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Nonunique best proximity point results with an application to nonlinear fractional differential equations

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Abstract: In this paper, we point out an error in proving famous Achari type nonunique fixed point results. Also, we prove some best proximity point results in b-metric spaces by introducing new concepts. Hence, we develop some results existing in the literature. Finally, we give a result for the existence of the solution of nonlinear fractional differential equations.

Key words: Nonlinear fractional differential equation, comparison function, best proximity point, b-metric space

1. Introduction

The fixed point theory is a very significant tool to solve various problems in approximation theory, nonlinear analysis, differential equations, control systems and game theory. Therefore, the theory has been improved by many authors. In this context, the Banach contraction principle [9] accepted as the beginning of the fixed point theory on metric spaces was proved. Let (Υ, σ) be a complete metric space and $\varphi : \Upsilon \to \Upsilon$ be a contraction mapping, then φ has a unique fixed point. It is known that the mapping φ has to be continuous and φ has a unique fixed point in this principle. Hence, a great number of results have been proved to obtain the existence and uniqueness of fixed points in this field [2, 17, 18, 22]. However, the solution of nonlinear systems used to solve real-life problems may not be unique. In this sense, Ćirić [14] obtained some nonunique fixed point results for a self-mapping φ satisfying

$$\min\{\sigma(\varphi\varrho, \varphi\xi), \sigma(\varrho, \varphi\varrho), \sigma(\xi, \varphi\xi)\} - \min\{\sigma(\varrho, \varphi\xi), \sigma(\xi, \varphi\varrho)\} \leq k\sigma(\varrho, \xi)$$

where $k \in [0, 1)$ and σ is a metric on Υ. Also, the mapping φ may not be continuous in the result of Ćirić. After that, many authors have studied nonunique fixed point theory [4, 15, 20, 21].

Recently, the fixed point theory has been improved by considering nonself mappings $\varphi : \wp \to \Re$ where $\wp, \Re$ are nonempty subsets of a metric space (Υ, σ). If the intersection of $\wp$ and $\Re$ is empty, then φ cannot have a solution to the equation $\varphi\varrho = \varrho$. Because of the fact that $\sigma(\varrho, \varphi\varrho) \geq \sigma(\wp, \Re)$ for all $\varrho \in \wp$, it is sensible to find a point ϱ satisfying $\sigma(\varrho, \varphi\varrho) = \sigma(\wp, \Re)$ which is called a best proximity point [10]. Since every best proximity point is a natural generalization of a fixed point in the case of $\wp = \Re = \Upsilon$, many authors have studied this topic in the literature [5, 6, 8, 24–26].

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On the other hand, introducing a nice concept of a b-metric, Czerwik [16] obtained a generalization of the Banach contraction principle in a different way from the results existing in the literature.

Definition 1.1 *[16]Let Υ be a nonempty set and $\sigma : \Upsilon \times \Upsilon \to [0, \infty)$ be a function such that for all $\varrho, \xi, z \in \Upsilon$,*

b1) $\sigma(\varrho, \xi) = 0$ *if and only if* $\varrho = \xi$,

b2) $\sigma(\varrho, \xi) = \sigma(\xi, \varrho)$,

b3) $\sigma(\varrho, z) \leq s[\sigma(\varrho, \xi) + \sigma(\xi, z)]$ *where* $s \geq 1$.

Then, σ is said to be a b-metric on Υ. Also, (Υ, σ) is said to be a b-metric space.

It is clear that every metric space is a b-metric space. However, the converse may not be true. Indeed, the following well-known example of b-metric spaces shows this fact. Let $\Upsilon = \mathbb{R}$ and $\sigma : \Upsilon \times \Upsilon \to [0, \infty)$ be a function defined as $\sigma(\varrho, \xi) = (\varrho - \xi)^2$ for all $\varrho, \xi \in \Upsilon$. Then (Υ, σ) is a b-metric space with the coefficient $s = 2$. If we take $\varrho = 8$, $\xi = 5$, and $z = 2$, then

$$\sigma(8, 2) = 36 > \sigma(8, 5) + \sigma(5, 2).$$

Hence, it is not a metric space.

Let (Υ, σ) be a b-metric space with the coefficient $s \geq 1$. We denote the family of all open subsets of Υ by τ_σ which has, as a base, the family

$$\{B(\varrho, r) : \varrho \in \Upsilon \text{ and } r > 0\}$$

where

$$B(\varrho, r) = \{\xi \in \Upsilon : \sigma(\varrho, \xi) < r\}.$$

Let $\{\varrho_n\}$ be sequence in Υ and $\varrho \in \Upsilon$. It can be seen that the sequence $\{\varrho_n\}$ converges to ϱ with respect to τ_σ if and only if

$$\lim_{n\to\infty} \sigma(\varrho_n, \varrho) = 0.$$

The sequence $\{\varrho_n\}$ in Υ is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\sigma(\varrho_n, \varrho_m) < \varepsilon$ for all $n, m \geq n_0$. (Υ, σ) is said to be a complete b-metric space if every Cauchy sequence in Υ converges to some $\varrho \in \Upsilon$ with respect to τ_σ.

Let $\varphi : X \to X$ be a mapping. Then, φ is called a continuous mapping at a point ϱ in X if for every sequence $\{\varrho_n\}$ in X satisfying $\sigma(\varrho_n, \varrho\ ) = 0$ as $n \to \infty$, $\varphi\varrho_n$ converges to $\varphi\varrho$ with respect to σ.

Note that, unlike ordinary metric, b-metric may not be continuous. To overcome this disadvantage, we give the following definition which is very important for our main results

Definition 1.2 ([7]) *Let (Υ, σ) be a b-metric space with the coefficient $s \geq 1$ and $\emptyset \neq \wp, \Re \subseteq \Upsilon$. The pair $(\wp, \Re)$ satisfies the property (M_C) if for all sequences $\{\varrho_n\}$ in $\wp$, $\{\xi_n\}$ in $\Re$ and $\varrho \in \wp$, $\xi \in \Re$, we have*

$$\lim_{n\to\infty} \sigma(\varrho_n, \varrho) = \lim_{n\to\infty} \sigma(\xi_n, \xi) = 0 \implies \lim_{n\to\infty} \sigma(\varrho_n, \xi_n) = \sigma(\varrho, \xi).$$

Now, we recall that definition of comparison functions and its properties.

Definition 1.3 *A mapping* $\psi : [0,\infty) \to [0,\infty)$ *is called a comparison function if it satisfies the following*

- ψ is increasing,
- $\psi^n(\gamma) \to 0$ as $n \to \infty$ for all $\gamma \in [0,\infty)$.

We denote the set of all comparison functions by Φ. For details, we refer the reader to [13, 23].

The following lemma is an important property of comparison functions.

Lemma 1.4 ([13, 23]) *If* $\psi : [0,\infty) \to [0,\infty)$ *is a comparison function, then*

- *for all* $k \geq 1$ *each iterate* ψ^k *of* ψ *is a comparison function,*
- $\psi(\gamma) < \gamma$ *for any* $\gamma > 0$,
- ψ *is continuous at* 0.

Later, Berinde [12] defined the notion of (b)-comparison function to obtain some fixed point results in the setting of b-metric spaces.

Definition 1.5 *Let* $\psi : [0,\infty) \to [0,\infty)$ *be a function and* $s \geq 1$ *be a real number. If it satisfies the following*

- ψ *is increasing,*
- *there exists* $k_0 \in \mathbb{N}$, $a \in (0,1)$ *and a convergent nonnegative series* $\sum_{k=1}^{\infty} \xi_k$ *such that* $s^{k+1}\psi^{k+1}(\gamma) \leq as^k\psi^k(\gamma) + \xi_k$ *for* $k \geq k_0$ *and any* $\gamma \in [0,\infty)$,

then the function ψ *is said to be a* (b)*-comparison function.*

Now, we present the following lemma.

Lemma 1.6 ([11]) *If* $\psi : [0,\infty) \to [0,\infty)$ *is* b*-comparison function, then*

- *The series* $\sum_{k=1}^{\infty} s^k\psi^k(t)$ *converges to* $t \in \mathbb{R}$.
- *The function* $b_s : [0,\infty) \to [0,\infty)$ *defined by* $b_s(t) = \sum_{k=1}^{\infty} s^k\psi^k(t)$ *converges to* $t \in \mathbb{R}$ *is increasing and continuous at* 0.

Remark 1.7 ([19]) *From Lemma 1.6, it can be seen that every* b*-comparison function is a comparison function, and so from Lemma 1.4 every* b*-comparision function* ψ *holds* $\psi(\gamma) < \gamma$ *for any* $\gamma > 0$,

The set of all b-comparison functions will be denoted by Φ_b. Now, we restate the related fundamental concepts and notations of best proximity point theory in the realm of b-metric spaces.

Let (Υ, σ) be a b-metric space with the coefficient $s \geq 1$ and $\emptyset \neq \wp, \Re \subseteq \Upsilon$. We will use the subsets of $\wp$ and $\Re$, respectively:

$$\wp_0 = \{\varrho \in \wp : \sigma(\varrho, \xi) = \sigma(\wp, \Re) \text{ for some } \xi \in \Re\}$$

and

$$\Re_0 = \{\xi \in \Re : \sigma(\varrho, \xi) = \sigma(\wp, \Re) \text{ for some } \varrho \in \wp\}$$

where $\sigma(\wp, \Re) = \inf\{\sigma(\varrho, \xi) : \varrho \in \wp \text{ and } \xi \in \Re\}$.

Definition 1.8 *Let* (Υ,σ) *be a* b*-metric space,* $\emptyset\neq\wp,\Re\subseteq\Upsilon$*,* $\varphi:\wp\to\Re$ *be a mapping. The pair* $(\wp,\Re)$ *is said to have* P*-property if and only if*

$$\begin{array}{l}\sigma(\varrho_1,\xi_1)=\sigma(\wp,\Re)\\ \sigma(\varrho_2,\xi_2)=\sigma(\wp,\Re)\end{array}\quad \textit{implies } \sigma(\varrho_1,\varrho_2)=\sigma(\xi_1,\xi_2)$$

for all $\varrho_1,\varrho_2\in\wp_0$ *and* $\xi_1,\xi_2\in\Re_0$*.*

In this paper, we point out an error in proving famous Achari type nonunique fixed point results. Also, we obtain some best proximity point results in b-metric spaces by introducing new concepts. Hence, we extend and develop some results existing in the literature. Finally, we give a result for the existence of the solution of nonlinear fractional differential equations.

2. Note on Achari type nonunique fixed point results

In this section, we point out an error in proving Achari type nonunique fixed point results. Achari [3] proved a nonunique fixed point result for a self-mapping φ on a metric space (Υ,σ) satisfying the following condition:

$$\frac{P_A(\varrho_1,\varrho_2)-Q_A(\varrho_1,\varrho_2)}{R_A(\varrho_1,\varrho_2)}\leq k\sigma(\varrho_1,\varrho_2) \tag{2.1}$$

for all distinct $\varrho_1,\varrho_2\in\Upsilon$ where $k\in[0,1)$,

$$\begin{aligned} P_A(\varrho_1,\varrho_2) &= \min\left\{\sigma(\varphi\varrho_1,\varphi\varrho_2)\sigma(\varrho_1,\varrho_2),\sigma(\varrho_1,\varphi\varrho_1)\sigma(\varrho_2,\varphi\varrho_2)\right\},\\ Q_A(\varrho_1,\varrho_2) &= \min\{\sigma(\varrho_1,\varphi\varrho_1)\sigma(\varrho_1,\varphi\varrho_2),\sigma(\varrho_2,\varphi\varrho_2)\sigma(\varrho_2,\varphi\varrho_1)\}\end{aligned}$$

and

$$R_A(\varrho_1,\varrho_2)=\min\left\{\sigma(\varrho_1,\varphi\varrho_1),\sigma(\varrho_2,\varphi\varrho_2)\right\}.$$

Until this time, many authors have obtained a generalization of this result by taking nonlinear function instead of constant k or using more general space. In all of these results, the authors proved that φ has a fixed point in Υ under the assumptions completeness of Υ and continuity of φ. However, we will show φ satisfying (2.1) has a fixed point in Υ without any restriction on Υ and φ.

We claim that if φ satisfies (2.1), then for every sequence $\{\varrho_n\}$ defined by $\varrho_{n+1}=\varphi\varrho_n$ for all $n\geq 1$ with the any initial point $\varrho_0\in\Upsilon$, there exists $n_0\in\mathbb{N}$ such that $\varrho_{n_0}=\varrho_{n_0+1}$. Indeed, assume that $\varrho_n\neq\varrho_{n+1}$ for all $n\geq 1$. Then, we have

$$\begin{aligned} P_A(\varrho_n,\varrho_{n+1}) &= \min\left\{\sigma(\varphi\varrho_n,\varphi\varrho_{n+1})\sigma(\varrho_n,\varrho_{n+1}),\sigma(\varrho_n,\varphi\varrho_n)\sigma(\varrho_{n+1},\varphi\varrho_{n+1})\right\}\\ &= \min\left\{\sigma(\varrho_{n+1},\varrho_{n+2})\sigma(\varrho_n,\varrho_{n+1}),\sigma(\varrho_n,\varrho_{n+1})\sigma(\varrho_{n+1},\varrho_{n+2})\right\}\\ &= \sigma(\varrho_n,\varrho_{n+1})\sigma(\varrho_{n+1},\varrho_{n+2}),\end{aligned}$$

$$\begin{aligned} Q_A(\varrho_n,\varrho_{n+1}) &= \min\{\sigma(\varrho_n,\varphi\varrho_n)\sigma(\varrho_n,\varphi\varrho_{n+1}),\sigma(\varrho_{n+1},\varphi\varrho_{n+1})\sigma(\varrho_{n+1},\varphi\varrho_n)\}\\ &= \min\left\{\sigma(\varrho_n,\varrho_{n+1})\sigma(\varrho_n,\varrho_{n+2}),\sigma(\varrho_{n+1},\varrho_{n+2})\sigma(\varrho_{n+1},\varrho_{n+1})\right\}\\ &= 0,\end{aligned}$$

and

$$\begin{aligned} R_A(\varrho_n, \varrho_{n+1}) &= \min\{\sigma(\varrho_n, \varphi\varrho_n), \sigma(\varrho_{n+1}, \varphi\varrho_{n+1})\} \\ &= \min\{\sigma(\varrho_n, \varrho_{n+1}), \sigma(\varrho_{n+1}, \varrho_{n+2})\} \end{aligned}$$

for all $n \geq 1$. Since φ satisfies (2.1), we get

$$\frac{\sigma(\varrho_n, \varrho_{n+1})\sigma(\varrho_{n+1}, \varrho_{n+2})}{\min\{\sigma(\varrho_n, \varrho_{n+1}), \sigma(\varrho_{n+1}, \varrho_{n+2})\}} \leq k\sigma(\varrho_n, \varrho_{n+1}) \tag{2.2}$$

for all $n \geq 1$. If $\sigma(\varrho_{n_0+1}, \varrho_{n_0+2}) \leq \sigma(\varrho_{n_0}, \varrho_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then we have $R_A(\varrho_{n_0}, \varrho_{n_0+1}) = \sigma(\varrho_{n_0+1}, \varrho_{n_0+2})$, and so from (2.2), we get

$$\begin{aligned} \sigma(\varrho_{n_0}, \varrho_{n_0+1}) &\leq k\sigma(\varrho_{n_0}, \varrho_{n_0+1}) \\ &< \sigma(\varrho_{n_0}, \varrho_{n_0+1}) \end{aligned}$$

which is a contradiction. If $\sigma(\varrho_{n_0}, \varrho_{n_0+1}) \leq \sigma(\varrho_{n_0+1}, \varrho_{n_0+2})$ for some $n_0 \in \mathbb{N}$, then we have $R_A(\varrho_{n_0}, \varrho_{n_0+1}) = \sigma(\varrho_{n_0}, \varrho_{n_0+1})$, and so from (2.2), we get

$$\begin{aligned} \sigma(\varrho_{n_0}, \varrho_{n_0+1}) &\leq \sigma(\varrho_{n_0+1}, \varrho_{n_0+2}) \\ &\leq k\sigma(\varrho_{n_0}, \varrho_{n_0+1}) \\ &< \sigma(\varrho_{n_0}, \varrho_{n_0+1}) \end{aligned}$$

which is a contradiction. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\varrho_{n_0} = \varrho_{n_0+1} = \varphi\varrho_{n_0}.$$

Hence, φ has a fixed point in Υ without any restriction on Υ and φ.

3. Nonunique best proximity point results

3.1. Ćirić type nonunique best proximity point results

Firstly, we introduce the definitions of s-approximately compact and proximal ψ_b-Ćirić type nonunique contraction mapping.

Definition 3.1 *Let (Υ, σ) be a b-metric space with the coefficient $s \geq 1$. If every sequence $\{\xi_n\}$ in $\Re$ satisfying $\sigma(\varrho, \Re) \leq \lim_{n\to\infty} \sigma(\varrho, \xi_n) \leq s\sigma(\varrho, \Re)$ for some $\varrho \in \wp$ has a convergent subsequence in $\Re$, then $\Re$ is called an s-approximately compact with respect to $\wp$.*

Definition 3.2 *Let (Υ, σ) be a b-metric space with the coefficient $s \geq 1$ and $\emptyset \neq \wp, \Re \subseteq \Upsilon$. A mapping $\varphi : \wp \to \Re$ is said to be proximal ψ_b-Ćirić type nonunique contraction mapping if there exists $\psi \in \Phi_b$ such that*

$$\begin{matrix} \sigma(u_1, \varphi\varrho_1) = \sigma(\wp, \Re) \\ \sigma(u_2, \varphi\varrho_2) = \sigma(\wp, \Re) \end{matrix}$$

implies

$$P_{\acute{C}}(\varrho_1, \varrho_2, u_1, u_2) - R_{\acute{C}}(\varrho_1, \varrho_2, u_1, u_2) \leq \psi(\sigma(\varrho_1, \varrho_2)) \tag{3.1}$$

for all $u_1, u_2, \varrho_1, \varrho_2 \in \wp$, *where*

$$P_{\acute{C}}(\varrho_1, \varrho_2, u_1, u_2) = \min\{\sigma(u_1, u_2), \sigma(\varrho_1, u_1), \sigma(\varrho_2, u_2)\}$$

and

$$R_{\acute{C}}(\varrho_1, \varrho_2, u_1, u_2) = \min\{\sigma(\varrho_2, u_1), \sigma(\varrho_1, u_2)\}.$$

Theorem 3.3 *Let* (Υ, σ) *be a complete* b*-metric space with the coefficient* $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \Upsilon$ *where* $\wp$ *is closed and* $\Re$ *is an* s*-approximately compact w.r.t.* $\wp$. *Assume that the pair* $(\wp, \Re)$ *satisfies the property* (M_C) *and* $\wp_0 \neq \emptyset$. *If* $\varphi : \wp \to \Re$ *is a proximal* ψ_b*-Ćirić type nonunique contraction mapping satisfying* $\varphi(\wp_0) \subseteq \Re_0$ *and* $g(\varrho) = \sigma(\varrho, \varphi\varrho)$ *is a lower semicontinuous on* $\wp$, *then* φ *has a best proximity point in* $\wp$.

Proof Let $\varrho_0 \in \wp_0$ be an arbitrary point. Since $\varphi\varrho_0 \in \varphi(\wp_0) \subseteq \Re_0$, there exists $\varrho_1 \in \wp_0$ such that

$$\sigma(\varrho_1, \varphi\varrho_0) = \sigma(\wp, \Re).$$

Similarly, there exists $\varrho_2 \in \wp_0$ such that

$$\sigma(\varrho_2, \varphi\varrho_1) = \sigma(\wp, \Re).$$

Repeating this process, we can construct a sequence $\{\varrho_n\}$ such that

$$\sigma(\varrho_{n+1}, \varphi\varrho_n) = \sigma(\wp, \Re) \tag{3.2}$$

for all $n \geq 1$. If there exists $n_0 \in \mathbb{N}$ such that $\sigma(\varrho_{n_0}, \varrho_{n_0+1}) = 0$, then ϱ_{n_0} is a best proximity point of φ. So we assume $\sigma(\varrho_n, \varrho_{n+1}) > 0$ for all $n \geq 1$. Then, we have

$$\begin{aligned} P_{\acute{C}}(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) &= \min\left\{ \begin{array}{c} \sigma(\varrho_n, \varrho_{n+1}), \\ \sigma(\varrho_{n-1}, \varrho_n), \sigma(\varrho_n, \varrho_{n+1}) \end{array} \right\} \\ &= \min\{\sigma(\varrho_n, \varrho_{n+1}), \sigma(\varrho_{n-1}, \varrho_n)\}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} R_{\acute{C}}(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) &= \min\{\sigma(\varrho_n, \varrho_n), \sigma(\varrho_{n-1}, \varrho_{n+1})\} \\ &= 0. \end{aligned} \tag{3.4}$$

for all $n \geq 1$. Further, since φ is a proximal ψ_b-Ćirić type nonunique contraction mapping, we have

$$P_{\acute{C}}(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) - R_{\acute{C}}(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) \leq \psi(\sigma(\varrho_{n-1}, \varrho_n))$$

for all $n \geq 1$, and so from (3.3) and (3.4), we have

$$\min\{\sigma(\varrho_n, \varrho_{n+1}), \sigma(\varrho_{n-1}, \varrho_n)\} \leq \psi(\sigma(\varrho_{n-1}, \varrho_n))$$

for all $n \geq 1$. Since $\sigma(\varrho_{n-1}, \varrho_n) \leq \psi(\sigma(\varrho_{n-1}, \varrho_n)) < \sigma(\varrho_{n-1}, \varrho_n)$ is impossible, we get

$$\sigma(\varrho_n, \varrho_{n+1}) \leq \psi(\sigma(\varrho_{n-1}, \varrho_n))$$

for all $n \geq 1$. Therefore, we have

$$\begin{aligned} \sigma(\varrho_n, \varrho_{n+1}) &\leq \psi\left(\sigma(\varrho_{n-1}, \varrho_n)\right) \\ &\leq \psi^2\left(\sigma(\varrho_{n-2}, \varrho_{n-1})\right) \\ &\quad \vdots \\ &\leq \psi^n\left(\sigma(\varrho_0, \varrho_1)\right). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \sigma(\varrho_n, \varrho_{n+p}) &\leq s\sigma(\varrho_n, \varrho_{n+1}) + s^2\sigma(\varrho_{n+1}, \varrho_{n+2}) + \cdots + s^p\sigma(\varrho_{n+p-1}, \varrho_{n+p}) \\ &\leq \frac{1}{s^{n-1}}\left\{ \begin{array}{c} s^n\psi^n\left(\sigma(\varrho_0, \varrho_1)\right) + s^{n+1}\psi^{n+1}\left(\sigma(\varrho_0, \varrho_1)\right) \\ + \cdots + s^{n+p-1}\psi^{n+p-1}\left(\sigma(\varrho_0, \varrho_1)\right) \end{array} \right\} \\ &\leq \frac{1}{s^{n-1}}\sum_{k=0}^{\infty} s^k\psi^k\left(\sigma(\varrho_0, \varrho_1)\right). \end{aligned} \tag{3.5}$$

Since $\sum_{k=0}^{\infty} s^k\psi^k\left(\sigma(\varrho_0, \varrho_1)\right)$ is convergent, we have

$$\sum_{k=0}^{\infty} s^k\psi^k\left(\sigma(\varrho_0, \varrho_1)\right) < \infty.$$

Taking limit $n \to \infty$ in the inequality (3.5), $\{\varrho_n\}$ is a Cauchy sequence in $\wp$. Since (Υ, σ) is a complete b-metric space and $\wp$ is a closed subset of Υ, there exists $\varrho^* \in \wp$ such that $\varrho_n \to \varrho^*$ as $n \to \infty$. On the other hand, we have

$$\begin{aligned} \sigma(\varrho^*, \Re) &\leq \sigma(\varrho^*, \varphi\varrho_n) \\ &\leq s\sigma(\varrho^*, \varrho_{n+1}) + s\sigma(\varrho_{n+1}, \varphi\varrho_n) \\ &= s\sigma(\varrho^*, \varrho_{n+1}) + s\sigma(\wp, \Re) \\ &\leq s\sigma(\varrho^*, \varrho_{n+1}) + s\sigma(\varrho^*, \Re). \end{aligned}$$

Therefore, we have $\sigma(\varrho^*, \Re) \leq \lim_{n\to\infty} \sigma(\varrho^*, \varphi\varrho_n) \leq s\sigma(\varrho^*, \Re)$. Since $\Re$ is an s-approximately compact w.r.t. $\wp$, there exists a convergent subsequence $\varphi\varrho_{n_k}$ of $\varphi\varrho_n$ such that $\varphi\varrho_{n_k} \to \xi^*$ for some $\xi^* \in \Re$. Also, since the pair $(\wp, \Re)$ satisfies the property (M_C), from (3.2), we get

$$\sigma(\varrho^*, \xi^*) = \sigma(\wp, \Re). \tag{3.6}$$

Further, since $g(\varrho) = \sigma(\varrho, \varphi\varrho)$ is a lower semicontinuous on $\wp$, we have

$$\begin{aligned} \sigma(\wp, \Re) &\leq \sigma(\varrho^*, \varphi\varrho^*) \\ &= g(\varrho^*) \\ &\leq \liminf g(\varrho_{n_k}) \\ &= \liminf \sigma(\varrho_{n_k}, \varphi\varrho_{n_k}) \\ &= \sigma(\varrho^*, \xi^*) \\ &= \sigma(\wp, \Re). \end{aligned}$$

Therefore, we have $\sigma(\varrho^*, \varphi\varrho^*) = \sigma(\wp, \Re)$. Hence, ϱ^* is a best proximity point of φ. □

If φ is a continuous mapping on $\wp$ in Theorem 3.3, we can omit the condition s-approximately compact of $\Re$ and lower semicontinuity of g. Hence, we obtain the following best proximity point result.

Theorem 3.4 *Let (Υ, σ) be a complete b-metric space with the coefficient $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \Upsilon$ where $\wp$ is closed. Assume that the pair $(\wp, \Re)$ satisfies the property (M_C) and $\wp_0 \neq \emptyset$. If $\varphi : \wp \to \Re$ is a continuous proximal ψ_b-Ćirić type nonunique contraction mapping satisfying $\varphi(\wp_0) \subseteq \Re_0$, then φ has a best proximity point in $\wp$.*

Proof Let $\varrho_0 \in \wp_0$ be an arbitrary point. As in the proof of Theorem 3.3, we can construct a sequence $\{\varrho_n\}$ in $\wp$ and it can be seen that $\{\varrho_n\}$ is a Cauchy sequence. Because of the fact that $\wp$ is closed, there is $\varrho^* \in \wp$ satisfying $\varrho_n \to \varrho^*$. Also, since φ is a continuous mapping, we have $\varphi\varrho_n \to \varphi\varrho^*$. Because of the fact that the pair $(\wp, \Re)$ satisfies the property (M_C), we get

$$\sigma(\varrho^*, \varphi\varrho^*) = \lim_{n\to\infty} \sigma(\varrho_{n+1}, \varphi\varrho_n) = \sigma(\wp, \Re).$$

□

If we take $\wp = \Re = \Upsilon$ in Theorem 3.4, then we obtain the following fixed point result.

Corollary 3.5 *Let (Υ, σ) be a complete b-metric space with the coefficient $s \geq 1$ and $\varphi : \Upsilon \to \Upsilon$ be a continuous mapping. If there exists $\psi \in \Phi_b$ such that*

$$\min\{\sigma(\varphi\varrho, \varphi\xi), \sigma(\varrho, \varphi\varrho), \sigma(\xi, \varphi\xi)\} - \min\{\sigma(\varrho, \varphi\xi), \sigma(\xi, \varphi\varrho)\} \leq \psi(\sigma(\varrho, \xi)) \tag{3.7}$$

for all $\varrho, \xi \in \Upsilon$, then φ has a fixed point in Υ.

3.2. Pachpatte type nonunique best proximity point results

Now, we introduce the definition of proximal ψ_b-Pachpatte type nonunique contraction mapping

Definition 3.6 *Let (Υ, σ) be a b-metric space with the coefficient $s \geq 1$ and $\emptyset \neq \wp, \Re \subseteq \Upsilon$. A mapping $\varphi : \wp \to \Re$ is said to be proximal ψ_b-Pachpatte type nonunique contraction mapping if there exists $\psi \in \Phi_b$ such that*

$$\psi(ab) \leq a\psi(b) \tag{3.8}$$

for all $a, b > 0$ and

$$\begin{array}{c} \sigma(u_1, \varphi\varrho_1) = \sigma(\wp, \Re) \\ \sigma(u_2, \varphi\varrho_2) = \sigma(\wp, \Re) \end{array}$$

implies

$$P_P(\varrho_1, \varrho_2, u_1, u_2) - Q_P(\varrho_1, \varrho_2, u_1, u_2) \leq \psi(\sigma(\varrho_1, u_1)\sigma(\varrho_2, u_2)) \tag{3.9}$$

for all $u_1, u_2, \varrho_1, \varrho_2 \in \wp$, where

$$P_P(\varrho_1, \varrho_2, u_1, u_2) = \min\left\{(\sigma(u_1, u_2))^2, \sigma(\varrho_1, \varrho_2)\sigma(u_1, u_2), (\sigma(\varrho_2, u_2))^2\right\},$$

and

$$Q_P(\varrho_1, \varrho_2, u_1, u_2) = \min\{\sigma(\varrho_1, u_1)\sigma(\varrho_2, u_2), \sigma(\varrho_1, u_2)\sigma(\varrho_2, u_1)\}.$$

Theorem 3.7 *Let (Υ, σ) be a complete b-metric space with the coefficient $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \Upsilon$ where $\wp$ is closed and $\Re$ is an s-approximately compact w.r.t. $\wp$. Assume that the pair $(\wp, \Re)$ satisfies the property (M_C) and $\wp_0 \neq \emptyset$. If $\varphi : \wp \to \Re$ is a proximal ψ_b-Pachpatte type nonunique contraction mapping satisfying $\varphi(\wp_0) \subseteq \Re_0$ and $g(\varrho) = \sigma(\varrho, \varphi\varrho)$ is a lower semicontinuous on $\wp$, then φ has a best proximity point in $\wp$.*

Proof Let $\varrho_0 \in \wp_0$ be an arbitrary point. Since $\varphi\varrho_0 \in \varphi(\wp_0) \subseteq \Re_0$, there exists $\varrho_1 \in \wp_0$ such that

$$\sigma(\varrho_1, \varphi\varrho_0) = \sigma(\wp, \Re).$$

Similarly, there exists $\varrho_2 \in \wp_0$ such that

$$\sigma(\varrho_2, \varphi\varrho_1) = \sigma(\wp, \Re).$$

Repeating this process, we can construct a sequence $\{\varrho_n\}$ such that

$$\sigma(\varrho_{n+1}, \varphi\varrho_n) = \sigma(\wp, \Re) \tag{3.10}$$

for all $n \geq 1$. If there exists $n_0 \in \mathbb{N}$ such that $\sigma(\varrho_{n_0}, \varrho_{n_0+1}) = 0$, then ϱ_{n_0} is a best proximity point of φ. Therefore, we assume $\sigma(\varrho_n, \varrho_{n+1}) > 0$ for all $n \geq 1$. From (3.10), we have

$$\begin{aligned} P_P(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) &= \min \left\{ \begin{array}{c} (\sigma(\varrho_n, \varrho_{n+1}))^2, \\ \sigma(\varrho_{n-1}, \varrho_n)\sigma(\varrho_n, \varrho_{n+1}), \\ (\sigma(\varrho_n, \varrho_{n+1}))^2 \end{array} \right\} \\ &= \min \left\{ (\sigma(\varrho_n, \varrho_{n+1}))^2, \sigma(\varrho_{n-1}, \varrho_n)\sigma(\varrho_n, \varrho_{n+1}) \right\} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} Q_P(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) &= \min \left\{ \begin{array}{c} \sigma(\varrho_{n-1}, \varrho_n)\sigma(\varrho_n, \varrho_{n+1}), \\ \sigma(\varrho_{n-1}, \varrho_{n+1})\sigma(\varrho_n, \varrho_n) \end{array} \right\} \\ &= 0. \end{aligned} \tag{3.12}$$

for all $n \geq 1$. Further, since φ is a ψ_b-Pachpatte type nonunique contraction mapping, we have

$$P_P(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) - Q_P(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) \leq \psi\left(\sigma(\varrho_{n-1}, \varrho_n)\sigma(\varrho_n, \varrho_{n+1})\right)$$

for all $n \geq 1$, and so from (3.11) and (3.12), we have

$$\min \left\{ (\sigma(\varrho_n, \varrho_{n+1}))^2, \sigma(\varrho_{n-1}, \varrho_n)\sigma(\varrho_n, \varrho_{n+1}) \right\} \leq \psi\left(\sigma(\varrho_{n-1}, \varrho_n)\sigma(\varrho_n, \varrho_{n+1})\right)$$

for all $n \geq 1$. Since $P_P(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) = \sigma(\varrho_{n-1}, \varrho_n)\sigma(\varrho_n, \varrho_{n+1})$ is impossible, we get

$$(\sigma(\varrho_n, \varrho_{n+1}))^2 \leq \psi\left(\sigma(\varrho_{n-1}, \varrho_n)\sigma(\varrho_n, \varrho_{n+1})\right) \tag{3.13}$$

for all $n \geq 1$. From (3.8) and(3.13), we have

$$\begin{aligned} (\sigma(\varrho_n, \varrho_{n+1}))^2 &\leq \psi\left(\sigma(\varrho_{n-1}, \varrho_n)\sigma(\varrho_n, \varrho_{n+1})\right) \\ &\leq \psi\left(\sigma(\varrho_{n-1}, \varrho_n)\right)\sigma(\varrho_n, \varrho_{n+1}) \end{aligned}$$

and so

$$\sigma(\varrho_n, \varrho_{n+1}) \leq \psi(\sigma(\varrho_{n-1}, \varrho_n))$$

for all $n \geq 1$. As in the proof of Theorem 3.3, we can obtain φ has a best proximity point in $\wp$. □

Similarly to Theorem 3.4, we present the following best proximity point result.

Theorem 3.8 *Let (Υ, σ) be a complete b-metric space with the coefficient $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \Upsilon$ where $\wp$ is closed. Assume that the pair $(\wp, \Re)$ satisfies the property (M_C) and $\wp_0 \neq \emptyset$. If $\varphi : \wp \to \Re$ is a continuous proximal ψ_b-Pachpatte type nonunique contraction mapping satisfying $\varphi(\wp_0) \subseteq \Re_0$, then φ has a best proximity point in $\wp$.*

If we take $\wp = \Re = \Upsilon$ in Theorem 3.8, then we obtain the following fixed point result.

Corollary 3.9 *Let (Υ, σ) be a complete b-metric space with the coefficient $s \geq 1$ and $\varphi : \Upsilon \to \Upsilon$ be a continuous mapping. If there exists $\psi \in \Phi_b$ such that*

$$\psi(ab) \leq a\psi(b)$$

for all $a, b > 0$ and

$$P_P(\varrho_1, \varrho_2) - Q_P(\varrho_1, \varrho_2) \leq \psi(\sigma(\varrho_1, \varphi\varrho_1)\sigma(\varrho_2, \varphi\varrho_2))$$

for all $\varrho_1, \varrho_2 \in \Upsilon$ where

$$P_P(\varrho_1, \varrho_2) = \min\left\{(\sigma(\varphi\varrho_1, \varphi\varrho_2))^2, \sigma(\varrho_1, \varrho_2)\sigma(\varphi\varrho_1, \varphi\varrho_2), (\sigma(\varrho_2, \varphi\varrho_2))^2\right\}$$

and

$$Q_P(\varrho_1, \varrho_2) = \min\{\sigma(\varrho_1, \varphi\varrho_1)\sigma(\varrho_2, \varphi\varrho_2), \sigma(\varrho_1, \varphi\varrho_2)\sigma(\varrho_2, \varphi\varrho_1)\},$$

then φ has a fixed point in Υ.

3.3. Ćirić-Jotić type nonunique best proximity point results

Now, we introduce the definition of proximal ψ_b-Ćirić-Jotić type nonunique contraction mapping.

Definition 3.10 *Let (Υ, σ) be a b-metric space with the coefficient $s \geq 1$ and $\emptyset \neq \wp, \Re \subseteq \Upsilon$. A mapping $\varphi : \wp \to \Re$ is said to be proximal ψ_b-Ćirić-Jotić type nonunique contraction mapping if there exist $\psi \in \Phi_b$ and $a \geq 0$ such that*

$$\begin{array}{c} \sigma(u_1, \varphi\varrho_1) = \sigma(\wp, \Re) \\ \sigma(u_2, \varphi\varrho_2) = \sigma(\wp, \Re) \end{array}$$

implies

$$P_{\acute{C}J}(\varrho_1, \varrho_2, u_1, u_2) - aQ_{\acute{C}J}(\varrho_1, \varrho_2, u_1, u_2) \leq \psi(R_{\acute{C}J}(\varrho_1, \varrho_2, u_1, u_2)) \tag{3.14}$$

for all $u_1, u_2, \varrho_1, \varrho_2 \in \wp$, where

$$P_{\acute{C}J}(\varrho_1, \varrho_2, u_1, u_2) = \min\left\{\begin{array}{c} \sigma(u_1, u_2), \sigma(\varrho_1, \varrho_2), \sigma(\varrho_1, u_1), \\ \sigma(\varrho_2, u_2), \frac{\sigma(\varrho_1, u_1)[1+\sigma(\varrho_2, u_2)]}{1+\sigma(\varrho_1, \varrho_2)} \\ \frac{\sigma(\varrho_2, u_2)[1+\sigma(\varrho_1, u_1)]}{1+\sigma(\varrho_1, \varrho_2)}, \frac{\min\{\sigma^2(u_1, u_2), \sigma^2(\varrho_1, u_1), \sigma^2(\varrho_2, u_2)\}}{\psi(\sigma(\varrho_1, \varrho_2))} \end{array}\right\},$$

$$Q_{\acute{C}J}(\varrho_1,\varrho_2,u_1,u_2)=\min\{\sigma(\varrho_1,u_2),\sigma(\varrho_2,u_1)\}$$

and

$$R_{\acute{C}J}(\varrho_1,\varrho_2,u_1,u_2)=\max\left\{\sigma(\varrho_1,\varrho_2),\sigma(\varrho_1,u_1)\right\}.$$

Theorem 3.11 *Let (Υ,σ) be a complete b-metric space with the coefficient $s\geq 1$, $\emptyset\neq\wp,\Re\subseteq\Upsilon$ where $\wp$ is closed and $\Re$ is an s-approximately compact w.r.t. $\wp$. Assume that the pair $(\wp,\Re)$ satisfies the property (M_C) and $\wp_0\neq\emptyset$. If $\varphi:\wp\to\Re$ is a proximal ψ_b-Ćirić-Jotić type nonunique contraction mapping satisfying $\varphi(\wp_0)\subseteq\Re_0$ and $g(\varrho)=\sigma(\varrho,\varphi\varrho)$ is a lower semicontinuous on $\wp$, then φ has a best proximity point in $\wp$.*

Proof Let $\varrho_0\in\wp_0$ be an arbitrary point. Since $\varphi\varrho_0\in\varphi(\wp_0)\subseteq\Re_0$, there exists $\varrho_1\in\wp_0$ such that

$$\sigma(\varrho_1,\varphi\varrho_0)=\sigma(\wp,\Re).$$

Similarly, there exists $\varrho_2\in\wp_0$ such that

$$\sigma(\varrho_2,\varphi\varrho_1)=\sigma(\wp,\Re).$$

Repeating this process, we can construct a sequence $\{\varrho_n\}$ such that

$$\sigma(\varrho_{n+1},\varphi\varrho_n)=\sigma(\wp,\Re) \tag{3.15}$$

for all $n\geq 1$. If there exists $n_0\in\mathbb{N}$ such that $\sigma(\varrho_{n_0},\varrho_{n_0+1})=0$, then ϱ_{n_0} is a best proximity point of φ. Therefore, we assume $\sigma(\varrho_n,\varrho_{n+1})>0$ for all $n\geq 1$. Then, we have

$$\begin{aligned} P_{\acute{C}J}(\varrho_{n-1},\varrho_n,\varrho_n,\varrho_{n+1}) &= \min\left\{\begin{array}{c} \sigma(\varrho_n,\varrho_{n+1}),\sigma(\varrho_{n-1},\varrho_n),\\ \sigma(\varrho_{n-1},\varrho_n),\sigma(\varrho_n,\varrho_{n+1}),\\ \frac{\sigma(\varrho_{n-1},\varrho_n)[1+\sigma(\varrho_n,\varrho_{n+1})]}{1+\sigma(\varrho_{n-1},\varrho_n)}\\ \frac{\sigma(\varrho_n,\varrho_{n+1})[1+\sigma(\varrho_{n-1},\varrho_n)]}{1+\sigma(\varrho_{n-1},\varrho_n)},\\ \frac{\min\left\{\begin{array}{c}\sigma^2(\varrho_n,\varrho_{n+1}),\sigma^2(\varrho_{n-1},\varrho_n),\\ \sigma^2(\varrho_n,\varrho_{n+1})\end{array}\right\}}{\psi(\sigma(\varrho_{n-1},\varrho_n))} \end{array}\right\} \\ &= \min\left\{\begin{array}{c} \sigma(\varrho_n,\varrho_{n+1}),\sigma(\varrho_{n-1},\varrho_n),\\ \frac{\sigma(\varrho_{n-1},\varrho_n)[1+\sigma(\varrho_n,\varrho_{n+1})]}{1+\sigma(\varrho_{n-1},\varrho_n)},\\ \frac{\min\left\{\sigma^2(\varrho_n,\varrho_{n+1}),\sigma^2(\varrho_{n-1},\varrho_n)\right\}}{\psi(\sigma(\varrho_{n-1},\varrho_n))} \end{array}\right\}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} Q_{\acute{C}J}(\varrho_{n-1},\varrho_n,\varrho_n,\varrho_{n+1}) &= \min\left\{\sigma(\varrho_{n-1},\varrho_{n+1}),\sigma(\varrho_n,\varrho_n)\right\} \\ &= 0, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} R_{\acute{C}J}(\varrho_{n-1},\varrho_n,\varrho_n,\varrho_{n+1}) &= \max\{\sigma(\varrho_{n-1},\varrho_n),\sigma(\varrho_{n-1},\varrho_n)\} \\ &= \sigma(\varrho_{n-1},\varrho_n) \end{aligned} \tag{3.18}$$

for all $n\geq 1$.

Case1 : Let $P_{\acute{C}J}(\varrho_{n-1},\varrho_n,\varrho_n,\varrho_{n+1})=\sigma(\varrho_{n-1},\varrho_n)$. Since φ is a ψ_b-Ćirić-Jotić type nonunique contraction mapping, we have

$$\begin{aligned}\sigma(\varrho_{n-1},\varrho_n) &\leq \psi(\sigma(\varrho_{n-1},\varrho_n))\\ &< \sigma(\varrho_{n-1},\varrho_n)\end{aligned}$$

which is a contradiction.

Case2 : Let $P_{\acute{C}J}(\varrho_{n-1},\varrho_n,\varrho_n,\varrho_{n+1})=\frac{\sigma(\varrho_{n-1},\varrho_n)[1+\sigma(\varrho_n,\varrho_{n+1})]}{1+\sigma(\varrho_{n-1},\varrho_n)}$. Since φ is a ψ_b-Ćirić-Jotić type nonunique contraction mapping, we have

$$\frac{\sigma(\varrho_{n-1},\varrho_n)[1+\sigma(\varrho_n,\varrho_{n+1})]}{1+\sigma(\varrho_{n-1},\varrho_n)}\leq\psi(\sigma(\varrho_{n-1},\varrho_n))$$

which implies that

$$\begin{aligned}\sigma(\varrho_{n-1},\varrho_n)+\sigma(\varrho_{n-1},\varrho_n)\sigma(\varrho_n,\varrho_{n+1}) &\leq \psi(\sigma(\varrho_{n-1},\varrho_n)\\ &\quad+\psi(\sigma(\varrho_{n-1},\varrho_n)\sigma(\varrho_{n-1},\varrho_n)\\ &< \sigma(\varrho_{n-1},\varrho_n)\\ &\quad+\psi(\sigma(\varrho_{n-1},\varrho_n)\sigma(\varrho_{n-1},\varrho_n),\end{aligned}$$

and so we get

$$\sigma(\varrho_n,\varrho_{n+1})\leq\psi(\sigma(\varrho_{n-1},\varrho_n)$$

for all $n\geq 1$.

Case3 : Let $P_{\acute{C}J}(\varrho_{n-1},\varrho_n,\varrho_n,\varrho_{n+1})=\frac{\min\left\{\sigma^2(\varrho_n,\varrho_{n+1}),\sigma^2(\varrho_{n-1},\varrho_n)\right\}}{\psi(\sigma(\varrho_{n-1},\varrho_n))}$. Now, if

$$\min\left\{\sigma^2(\varrho_n,\varrho_{n+1}),\sigma^2(\varrho_{n-1},\varrho_n)\right\}=\sigma^2(\varrho_{n-1},\varrho_n),$$

then since φ is a ψ_b-Ćirić-Jotić type nonunique contraction mapping, we have

$$\frac{\sigma^2(\varrho_{n-1},\varrho_n)}{\psi(\sigma(\varrho_{n-1},\varrho_n))}\leq\psi\left(\sigma(\varrho_{n-1},\varrho_n)\right)$$

which implies that

$$\sigma(\varrho_{n-1},\varrho_n)\leq\psi\left(\sigma(\varrho_{n-1},\varrho_n)\right)$$

for all $n\geq 1$. This is a contradiction. Hence, we assume that

$$\min\left\{\sigma^2(\varrho_n,\varrho_{n+1}),\sigma^2(\varrho_{n-1},\varrho_n)\right\}=\sigma^2(\varrho_n,\varrho_{n+1}).$$

Since φ is a ψ_b-Ćirić-Jotić type nonunique contraction mapping, we have

$$\frac{\sigma^2(\varrho_n,\varrho_{n+1})}{\psi(\sigma(\varrho_{n-1},\varrho_n))}\leq\psi\left(\sigma(\varrho_{n-1},\varrho_n)\right)$$

which implies that

$$\sigma(\varrho_n,\varrho_{n+1})\leq\psi(\sigma(\varrho_{n-1},\varrho_n)$$

for all $n \geq 1$.

Case4 : Let $P_{\acute{C}J}(\varrho_{n-1}, \varrho_n, \varrho_n, \varrho_{n+1}) = \sigma(\varrho_n, \varrho_{n+1})$. Since φ is a ψ_b-Ćirić-Jotić type nonunique contraction mapping, we have

$$\sigma(\varrho_n, \varrho_{n+1}) \leq \psi(\sigma(\varrho_{n-1}, \varrho_n)$$

for all $n \geq 1$. As in the proof of Theorem 3.3, we can obtain φ has a best proximity point in $\wp$. □

Similarly to Theorem 3.4, we present the following best proximity point result.

Theorem 3.12 *Let (Υ, σ) be a complete b-metric space with the coefficient $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \Upsilon$ where $\wp$ is closed. Assume that the pair $(\wp, \Re)$ satisfies the property (M_C) and $\wp_0 \neq \emptyset$. If $\varphi : \wp \to \Re$ is a continuous proximal ψ_b-Ćirić-Jotić type nonunique contraction mapping satisfying $\varphi(\wp_0) \subseteq \Re_0$, then φ has a best proximity point in $\wp$.*

If we take $\wp = \Re = \Upsilon$ in Theorem 3.12, then we present the following fixed point result.

Corollary 3.13 *Let φ be a continuous self mapping on a complete b-metric space (Υ, σ) with the coefficient $s \geq 1$. If there exists $\psi \in \Phi_b$ such that*

$$P_{\acute{C}J}(\varrho_1, \varrho_2) - aQ_{\acute{C}J}(\varrho_1, \varrho_2) \leq \psi(R_{\acute{C}J}(\varrho_1, \varrho_2))$$

for all $u_1, u_2, \varrho_1, \varrho_2 \in \wp$, where

$$P_{\acute{C}J}(\varrho_1, \varrho_2) = \min\left\{\begin{array}{c} \sigma(\varphi\varrho_1, \varphi\varrho_2), \sigma(\varrho_1, \varrho_2), \\ \sigma(\varrho_1, \varphi\varrho_1), \sigma(\varrho_2, \varphi\varrho_2), \\ \frac{\sigma(\varrho_1, \varphi\varrho_1)[1+\sigma(\varrho_2, \varphi\varrho_2)]}{1+\sigma(\varrho_1, \varrho_2)} \\ \frac{\sigma(\varrho_2, \varphi\varrho_2)[1+\sigma(\varrho_1, \varphi\varrho_1)]}{1+\sigma(\varrho_1, \varrho_2)}, \\ \frac{\min\{\sigma^2(\varphi\varrho_1, \varphi\varrho_2), \sigma^2(\varrho_1, \varphi\varrho_1), \sigma^2(\varrho_2, \varphi\varrho_2)\}}{\psi(\sigma(\varrho_1, \varrho_2))} \end{array}\right\},$$

$$Q_{\acute{C}J}(\varrho_1, \varrho_2) = \min\{\sigma(\varrho_1, \varphi\varrho_2), \sigma(\varrho_2, \varphi\varrho_1)\},$$

and

$$R_{\acute{C}J}(\varrho_1, \varrho_2,) = \max\left\{\sigma(\varrho_1, \varrho_2), \sigma(\varrho_1, \varphi\varrho_1)\right\},$$

then φ has a fixed point in Υ.

4. Application

In this section, we give sufficient conditions for the existence and uniqueness of the solution of nonlinear fractional differential equations by taking into account Corollary 3.5. The Caputo derivative of a continuous function $h : [0, \infty) \to \mathbb{R}$, of order $\alpha > 0$ is defined as

$${}^C D^\alpha(h(\gamma)) = \frac{1}{\Gamma(n-\alpha)} \int_0^\gamma (\gamma - s)^{n-\alpha-1} h^{(n)}(s) ds, \alpha > 0, n-1 < \alpha < n$$

where Γ is the gamma function and n is an integer.

The following nonlinear fractional differential equation of Caputo type

$${}^C D^\alpha(\varrho(\gamma)) = g(\gamma, \varrho(\gamma)) \tag{4.1}$$

with integral boundary conditions

$$\varrho(0) = 0 \text{ and } \varrho(1) = \int_0^{\varsigma} \varrho(u)du$$

where $1 < \alpha \leq 2,\ 0 < \ \gamma, \varsigma < 1,\ \varrho \in C[0,1]$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function. Because of the fact that g is a continuous, we can see that the equation (4.1) is equivalent to the integral equation [1].

$$\begin{aligned}\varrho(\gamma) \quad &= \quad \frac{1}{\Gamma(\alpha)} \int_0^{\gamma} (\gamma - u)^{\alpha-1} g(u, \varrho(u))du \\ &- \frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} g(u, \varrho(u))du \\ &+ \frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \int_0^{\varsigma} \left(\int_0^u (u-r)^{\alpha-1} g(r, \varrho(r))dr \right) du\end{aligned} \tag{4.2}$$

Theorem 4.1 *Suppose the following conditions hold:*

(i) the mapping $\varphi : \Upsilon \to \Upsilon$

$$\begin{aligned}\varphi\varrho(\gamma) \quad &= \quad \frac{1}{\Gamma(\alpha)} \int_0^{\gamma} (\gamma - u)^{\alpha-1} g(u, \varrho(u))du \\ &- \frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} g(u, \varrho(u))du \\ &+ \frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \int_0^{\varsigma} \left(\int_0^u (u-r)^{\alpha-1} g(r, \varrho(r))dr \right) du\end{aligned}$$

for all $\varrho \in C[0,1]$ *and* $\gamma \in [0,1]$, *is a continuous mapping.*

(ii) there exists q *in* $[0,1)$ *such that*

$$|g(u, \varrho(u)) - g(u, \xi(u))| \leq \frac{\Gamma(\alpha+1)}{5} \left\{ \ \psi\left(|\varrho(u) - \xi(u)|^2\right) + N(\varrho, \xi) \ \right\}^{\frac{1}{2}}$$

where

$$N(\varrho, \xi) = \min\{|\varrho(u) - \varphi\xi(u)|^2, |\xi(u) - \varphi\varrho(u)|^2\}.$$

Then, the problem (4.1) has a unique solution

Proof Let $\Upsilon = C[0,1]$ and a function $p : \Upsilon \times \Upsilon \to [0,\infty)$ defined as

$$\sigma(u,v) = \sup_{\gamma \in [0,1]} |u(\gamma) - v(\gamma)|^2$$

for all $u, v \in \Upsilon$ and $\gamma \in [0,1]$. Then, (Υ, σ) is a complete b-metric space with the coefficient $s = 2$. Now, we shall show that φ satisfies the inequality (3.7). For all $\varrho, \xi \in \Upsilon$ and $\gamma \in [0,1]$, we have

$$\begin{aligned}
|\varphi\varrho(\gamma) - \varphi\xi(\gamma)| &= \left| \begin{array}{c} \frac{1}{\Gamma(\alpha)} \int_0^\gamma (\gamma-u)^{\alpha-1} g(u,\varrho(u))du \\ -\frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} g(u,\varrho(u))du \\ +\frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \int_0^\varsigma \left(\int_0^u (u-r)^{\alpha-1} g(r,\varrho(r))dr\right) du \\ -\frac{1}{\Gamma(\alpha)} \int_0^\gamma (\gamma-u)^{\alpha-1} g(u,\xi(u))du \\ +\frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} g(u,\xi(u))du \\ -\frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \int_0^\varsigma \left(\int_0^u (u-r)^{\alpha-1} g(r,\xi(r))dr\right) du \end{array} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_0^\gamma |\gamma-u|^{\alpha-1} \left(|g(u,\varrho(u)) - g(u,\xi(u))|\right) du \right\} \\
&\quad + \frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \left\{ \int_0^1 (1-u)^{\alpha-1} \left(|g(u,\varrho(u)) - g(u,\xi(u))|\right) du \right\} \\
&\quad + \frac{2\gamma}{(2-\varsigma^2)\Gamma(\alpha)} \left\{ \int_0^\varsigma \left(\int_0^u |u-r|^{\alpha-1} \left(|g(r,\varrho(r)) - g(r,\xi(r))|\right) dr \right) du \right\} \\
&\leq \sup_{\gamma\in[0,1]} \left[\int_0^\gamma \left(\begin{array}{c} \frac{|\gamma-u|^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{5} \\ \times \left\{ \psi\left(|\varrho(u)-\xi(u)|^2 \right) + N(\varrho,\xi) \right\}^{\frac{1}{2}} \end{array} \right) du \right. \\
&\quad + \frac{2\gamma}{(2-\varsigma^2)} \left\{ \int_0^1 \left(\begin{array}{c} \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{5} \\ \times \left\{ \psi\left(|\varrho(u)-\xi(u)|^2 \right) + N(\varrho,\xi) \right\}^{\frac{1}{2}} \end{array} \right) du \right\} \\
&\quad \left. + \frac{2\gamma}{(2-\varsigma^2)} \left\{ \int_0^\varsigma \left(\int_0^u \left[\begin{array}{c} \frac{|u-r|^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{5} \\ \times \left\{ \begin{array}{c} \psi\left(|\varrho(r)-\xi(r)|^2 \right) \\ +N(\varrho,\xi) \end{array} \right\}^{\frac{1}{2}} \end{array} \right] dr \right) du \right\} \right] \\
&\leq \frac{\Gamma(\alpha+1)}{5} \left\{ \psi(\sigma(\varrho,\xi)) + \min\left\{\sigma(\varrho,T\xi), \sigma(\xi,T\varrho)\right\} \right\}^{\frac{1}{2}} \\
&\quad \times \sup_{\gamma\in[0,1]} \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{2\gamma}{(2-\varsigma^2)} \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \right) \right\} \\
&\leq \left(\psi(\sigma(\varrho,\xi) + \min\left\{\sigma(\varrho,T\xi), \sigma(\xi,T\varrho)\right\} \right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, we have

$$|T\varrho(\gamma) - T\xi(\gamma)|^2 \leq \psi(\sigma(\varrho,\xi) + \min\left\{\sigma(\varrho,T\xi), \sigma(\xi,T\varrho)\right\}.$$

and so

$$\sigma(T\varrho, T\xi) = \sup_{\gamma\in[0,1]} |T\varrho(\gamma) - T\xi(\gamma)|^2 \leq \psi(\sigma(\varrho,\xi) + \min\left\{\sigma(\varrho,T\xi), \sigma(\xi,T\varrho)\right\}$$

Therefore, we get

$$\min\left\{\sigma(T\varrho,T\xi), \sigma(\varrho,T\varrho), \sigma(\xi,T\xi)\right\} - \min\left\{\sigma(\varrho,T\xi), \sigma(\xi,T\varrho)\right\} \leq \psi(\sigma(\varrho,\xi).$$

Then, all hypothesis of Corollary 3.5 are satisfied, and so T has a fixed point. Hence, nonlinear fractional differential equation of Caputo type (4.1) has a solution. □

References

[1] Agarwal RP, Benchohra M,Hamani S. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Applicandae Mathematicae 2010; 109 (3): 973-1033.

[2] Abbas M, Ali B, Romaguera S. Fixed and periodic points of generalized contractions in metric spaces. Fixed Point Theory and Applications 2013; 2013 (1): 1-11.

[3] Achari J. On Ćirić's non-unique fixed points. Matematički Vesnik 1976; 13: 255-257.

[4] Alsulami HH, Karapınar E, Rakočević V. Ćirić type nonunique fixed point theorems on b-metric spaces. Filomat 2017; 31 (11): 3147-3156.

[5] Altun I, Sahin H, and Aslantas M. A new approach to fractals via best proximity point. Chaos, Solitons and Fractals 2021; 146: 1-7.

[6] Aslantas M. Some best proximity point results via a new family of F-contraction and an application to homotopy theory, Journal of Fixed Point Theory and Applications 2021; 23 (4): 1-20.

[7] Aslantaş M. Best proximity point theorems for proximal b-cyclic contractions on b-metric spaces. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 2021; 70 (1): 483-496.

[8] Aslantas M. A new contribution to best proximity point theory on quasi metric spaces and an application to nonlinear integral equations. Optimization 2022; 1-14.

[9] Banach S. Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales. Fundamenta Mathematicae 1922; 3: 133-181 (in French).

[10] Basha SS, Veeramani P. Best approximations and best proximity pairs. Acta Scientiarum Mathematicarum 1977; 63: 289-300.

[11] Berinde V. Generalized contractions in quasi-metric spaces. Seminaron Fixed Point Theory, Babes -Bolyai University,Research Seminars 1993; 3-9.

[12] Berinde V. Sequences of operators and fixed points in quasimetric spaces. Studia Universitatis Babeș-Bolyai Mathematica 1996; 16 (4): 23-27.

[13] Berinde V. Contracţii generalizate şi aplicaţii. Editura Club Press 22, Baia Mare, 1997 (in Romanian).

[14] Ćirić LB. On some maps with a nonunique fixed point. Institut Mathématique 1974; 17: 52-58.

[15] Ćirić LB, Jotić N. A further extension of maps with non-unique fixed points. Matematički Vesnik 1998; 50 (210): 1-4.

[16] Czerwik S. Contraction mappings in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensi 1993; 1: 5-11.

[17] Hussain N, Mitrović ZD, Radenović S. A common fixed point theorem of Fisher in b-metric spaces. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2019; 113 (2): 949-956.

[18] Jleli M, Samet B. A new generalization of the Banach contraction principle. Journal of Inequalities and Applications 2014; 2014: 1-8.

[19] Karapinar E. A short survey on the recent fixed point results on b-metric spaces. Constructive Mathematical Analysis 2018; 1 (1): 15-44.

[20] Karapınar E, Romaguera S. Nonunique fixed point theorems in partial metric spaces. Filomat 2013; 27 (7): 1305-1314.

[21] Pachpatte BG. On Ćirić type maps with a non-unique fixed point,Indian Journal of Pure and Applied Mathematics 1979; 10: 1039-1043.

[22] Reich S, Zaslavski AJ. Existence of a unique fixed point for nonlinear contractive mappings. Mathematics 2020; 8: 1-7.

[23] Rus IA. Generalized contractions and applications. Cluj University Press, Cluj-Napoca 2001.

[24] Sahin H. A New Type of F F-Contraction and Their Best Proximity Point Results with Homotopy Application. Acta Applicandae Mathematicae 2022; 179(1): 1-15.

[25] Sahin H. A New Best Proximity Point Result with an Application to Nonlinear Fredholm Integral Equations. Mathematics 2022; 10 (4): 1-14.

[26] Sankar Raj V. Best proximity point theorems for non-self mappings. Fixed Point Theory 2013; 14: 447-454.