1})+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)),\ t\in\{a+n_N+1,...\}, \end{cases}$$

where $l=1,...,N-1$ and $N\to\infty$ and $I_i:\mathbb{R}\to\mathbb{R}$.

Proof For $t\in\{a+n_0+1,...,a+n_1\}$, we get

$$u(t)=u_0+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)),$$

$$\overline{u}_{n_1+1}=u(a+n_1+1)=u_0+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta))|_{t=a+n_1+1}. \tag{3.1}$$

For $t \in \{a+n_1+1,...,a+n_2\}$, we use Lemma 2.5 and Eq. (3.1) to obtain the following

$$\begin{aligned} u(t) &= u_{n_1+1} - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta))|_{t=a+n_1+1} + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)) \\ &= \overline{u}_{n_1+1} + I_1(\overline{u}_{n_1+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta))|_{t=a+n_1+1} + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)) \\ &= u_0 + I_1(\overline{u}_{n_1+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)). \end{aligned}$$

Let us suppose, for $t \in \{a+n_k+1,...,a+n_{k+1}\}$, following holds

$$u(t) = u_0 + \sum_{i=1}^{k} I_i(\overline{u}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)).$$

For $t \in \{a+n_{k+1}+1,...,a+n_{k+2}\}$,we have

$$\begin{aligned} u(t) &= u_{n_{k+1}+1} - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta))|_{t=a+n_{k+1}+1} + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)) \\ &= \overline{u}_{n_{k+1}+1} + I_{k+1}(\overline{u}_{n_{k+1}+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta))|_{t=a+n_{k+1}+1} + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)) \\ &= u_0 + I_{k+1}(\overline{u}_{n_{k+1}+1}) + \sum_{i=1}^{k} I_i(\overline{u}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)) \\ &= u_0 + \sum_{i=1}^{k+1} I_i(\overline{u}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)). \end{aligned}$$

Hence by mathematical induction, the proof is complete. □

Now we will prove the existence and uniqueness of the solution using fixed point theorems. Let us consider a Banach space of all continuous functions $u : \mathbb{N}_a \to \mathbb{R}$ with norm defined as $\|u\| = \sup_{t\in\mathbb{N}_a} |u(t)|$. First of all we will use Banach fixed point theorem.

Theorem 3.2 *Let us assume that:*
(A_1) *there exists a constant* $\mathcal{L} > 0$ *such that*

$$|\mathscr{H}(t,u) - \mathscr{H}(t,v)| \leq \mathcal{L}|u-v|,$$

(A_2) *there exists a constant* $\mathcal{L}^* > 0$ *such that*

$$|I_k(u) - I_k(v)| \leq \mathcal{L}^*|u-v|, \qquad k = 1,2,...N-1.$$

If

$$\left((N-1)\mathcal{L}^* + \mathcal{L}\frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\right) < 1, \tag{3.2}$$

then problem (1.1) *has a unique solution.*

Proof Consider the operator $\mathcal{T} : C(\mathbb{N}_a,\mathbb{R}) \to C(\mathbb{N}_a,\mathbb{R})$ defined by

$$\mathcal{T}(u)(t) = u_0 + \sum_{i=1}^{k} I_i(\overline{u}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)).$$

Clearly, fixed points of the operator $\mathcal{T}$ are solutions of problem (1.1). We will use Banach fixed point theorem to prove that $\mathcal{T}$ has unique fixed point. We shall show that $\mathscr{H}$ is a contraction. For u, v defined on $\mathbb{N}_a$, we have

$$
\begin{aligned}
&|\mathcal{T}(u)(t) - \mathcal{T}(v)(t)| \\
&\quad \leq \sum_{i=1}^{k} |I_i(\overline{u}_{n_i+1}) - I_i(\overline{v}_{n_i+1})| + \Delta_{a+1-\zeta}^{-\zeta} |\mathscr{H}(t+\zeta, u(t+\zeta) - \mathscr{H}(t+\zeta, v(t+\zeta))| \\
&\quad \leq \sum_{i=1}^{k} \mathcal{L}^* |\overline{u}_{n_i+1} - \overline{v}_{n_i+1}| + \mathcal{L}\Delta_{a+1-\zeta}^{-\zeta} |u(t+\zeta) - v(t+\zeta)| \\
&\quad \leq (N-1)\mathcal{L}^* \|u-v\| + \mathcal{L}\frac{(t-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)} \|u-v\| \\
&\quad \leq \left((N-1)\mathcal{L}^* + \mathcal{L}\frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)} \right) \|u-v\|, \qquad\qquad t < T.
\end{aligned}
$$

Therefore,

$$
\|\mathcal{T}(u) - \mathcal{T}(v)\| \leq \left((N-1)\mathcal{L}^* + \mathcal{L}\frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)} \right) \|u-v\|.
$$

Using inequality (3.2), we conclude that $\mathcal{T}$ is contraction. Hence the Banach contraction principle guarantee that the operator $\mathcal{T}$ has a unique fixed point which is in turn solution of problem (1.1).

□

In what follows, we develop a sufficient condition for the existence of at least one solution.

Theorem 3.3 *Assume that:*

(A_3) *the function* $\mathscr{H} : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$ *is continuous,*

(A_4) *there exists a constant* $\mathcal{M} > 0$ *such that* $|\mathscr{H}(t,u)| \leq \mathcal{M}$,

(A_5) *the functions* $I_k : \mathbb{R} \to \mathbb{R}$ *are continuous and there exists a constant* $\mathcal{M}^* > 0$ *such that* $|I_k(u)| \leq \mathcal{M}^*, \quad k = 1, 2, ...N-1$.

Then problem (1.1) *has at least one solution.*

Proof The conditions are developed so that the assumptions of Schaefer's fixed point theorem are fulfilled. The proof is given in several steps.
Step 1: $\mathcal{T}$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$.

$$
\begin{aligned}
&|\mathcal{T}(u_n)(t) - \mathcal{T}(u)(t)| \\
&\quad \leq \sum_{i=1}^{k} |I_i(\overline{u}_{n_{n_i+1}}) - I_i(\overline{u}_{n_i+1})| + \Delta_{a+1-\zeta}^{-\zeta} |\mathscr{H}(t+\zeta, u_n(t+\zeta) - \mathscr{H}(t+\zeta, u(t+\zeta)))|.
\end{aligned}
$$

Since $\mathscr{H}$ and I_k are continuous functions, we have

$$
\|\mathcal{T}(u_n) - \mathcal{T}(u)\| \to 0 \text{ as } n \to \infty.
$$

Step 2: $\mathcal{T}$ maps bounded sets into bounded sets. Indeed, it is enough to show that for any $\mu > 0$, there exists a positive constant l such that for each $u \in B_\mu = \{u \in C(\mathbb{N}_a, \mathbb{R}) : \|u\| \leq \mu\}$, we have $\|\mathcal{T}(u)\| \leq l$. Now using assumptions (A_4) and (A_5), we have

$$|\mathcal{T}(u)(t)| \leq |u_0| + \sum_{i=1}^{k} |I_i(\overline{u}_{n_i+1})| + \Delta_{a+1-\zeta}^{-\zeta} |\mathscr{H}(t+\zeta, u(t+\zeta))|$$

$$\leq |u_0| + (N-1)\mathcal{M}^* + \frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\mathcal{M} \leq l.$$

Thus

$$\|\mathcal{T}(u)\| \leq l.$$

Step 3: $\mathcal{T}$ maps bounded sets into equicontinuous sets. Let $t_1, t_2 \in \mathbb{N}_a$, $t_1 < t_2$, B_μ is bounded as mentioned in Step 2, and let $u \in B_\mu$. Then

$$\begin{aligned}
&|\mathcal{T}(u)(t_2) - \mathcal{T}(u)(t_1)| \\
&\leq \sum_{0<t_k<t_2-t_1} |I_i(\overline{u}_{n_i+1})| \\
&+ \frac{1}{\Gamma(\zeta)} \int_{a+1-\zeta}^{t_1-\zeta+1} \left|(t_2-\sigma(\tau))^{\underline{\zeta-1}} - (t_1-\sigma(\tau))^{\underline{\zeta-1}}\right| \left|\mathscr{H}(\tau+\zeta, u(\tau+\zeta))\right| \Delta\tau \\
&+ \frac{1}{\Gamma(\zeta)} \int_{t_1+1-\zeta}^{t_2-\zeta+1} \left|(t_2-\sigma(\tau))^{\underline{\zeta-1}}\right| \left|\mathscr{H}(\tau+\zeta, u(\tau+\zeta))\right| \Delta\tau \\
&\leq \sum_{0<t_k<t_2-t_1} |I_i(\overline{u}_{n_i+1})| + \frac{\mathcal{M}}{\Gamma(\zeta)} \int_{a+1-\zeta}^{t_1-\zeta+1} \left|(t_2-\sigma(\tau))^{\underline{\zeta-1}} - (t_1-\sigma(\tau))^{\underline{\zeta-1}}\right| \Delta\tau \\
&+ \frac{\mathcal{M}}{\Gamma(\zeta)} \int_{t_1+1-\zeta}^{t_2-\zeta+1} \left|(t_2-\sigma(\tau))^{\underline{\zeta-1}}\right| \Delta\tau.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $\mathcal{T}$ is completely continuous.
Step 4: Now it remains to show that the set

$$\mathcal{E} = \{u \in C(\mathbb{N}_a, \mathbb{R}) : u = \lambda\mathcal{T}(u) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $u \in \mathcal{E}$, then $u = \lambda\mathcal{T}(u)$ for some $0 < \lambda < 1$. Thus, we have

$$u(t) = \lambda u_0 + \lambda \sum_{i=1}^{k} I_i(\overline{u}_{n_i+1}) + \lambda \Delta_{a+1-\zeta}^{-\zeta} \mathscr{H}(t+\zeta, u(t+\zeta)).$$

Now using the assumptions (A_4) and (A_5) as in Step 2, we get

$$|u(t)| \leq |u_0| + (N-1)\mathcal{M}^* + \frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\mathcal{M}.$$

Therefore

$$\|u\| \leq |u_0| + (N-1)\mathcal{M}^* + \frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\mathcal{M} := \mathcal{R}.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $\mathcal{T}$ has a fixed point which is a solution of the problem (1.1). $\square$

In the following theorem, we give an existence result for the problem (1.1) by applying the nonlinear alternative of the Leray-Schauder type theorem.

Theorem 3.4 *Suppose that assumption (A_2) with the following assumptions hold:*

(A_6) *there exist continuous and nondecreasing functions $\varphi : \mathbb{N}_a \to \mathbb{R}^+$ and $\psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ such that*

$$|\mathscr{H}(t,u)| \leq \varphi(t)\psi(|u|),$$

(A_7) *there exist continuous and nondecreasing functions $\varphi^* : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ such that*

$$|I_k(u)| \leq \varphi^*(|u|),$$

(A_8) *there exists $\overline{M} > 0$ such that*

$$\frac{\overline{M}}{|u_0| + (N-1)\varphi^*(\overline{M}) + \overline{\varphi}\ \psi(\overline{M})\dfrac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}} > 1,$$

where $\overline{\varphi} = \sup\{\varphi(t) : t \in \mathbb{N}_a\}$.

Then problem (1.1) has at least one solution.

Proof Consider the operator $\mathcal{T}$ as defined in previous theorems. We can see easily that the operator $\mathcal{T}$ is continuous. For each $t \in \mathbb{N}_a$ and $\lambda \in [0,1]$, we have $u(t) = \lambda(\mathcal{T}u)(t)$. Then from $(A_6)-(A_7)$ we have

$$\begin{aligned}
|u(t)| &\leq |u_0| + \sum_{i=1}^{k} |I_i(\overline{u}_{n_i+1})| + \Delta_{a+1-\zeta}^{-\zeta}|\mathscr{H}(t+\zeta, u(t+\zeta))| \\
&\leq |u_0| + \sum_{i=1}^{k} \varphi^*(|\overline{u}_{n_i+1}|) + \Delta_{a+1-\zeta}^{-\zeta}\varphi(t+\zeta)\psi(|u(t+\zeta)|) \\
&\leq |u_0| + (N-1)\varphi^*(\|u\|) + \overline{\varphi}\ \psi(\|u\|)\frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}.
\end{aligned}$$

Thus

$$\frac{\|u\|}{|u_0| + (N-1)\varphi^*(\|u\|) + \overline{\varphi}\ \psi(\|u\|)\dfrac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}} \leq 1.$$

By assumption (A_8), there exists $\overline{M}$ such that $\|u\| \neq \overline{M}$. Let

$$\mathcal{V} = \left\{u \in C(\mathbb{N}_a, \mathbb{R}) : \|u\| < \overline{M}\right\}.$$

The operator $\mathcal{T} : \overline{\mathcal{V}} \to C(\mathbb{N}_a, \mathbb{R})$ is continuous and completely continuous. For some $\lambda \in (0,1)$ there is no $u \in \partial\mathcal{V}$ such that $u = \lambda\mathcal{T}(u)$. Hence using the nonlinear alternative of Leray-Schauder type, we can say that $\mathcal{T}$ has a fixed point u in $\overline{\mathcal{V}}$ which is a solution of the problem (1.1). This completes the proof. $\square$

3.2. Gronwall inequality

Theorem 3.5 *Let u be defined on $\mathbb{N}_a$ satisfying the inequality given below*

$$|u(t)| \leq \mathfrak{d}_1(t) + \mathfrak{d}_2 \Delta_{a+1-\zeta}^{-\zeta} u(t+\zeta) + \sum_{0<t_l<t} \theta_l \left|\overline{u}_{n_l+1}\right|, \tag{3.3}$$

where $\mathfrak{d}_1(t)$ is nonnegative discrete and nondecreasing on $\mathbb{N}_a$ and $\mathfrak{d}_2,\ \theta_l \geq 0$ are constants. Then

$$|u(t)| \leq \mathfrak{d}_1(t) \left(1 + \theta e_\zeta(\mathfrak{d}_2, t-a)\right)^l e_\zeta(\mathfrak{d}_2, t-a), \tag{3.4}$$

for $t \in \{a + n_l + 1, ..., a + n_{l+1}\}$, where $l = 1, 2, ..., N-1$ and $\theta = max\{\theta_l : l = 1, ...N-1\}$.

Proof We will prove the Gronwall inequality in (3.4), using mathematical induction. For $t \in \{a + n_0 + 1, ..., a + n_1\}$, using Lemma 2.9 we derive,

$$|u(t)| \leq \mathfrak{d}_1(t) e_\zeta(\mathfrak{d}_2, t-a). \tag{3.5}$$

For $t \in \{a + n_l + 1, ..., a + n_{l+1}\}$, $l = 1, 2, ..., N-1$,

$$|u(t)| \leq \left(\mathfrak{d}_1(t) + \sum_{i=1}^{l} \theta_i |\overline{u}_{n_i+1}| \right) e_\zeta(\mathfrak{d}_2, t-a). \tag{3.6}$$

For $l = 0$, inequality (3.4) becomes inequality (3.5).
Let us suppose that inequality (3.4) holds for l, where $l = 1, 2, ..., N-1$. Then by inequality (3.6) and since $\mathfrak{d}_1, e_\zeta(\mathfrak{d}_2, t-a)$ are nondecreasing for $t \in \{a + n_l + 1 + 1, ..., a + n_{l+2}\}$, we have following estimate for u,

$$\begin{aligned}
|u(t)| &\leq \left(\mathfrak{d}_1(t) + \sum_{i=1}^{l+1} \theta_i \left|\overline{u}_{n_i+1}\right| \right) e_\zeta(\mathfrak{d}_2, t-a) \\
&\leq \left(\mathfrak{d}_1(t) + \sum_{i=1}^{l+1} \theta_i \mathfrak{d}_1(a + n_i + 1) \left(1 + \theta e_\zeta(\mathfrak{d}_2, n_i + 1)\right)^{i-1} e_\zeta(\mathfrak{d}_2, n_i + 1) \right) e_\zeta(\mathfrak{d}_2, t-a) \\
&\leq \left(\mathfrak{d}_1(t) + \theta \sum_{i=1}^{l+1} \mathfrak{d}_1(t) (1 + \theta e_\zeta(\mathfrak{d}_2, t-a))^{i-1} e_\zeta(\mathfrak{d}_2, t-a) \right) e_\zeta(\mathfrak{d}_2, t-a) \\
&\leq \mathfrak{d}_1(t) \left(1 + \theta e_\zeta(\mathfrak{d}_2, t-a)\right)^{l+1} e_\zeta(\mathfrak{d}_2, t-a).
\end{aligned}$$

Hence completing the proof. □

4. Stability analysis

Remark 4.1 *Solution of inequality (2.1) is a discrete function v defined on $\mathbb{N}_a$ iff there exists a discrete function $\mathscr{G}$ defined on $\mathbb{N}_a$ and a sequence $\mathscr{G}_l$ satisfying*

(i) $|\mathscr{G}(t)| \leq \epsilon,\ t \in \mathbb{N}_a$ *and* $|\mathscr{G}_l| \leq \epsilon,\ l = 1, 2, ..., N-1$,

(ii) ${}^c\Delta_a^\zeta v(t) = \mathscr{H}(t+\zeta, v(t+\zeta)) + \mathscr{G}(t),\ 0 < \zeta \leq 1,\ t \in \mathbb{N}_{a+1-\zeta},\ t \neq a + n_l + 1 - \zeta$,

(iii) $v_{n_l+1} = \overline{v}_{n_l+1} + I_l(\overline{v}_{n_l+1}) + \mathscr{G}_l,\ t = a + n_l + 1 - \zeta,\ l = 1, 2, ..., N-1$.

Remark 4.2 *As v is a solution of inequality (2.1), so the following inequality must be satisfied by v*

$$|v(t)-v(a)-\sum_{i=1}^{l} I_i(\overline{v}_{n_i+1})-\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,v(t+\zeta))| \leq \left(N-1+\frac{(t-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\right)\epsilon.$$

By previous Remark, we have

$${}^{c}\Delta_{a}^{\zeta}v(t)=\mathscr{H}(t+\zeta,v(t+\zeta))+\mathscr{G}(t),\ 0<\zeta\leq 1,\ t\in\mathbb{N}_{a+1-\zeta},\ t\neq a+n_l+1-\zeta,$$

$$v_{n_l+1}=\overline{v}_{n_l+1}+I_l(\overline{v}_{n_l+1})+\mathscr{G}_l,\ t=a+n_l+1-\zeta,\ l=1,2,...,N-1.$$

Then for $t\in\{a+n_l+1,...,a+n_{l+1}\}$, where $l=1,2,...,N-1$,

$$v(t)=v(a)+\sum_{i=1}^{l} I_i(\overline{v}_{n_i+1})+\sum_{i=1}^{l}\mathscr{G}_i+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,v(t+\zeta))+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{G}(t),$$

$$\begin{aligned}|v(t)-v(a)-\sum_{i=1}^{l} I_i(\overline{v}_{n_i+1})-\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,v(t+\zeta))| &= |\sum_{i=1}^{l}\mathscr{G}_i+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{G}(t)|\\ \leq \sum_{i=1}^{l}|\mathscr{G}_i|+\Delta_{a+1-\zeta}^{-\zeta}|\mathscr{G}(t)| \leq (N-1)\epsilon+\Delta_{a+1-\zeta}^{-\zeta}\epsilon &= \left(N-1+\frac{(t-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\right)\epsilon.\end{aligned}$$

Theorem 4.3 *Assume $\mathscr{H}$ defined on $\mathbb{N}_a$ satisfies Lipschitz condition with respect to second variable i.e $|\mathscr{H}(t,p)-\mathscr{H}(t,q)|\leq L_{\mathscr{H}}|p-q|$ for all $p,q\in\mathbb{R}$ where $L_{\mathscr{H}}>0$ is Lipschitz constant. Moreover, there exists a constant $\rho_k>0$ and a function $I_k:\mathbb{R}\to\mathbb{R}$ such that $|I_k(p)-I_k(q)|\leq\rho_k|p-q|$ for all $p,q\in\mathbb{R}$. Furthermore, if for $\lambda_\varphi>0$ we have $\Delta_{a+1-\zeta}^{-\zeta}\varphi(t)\leq\lambda_\varphi\varphi(t)$, where $\varphi:\mathbb{N}_a\to\mathbb{R}^+$ is nondecreaing. Then problem (1.1) is generalized Ulam-Hyers-Rassias stable with respect to φ.*

Proof Let u be the unique solution of the following fractional difference equation with impulse condition, whereas v be a solution of inequality (2.2),

$$\begin{cases}{}^{c}\Delta_{a}^{\zeta}u(t) &= \mathscr{H}(t+\zeta,u(t+\zeta)),\ t\in\mathbb{N}_{a+1-\zeta},\ t\neq a+n_l+1-\zeta,\ 0<\zeta\leq 1,\\ u_{n_l+1} &= \overline{u}_{n_l+1}+I_l(\overline{u}_{n_l+1}),\ t=a+n_l+1-\zeta,\ l=1,2,...,N-1,\\ u(a) &= v(a).\end{cases} \tag{4.1}$$

Then we have

$$u(t)=\begin{cases}v(a)+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)),\ t\in\{a+n_0+1,...,a+n_1\},\\ \vdots\\ v(a)+\sum_{i=1}^{l} I_i(\overline{u}_{n_i+1})+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)),\ t\in\{a+n_l+1,...,a+n_{l+1}\},\\ \vdots\\ v(a)+\sum_{i=1}^{N} I_i(\overline{u}_{n_i+1})+\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta,u(t+\zeta)),\ t\in\{a+n_N+1,...\},\end{cases} \tag{4.2}$$

where $l = 1, ..., N-1$ *and* $N \to \infty$.
Like in Remark 4.2, for $t \in \{a+n_l+1, ..., a+n_{l+1}\}$ where $l = 1, 2, ..., N-1$, we have

$$\begin{aligned}|v(t) - v(a) - \sum_{i=1}^{l} I_i(\overline{v}_{n_i+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta, v(t+\zeta))| &\leq \sum_{i=1}^{l}|\mathscr{G}_i| + \Delta_{a+1-\zeta}^{-\zeta}\varphi(t)\\ &\leq (N-1+\lambda_\varphi)\varphi(t).\end{aligned}$$

Hence, for $t \in \{a+n_l+1, ..., a+n_{l+1}\}$, $l = 1, 2, ..., N-1$, it follows that,

$$\begin{aligned}|v(t) - u(t)| &= |v(t) - v(a) - \sum_{i=1}^{l} I_i(\overline{u}_{n_i+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta, u(t+\zeta))|\\
&\leq |v(t) - v(a) - \sum_{i=1}^{l} I_i(\overline{v}_{n_i+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta, v(t+\zeta))|\\
&\quad + |\sum_{i=1}^{l} I_i(\overline{v}_{n_i+1}) - \sum_{i=1}^{l} I_i(\overline{u}_{n_i+1})|\\
&\quad + |\Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta, v(t+\zeta)) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{H}(t+\zeta, u(t+\zeta))|\\
&\leq (N-1+\lambda_\varphi)\varphi(t) + \sum_{i=1}^{l}|I_i(\overline{v}_{n_i+1}) - I_i(\overline{u}_{n_i+1})|\\
&\quad + \Delta_{a+1-\zeta}^{-\zeta}|\mathscr{H}(t+\zeta, v(t+\zeta)) - \mathscr{H}(t+\zeta, u(t+\zeta))|\\
&\leq (N-1+\lambda_\varphi)\varphi(t) + \sum_{i=1}^{l}\rho_i|\overline{v}_{n_i+1} - \overline{u}_{n_i+1}| + \Delta_{a+1-\zeta}^{-\zeta}L_{\mathscr{H}}|v(t+\zeta) - u(t+\zeta)|.\end{aligned}$$

By Theorem 3.5 (Gronwall inequality), there exists a constant $\mathscr{M}^* > 0$ independent of $\lambda_\varphi\varphi(t)$, so we have $|v(t) - u(t)| \leq \mathscr{M}^*(N-1+\lambda_\varphi)\varphi(t) = \eta\varphi(t)$.
So problem (1.1) is generalized Ulam-Hyers-Rassias stable. Hence completing the proof. □

5. Example

Let us consider the fractional difference equation with impulsive effect

$$^{c}\Delta_0^{\zeta}u(t) = 0,\ t \in (0,1] \cap \mathbb{N}_0 \setminus \left\{\frac{1}{2}\right\},\ u\left(\frac{1}{2}^{+}\right) - u\left(\frac{1}{2}^{-}\right) = \frac{|u(\frac{1}{2}^{-})|^{\frac{1}{2}}}{1+|u(\frac{1}{2}^{-})|^{\frac{1}{2}}}. \tag{5.1}$$

Also consider the following inequalities

$$|^{c}\Delta_0^{\zeta}v(t)| \leq \epsilon,\ t \in (0,1] \cap \mathbb{N}_0 \setminus \left\{\frac{1}{2}\right\},\ \left|v\left(\frac{1}{2}^{+}\right) - v\left(\frac{1}{2}^{-}\right) - \frac{|v(\frac{1}{2}^{-})|^{\frac{1}{2}}}{1+|v(\frac{1}{2}^{-})|^{\frac{1}{2}}}\right| \leq \epsilon,\ \epsilon > 0. \tag{5.2}$$

Let discrete function v be a solution of inequality (5.2). Then there exists a discrete function $\mathscr{G}$ and $\mathscr{G}_1 \in \mathbb{R}$, so we have the following

(i) $|\mathscr{G}(t)| \leq \epsilon,\ t \in [0,1] \cap \mathbb{N}_0,\ |\mathscr{G}_1| \leq \epsilon,$

(ii) ${}^c\Delta_0^{\zeta} v(t) = \mathscr{G}(t),\ t \in [0,1] \cap \mathbb{N}_0 \setminus \left\{\frac{1}{2}\right\},$

(iii) $v\left(\frac{1}{2}^+\right) - v\left(\frac{1}{2}^-\right) = \dfrac{|v(\frac{1}{2}^-)|^{\frac{1}{2}}}{1+|v(\frac{1}{2}^-)|^{\frac{1}{2}}} + \mathscr{G}_1.$

For $(i), (ii)$ and (iii), using Remark 4.2, we have

$$v(t) = v(0) + \varphi_{(1/2,1]\cap\mathbb{N}_0}(t)\left(\frac{|v(\frac{1}{2}^-)|^{\frac{1}{2}}}{1+|v(\frac{1}{2}^-)|^{\frac{1}{2}}} + \mathscr{G}_1\right) + \Delta_{1-\zeta}^{-\zeta}\mathscr{G}(t),$$

where $\varphi_{(1/2,1]\cap\mathbb{N}_0}(t)$ is characteristic function of $(1/2,1]\cap\mathbb{N}_0$.
Let u be the unique solution of problem (5.1),

$u(t) = v(0) + \varphi_{(1/2,1]\cap\mathbb{N}_0}(t)\dfrac{|u(\frac{1}{2}^-)|^{\frac{1}{2}}}{1+|u(\frac{1}{2}^-)|^{\frac{1}{2}}}.$

$$\begin{aligned}|v(t)-u(t)| &= \left|\varphi_{(1/2,1]\cap\mathbb{N}_0}(t)\left(\frac{|v(\frac{1}{2}^-)|^{\frac{1}{2}}}{1+|v(\frac{1}{2}^-)|^{\frac{1}{2}}} - \frac{|u(\frac{1}{2}^-)|^{\frac{1}{2}}}{1+|u(\frac{1}{2}^-)|^{\frac{1}{2}}} + \mathscr{G}_1\right) + \Delta_{1-\zeta}^{-\zeta}\mathscr{G}(t)\right| \\ &\leq \varphi_{(1/2,1]\cap\mathbb{N}_0}(t)\left|v\left(\frac{1}{2}^-\right) - u\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}} + |\mathscr{G}_1| + \Delta_{1-\zeta}^{-\zeta}|\mathscr{G}(t)| \\ &\leq \varphi_{(1/2,1]\cap\mathbb{N}_0}(t)\left|v\left(\frac{1}{2}^-\right) - u\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}} + \epsilon + \epsilon,\ t \in [0,1]\cap\mathbb{N}_0 \\ &\leq \sqrt{2\epsilon} + 2\epsilon,\ t \in [0,1]\cap\mathbb{N}_0.\end{aligned}$$

Hence problem (5.1) is generalized Ulam-Hyers stable.

6. Conclusions

This paper deals with Ulam's type stability analysis of impulsive fractional difference equations. A new type of delta discrete Gronwall inequality is derived with an impulsive effect to obtain the stability criterion. An example is provided which shows that the concept of stability analysis using Gronwall inequality for ordinary/fractional differential equation can be extended to discrete cases.

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