

Existence and uniqueness of mild solutions for mixed Caputo and Riemann–Liouville semilinear fractional integrodifferential equations with nonlocal conditions

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Abstract: The purpose of this paper is to investigate the existence and uniqueness of the mild solution to a class of semilinear fractional integrodifferential equations with state-dependent nonlocal fractional conditions. Our problem includes both Caputo and Riemann–Liouville fractional derivatives. Continuous dependence of solutions on initial conditions and ϵ -approximate mild solutions of the considered problem will be discussed.

Key words: Fractional derivatives, state-dependent nonlocal conditions, continuous dependence, C_0 -semigroup, ϵ -approximate solutions, fixed points

1. Introduction

Fractional calculus was originated in the 17th century at the same time as the integer calculus. It describes the most diminutive details of natural phenomena, which is better than practicing the integer calculus. For the history and further details about applications and significant results on fractional calculus, see [8, 11, 19].

Many authors are interested in studying the effect of some type of fractional derivatives on another type. For instance, Ahmed et al. [5] studied a class of fractional differential equations including Caputo on Caputo derivatives of different orders with Caputo fractional boundary conditions of the form:

$$\begin{cases} {}^C D^\alpha [{}^C D^\beta x(t) - g(t, x(t))] = f(t, x(t)), & t \in [0, b]; \\ x(0) = 0, \quad ({}^C D^\gamma x)(b) = \lambda(I^\delta x)(b), \end{cases}$$

where I^δ is the Riemann–Liouville fractional integral of order δ , $0 < \alpha, \beta, \gamma < 1$, $\lambda \neq \Gamma(\beta + \delta + 1)/b^{\gamma + \delta} \Gamma(\beta - \gamma + 1)$, and $g, f : [0, b] \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions. In [17], Ntouyas et al. investigated the boundary value problem including Riemann–Liouville on Caputo fractional derivatives with nonlocal fractional integrodifferential boundary conditions:

$$\begin{cases} {}^R D^q ({}^C D^r x)(t) = f(t, x(t)), & t \in (0, b); \\ x'(\zeta) = \lambda {}^C D^\nu x(\eta), \quad x(b) = \mu (I^p x)(\beta), & \zeta, \eta, \beta \in (0, b), \quad \lambda, \mu \in \mathbb{R}, \end{cases}$$

where $q \in (0, 1)$ and $r \in (0, 1)$ and $\nu \in (0, q + r)$. I^p is the Riemann–Liouville fractional integral of order $p > 0$ and $f : [0, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function. For more recent papers about fractional integrodifferential equations contain mixed or multiterm fractional derivatives, see [3, 4, 20].

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The fractional Euler–Lagrange equation (FELE) is one of the most famous fractional differential equations that contain mixed fractional derivatives, e.g., Riemann–Liouville, and Caputo derivatives. The fractional calculus of variations deals with problems in which the functional, the constraint conditions, or both depend on some fractional operator. The main goal is to find functions that extremize (minimize or maximize) such a fractional by applying its corresponding FELE on the Lagrangian (the integrand in the functional equation). In [1, 15], Euler–Lagrange equations for some kinds of variational problems with fractional derivatives in the Riemann–Liouville sense are presented. In [2, 16], Euler–Lagrange equations are defined in the sense of both Riemann–Liouville and Caputo. Recently, in [13], Herzallah discusses the existence and uniqueness of solutions to a fractional Euler–Lagrange equation with both Riemann–Liouville and Caputo derivatives. There is rapid development and significant growth in the study of FELE with its applications, especially in quantum mechanics, field theory, and optimal control. For more details, see [12].

Nonlocal conditions describe some peculiarities processes occurring at various positions inside domain, which is not possible with the end-point initial conditions; furthermore, they give more precise measurements, accurate results, and better effect than classical conditions. Hernandez and O’Regan [7] inaugurated a new type of nonlocal conditions called state-dependent nonlocal conditions (SDNC’s) which have the form $u(0) = H(\sigma(u), u)$. SDNC’s generalize some famous forms of nonlocal conditions, see also [6]. Nonlocal conditions give more accurate results than the conventional conditions. In [14], Herzallah and Radwan discussed the existence and uniqueness of mild solutions for the semilinear fractional differential equation ${}^C D^\alpha u(t) = Au(t) + f(t, u(t), u(\gamma(t)))$, $t \in J$, $\alpha \in (0, 1)$ subject to SDNC’s where $\gamma(\cdot)$ is a deviated continuous function ($\gamma(t) \leq t$, for all $t \in J$).

Motivated by the previous efforts, we investigate the existence and uniqueness of the mild solution of a new kind of semilinear fractional integrodifferential equations involving Caputo on Riemann–Liouville fractional derivatives of different orders which has the form

$${}^C D^\alpha {}^R D^\beta u(t) = Au(t) + I^\beta F(t, u(t), {}^R D^\beta u(t)), \quad t \in J := [0, b], \quad b > 0; \quad (1.1)$$

subject to the state-dependent nonlocal fractional conditions

$${}^R D^\beta u(t)|_{t=0} = H(\sigma(u), u) \quad \text{and} \quad A(I^{1-\beta} u(t)|_{t=0}) = 0, \quad (1.2)$$

where ${}^C D^\alpha, {}^R D^\beta$ denote Caputo and Riemann–Liouville fractional derivatives of orders $\alpha, \beta \in (0, 1)$, respectively. The operator A is the (infinitesimal) generator of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ of uniformly bounded linear operators on a Banach space X . The functions $F(\cdot), H(\cdot)$, and $\sigma(\cdot)$ are appropriate given satisfying some hypotheses specified later.

The rest of this paper is organized as follows: In Section 2, we display some notations, main definitions, and theorems which are used throughout the paper. In Section 3, we study the existence and uniqueness of the mild solution of Problem (1.1)-(1.2) by the aid of Krasnosel’skii’s fixed point theorem [21] and the contraction mapping principle, respectively. In Section 4, we will derive some estimates on mild solutions. Continuous dependence of mild solutions on initial conditions, and ϵ -approximate mild solutions of (1.1) will be discussed; moreover, we explain the relationships between them and the uniqueness result.

2. Preliminaries

Here we introduce some notations, main definitions, and theorems which are crucial in what follows.

Let $(X, \|\cdot\|_X)$ be a Banach space, $\mathcal{B}(X)$ be the space of all bounded linear operators on X , $C(J, X)$ be the set of all continuous functions $u : J \rightarrow X$ with the norm $\|u\|_C = \sup\{\|u(t)\| : u \in C(J, X), t \in J\}$, $C^n(J, X)$ be the set of all n -differentiable functions with $u^{(n)} \in C(J, X)$, and $AC(J, X)$ be the set of all absolutely continuous functions from J into X . Let $\Gamma(\cdot)$ be the Euler gamma function, φ_α ($\alpha > 0$) be the function $\varphi_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ for $t > 0$ and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and the operator A be the generator of the C_0 -semigroup $\{T(t)\}_{t \geq 0}$ of uniformly bounded linear operators on X , with $M := \sup_{t \in [0, \infty)} \|T(t)\|_{\mathcal{B}(X)} < \infty$. Let $\rho(A)$ be the resolvent set of A , i.e. the set of all complex numbers λ for which $\lambda I - A$ is invertible, and the family $\{(\lambda I - A)^{-1}\}_{\lambda \in \rho(A)}$ of bounded linear operators be the resolvent of A .

Let us recall some basic definitions [10, 18] on fractional calculus.

Definition 2.1 *The fractional integral of order $\alpha > 0$ with the lower limit 0 of the function $u : (0, \infty) \rightarrow X$ is defined by $I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds = (\varphi_\alpha * u)(t)$, $t > 0$. provided that the right-hand side is point-wise defined, where the symbol $*$ stands for the convolution operation.*

Definition 2.2 *The Riemann–Liouville fractional derivative of order α where $0 < \alpha < 1$ with the lower limit 0 of the function $u : (0, \infty) \rightarrow X$ is defined by ${}^R D^\alpha u(t) = D^1 I^{1-\alpha} u(t)$, $t > 0$, $D^1 := d/dt$.*

Definition 2.3 *The Caputo fractional derivative of order α where $0 < \alpha < 1$ with the lower limit 0 of the function $u \in AC(J, X)$ is defined by ${}^C D^\alpha u(t) = I^{1-\alpha} D^1 u(t)$, $t > 0$.*

As a prelude to the presentation of Krasnoselskii’s fixed point theorem [21], we give the following auxiliary concepts.

Definition 2.4 *Let X and Y be Banach spaces, and $\mathcal{P} : X \rightarrow Y$.*

- (1) *A family \mathcal{F} in X is said to be uniformly bounded if there exists a positive constant ℓ such that $|f(t)| \leq \ell$ for all $f \in \mathcal{F}$ and all $t \in J$. \mathcal{F} is called equicontinuous, if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ for all $t_1, t_2 \in J$ with $|t_1 - t_2| < \delta$ and all $f \in \mathcal{F}$. Further, \mathcal{F} is said to be relatively compact if and only if it is uniformly bounded and equicontinuous on J .*
- (2) *The operator \mathcal{P} is said to be a compact operator if for every bounded subset B of X , the image $\mathcal{P}(B)$ of Y is relatively compact.*
- (3) *The operator \mathcal{P} is said to be a completely continuous operator if it is continuous and compact.*

Theorem 2.5 (Krasnoselskii’s fixed point theorem) *Let X be a Banach space, B be a bounded, closed, and convex subset of X and K, Q be operators of B into X such that $Ku + Qv \in B$ for every pair $u, v \in B$. If Q is a contraction and K is completely continuous, then the equation $Ku + Qu = u$ has a solution in B .*

3. Existence and uniqueness of mild solutions

Consider the following assumptions:

- (H_1) $T(t)$ is a compact operator for each $t > 0$;

(H₂) $F : J \times X^2 \rightarrow X$ is continuous, and there exist $p(t), q(t) \in C(J, \mathbb{R}^+)$ such that

$$\|F(t, u_1, v_1) - F(t, u_2, v_2)\| \leq p\|u_1 - u_2\| + q\|v_1 - v_2\|$$

where $p = \max\{p(t) : t \in J\}$, $q = \max\{q(t) : t \in J\}$, and $\max\{F(t, 0, 0) : t \in J\} = 0$;

(H₃) $\sigma : C(J, X) \rightarrow J$ is a Lipschitz function with Lipschitz constant L_σ ;

(H₄) $H : J \times C(J, X) \rightarrow X$ is continuous, and there exist $\kappa, \lambda > 0$ such that $\|H(\sigma(u), u)\| \leq \kappa\|u\|$ and

$$\|H(u_1, u_2) - H(v_1, v_2)\| \leq \|u_1 - v_1\| + \lambda\|u_2 - v_2\|.$$

Consider the one-sided stable probability density

$$\psi_\gamma(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-1-\gamma n} \frac{\Gamma(1+\gamma n)}{n!} \sin(\gamma n \pi), \quad \theta \in (0, \infty), \quad \gamma \in (0, 1). \tag{3.1}$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda \theta} \psi_\gamma(\theta) d\theta = e^{-\lambda^\gamma}, \quad \lambda > 0, \tag{3.2}$$

and consider the probability density function

$$h_\gamma(\theta) = \frac{1}{\alpha} \theta^{-1-1/\gamma} \psi_\gamma(\theta^{-1/\gamma}), \quad \theta \in (0, \infty). \tag{3.3}$$

For any $v \in X$ and $t \geq 0$, define the operators $S_\gamma(t)$ and $R_\gamma(t)$ such that

$$S_\gamma(t)v = \int_0^\infty h_\gamma(\theta) T(t^\gamma \theta)v d\theta \quad \text{and} \quad R_\gamma(t)v = \gamma \int_0^\infty \theta h_\gamma(\theta) T(t^\gamma \theta)v d\theta. \tag{3.4}$$

Based on the following two lemmas, we'll define a mild solution for Problem (1.1)-(1.2).

Lemma 3.1 *Let $\gamma := \alpha + \beta \in (0, 1)$, and $u \in C(J, X)$ be a solution of the semilinear fractional differential equation (1.1). Then $u(t)$ is a solution of the fractional implicit integrodifferential equation*

$${}^R D^\beta u(t) = S_\gamma(t) ({}^R D^\beta u(t)|_{t=0}) + \int_0^t (t-s)^{\gamma-1} R_\gamma(t-s) F(s, u(s), {}^R D^\beta u(s)) ds. \tag{3.5}$$

Proof Suppose that $\gamma := \alpha + \beta \in (0, 1)$, and $u \in C(J, X)$ is a solution of (1.1). Operating I^α on both sides of (1.1), we obtain

$${}^R D^\beta u(t) = {}^R D^\beta u(t)|_{t=0} + \varphi_\alpha(t) * Au(t) + \varphi_\gamma(t) * F(t, u(t), {}^R D^\beta u(t)). \tag{3.6}$$

For $\lambda > 0$, suppose that

$$U(\lambda) = \int_0^\infty e^{-\lambda s} u(s) ds, \tag{3.7}$$

$$V(\lambda) = \int_0^\infty e^{-\lambda s} {}^R D^\beta u(s) ds, \tag{3.8}$$

$$P(\lambda) = \int_0^\infty e^{-\lambda s} F(s, u(s), {}^R D^\beta u(s)) ds. \tag{3.9}$$

Taking Laplace transform for (3.6), we get

$$V(\lambda) = \frac{1}{\lambda} ({}^R D^\beta u(t)|_{t=0}) + \frac{1}{\lambda^\alpha} AU(\lambda) + \frac{1}{\lambda^\gamma} P(\lambda) \tag{3.10}$$

The relation between $AU(\lambda)$ and $AV(\lambda)$ can be inferred as follows:

$$\begin{aligned} V(\lambda) &= \int_0^\infty e^{-\lambda s} {}^R D^\beta u(s) ds \\ &= \int_0^\infty e^{-\lambda s} D^1 I^{1-\beta} u(s) ds \\ &= \lambda \int_0^\infty e^{-\lambda s} I^{1-\beta} u(s) ds - I^{1-\beta} u(t)|_{t=0} \\ &= \lambda \int_0^\infty e^{-\lambda s} (\varphi_{1-\beta} * u)(s) ds - I^{1-\beta} u(t)|_{t=0} \\ &= \lambda \left(\frac{1}{\lambda^{1-\beta}} U(\lambda) \right) - I^{1-\beta} u(t)|_{t=0}. \end{aligned}$$

Then,

$$U(\lambda) = \frac{1}{\lambda^\beta} V(\lambda) + I^{1-\beta} u(t)|_{t=0}.$$

For the continuous function $u(t)$, we have $I^{1-\beta} u(t)|_{t=0} = 0$. Then

$$AU(\lambda) = \frac{1}{\lambda^\beta} AV(\lambda). \tag{3.11}$$

From (3.11) and (3.10), we have

$$\begin{aligned} V(\lambda) &= \frac{1}{\lambda} ({}^R D^\beta u(t)|_{t=0}) + \frac{1}{\lambda^\gamma} AV(\lambda) + \frac{1}{\lambda^\gamma} P(\lambda) \\ &= \lambda^{\gamma-1} (\lambda^\gamma I - A)^{-1} ({}^R D^\beta u(t)|_{t=0}) + (\lambda^\gamma I - A)^{-1} P(\lambda) \\ &= \lambda^{\gamma-1} \int_0^\infty e^{-\lambda^\gamma s} T(s) ({}^R D^\beta u(t)|_{t=0}) ds + \left(\int_0^\infty e^{-\lambda^\gamma s} T(s) ds \right) P(\lambda), \end{aligned} \tag{3.12}$$

where I is the identity operator defined on X . As in [14], by using (3.2), direct calculation gives that

$$\lambda^{\gamma-1} \int_0^\infty e^{-\lambda^\gamma s} T(s) ({}^R D^\beta u(t)|_{t=0}) ds = \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \psi_\gamma(\theta) T\left(\frac{t^\gamma}{\theta^\gamma}\right) ({}^R D^\beta u(t)|_{t=0}) d\theta \right] dt \tag{3.13}$$

and

$$\begin{aligned} &\left(\int_0^\infty e^{-\lambda^\gamma s} T(s) ds \right) P(\lambda) \\ &= \int_0^\infty e^{-\lambda t} \left[\gamma \int_0^t \int_0^\infty \psi_\gamma(\theta) T\left(\frac{(t-s)^\gamma}{\theta^\gamma}\right) \frac{(t-s)^{\gamma-1}}{\theta^\gamma} F(s, u(s), {}^R D^\beta u(s)) d\theta ds \right] dt. \end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), we have

$$V(\lambda) = \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \psi_\gamma(\theta) T\left(\frac{t^\gamma}{\theta^\gamma}\right) ({}^R D^\beta u(t)|_{t=0}) d\theta \right] dt + \int_0^\infty e^{-\lambda t} \left[\gamma \int_0^t \int_0^\infty \psi_\gamma(\theta) T\left(\frac{(t-s)^\gamma}{\theta^\gamma}\right) \frac{(t-s)^{\gamma-1}}{\theta^\gamma} F(s, u(s), {}^R D^\beta u(s)) d\theta ds \right] dt.$$

Inverting the Laplace transform, then

$${}^R D^\beta u(t) = \int_0^\infty \psi_\gamma(\theta) T\left(\frac{t^\gamma}{\theta^\gamma}\right) ({}^R D^\beta u(t)|_{t=0}) d\theta + \gamma \int_0^t \int_0^\infty \psi_\gamma(\theta) T\left(\frac{(t-s)^\gamma}{\theta^\gamma}\right) \frac{(t-s)^{\gamma-1}}{\theta^\gamma} F(s, u(s), {}^R D^\beta u(s)) d\theta ds.$$

Using (3.3), we get

$${}^R D^\beta u(t) = \int_0^\infty h_\gamma(\theta) T(t^\gamma \theta) ({}^R D^\beta u(t)|_{t=0}) d\theta + \gamma \int_0^t \int_0^\infty \theta(t-s)^{\gamma-1} h_\gamma(\theta) T((t-s)^\gamma \theta) F(s, u(s), {}^R D^\beta u(s)) d\theta ds.$$

Applying (3.4), we get (3.5) and the proof is completed. □

Lemma 3.2 *Let $u \in C(J, X)$ be a solution of the fractional implicit integrodifferential equation (3.5). Then $u(t)$ is a solution of the integrodifferential equation*

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) ({}^R D^\beta u(t)|_{t=0}) dz + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s)) ds dz. \tag{3.15}$$

Proof Let $u(t)$ be a solution of (3.5), then

$$D^1 I^{1-\beta} u(t) = S_\gamma(t) ({}^R D^\beta u(t)|_{t=0}) + \int_0^t (t-s)^{\alpha-1} R_\gamma(t-s) F(s, u(s), {}^R D^\beta u(s)) ds.$$

Operating I^1 on both sides,

$$I^1 D^1 I^{1-\beta} u(t) = I^{1-\beta} u(t) - I^{1-\beta} u(t)|_{t=0} = I^1 S_\gamma(t) ({}^R D^\beta u(t)|_{t=0}) + I^1 \int_0^t (t-s)^{\alpha-1} R_\gamma(t-s) F(s, u(s), {}^R D^\beta u(s)) ds,$$

then

$$I^{1-\beta} u(t) = I^1 S_\gamma(t) ({}^R D^\beta u(t)|_{t=0}) + I^1 \int_0^t (t-s)^{\gamma-1} R_\gamma(t-s) F(s, u(s), {}^R D^\beta u(s)) ds.$$

Operating I^β on both sides,

$$\begin{aligned} I^1 u(t) &= I^\beta I^{1-\beta} u(t) \\ &= I^{\beta+1} S_\gamma(t) ({}^R D^\beta u(t)|_{t=0}) + I^{\beta+1} \int_0^t (t-s)^{\gamma-1} R_\gamma(t-s) F(s, u(s), {}^R D^\beta u(s)) ds, \end{aligned}$$

Operating D^1 on both sides,

$$u(t) = I^\beta S_\gamma(t) ({}^R D^\beta u(t)|_{t=0}) + I^\beta \int_0^t (t-s)^{\gamma-1} R_\gamma(t-s) F(s, u(s), {}^R D^\beta u(s)) ds.$$

Therefore, we get the required. □

By summing up the previous discussion, we define a mild solution of Problem (1.1)-(1.2).

Definition 3.3 A function $u \in C(J, X)$ is said to be a mild solution of Problem (1.1)-(1.2), if it satisfies the integrodifferential equation

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) H(\sigma(u), u) dz \\ &+ \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s)) ds dz. \end{aligned} \tag{3.16}$$

$S_\alpha(t)$ and $R_\alpha(t)$ have some basic properties given in the following lemma [9].

Lemma 3.4 The operators $S_\alpha(t)$, $t \geq 0$, and $R_\alpha(t)$, $t \geq 0$ have the following properties:

- 1) For any fixed $t \geq 0$, the operators $S_\alpha(t)$ and $R_\alpha(t)$ are linear and bounded operators, which means that for any $u \in X$,

$$\|S_\alpha(t)u\| \leq M\|u\| \quad \text{and} \quad \|R_\alpha(t)u\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|u\| \quad \text{for all } t \in J. \tag{3.17}$$

- 2) For every $u \in X$, $t \mapsto S_\alpha(t)u$ and $t \mapsto R_\alpha(t)u$ are continuous functions from $[0, \infty)$ into X .
- 3) The operators $S_\alpha(t)$, $t \geq 0$ and $R_\alpha(t)$, $t \geq 0$ are strongly continuous in $[0, \infty)$, which means that for all $u \in X$ and $0 \leq t_1 < t_2 \leq b$, we have

$$\|S_\alpha(t_2)u - S_\alpha(t_1)u\| \rightarrow 0 \quad \text{and} \quad \|R_\alpha(t_2)u - R_\alpha(t_1)u\| \rightarrow 0 \quad \text{as } t_2 \rightarrow t_1.$$

- 4) If $T(t)$ is a compact operator for every $t > 0$, then the operators $S_\alpha(t)$ and $R_\alpha(t)$ are also compact for every $t > 0$.

Lemma 3.5 If

$$\frac{qMb^\gamma}{\Gamma(\gamma+1)} < 1, \tag{3.18}$$

then

$$\|F(\cdot, u(\cdot), {}^R D^\beta u(\cdot))\| \leq \left[p + qM \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \right] \|u\|. \tag{3.19}$$

Proof From (H_2) ,

$$\begin{aligned} \|F(t, u, v)\| &\leq \|F(t, u, v) - F(t, 0, 0)\| + \|F(t, 0, 0)\| \\ &\leq p \|u\| + q \|v\|. \end{aligned} \tag{3.20}$$

Using (3.5) with applying (3.20), (H_4) and Lemma 3.4 (part 1), we have

$$\begin{aligned} \|{}^R D^\beta u(t)\| &\leq \|S_\gamma(t) H(\sigma(u), u)\| + \int_0^t (t-s)^{\gamma-1} \|R_\gamma(t-s) F(s, u(s), {}^R D^\beta u(s))\| ds \\ &\leq \kappa M \|u\| + \frac{\gamma M}{\Gamma(\gamma+1)} \int_0^t (t-s)^{\gamma-1} (p\|u(s)\| + q\|{}^R D^\beta u(s)\|) ds, \end{aligned}$$

and

$$\|{}^R D^\beta u\| \leq M \left[\kappa \|u\| + \frac{b^\gamma}{\Gamma(\gamma+1)} (p\|u\| + q\|{}^R D^\beta u\|) \right].$$

Then,

$$\|{}^R D^\beta u\| \leq M \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \|u\|. \tag{3.21}$$

Using (3.20) and (3.21), we get the required. □

Theorem 3.6 *Let the assumptions (H_1) - (H_4) be satisfied. Then Problem (1.1)-(1.2) has at least one mild solution $u \in C(J, X)$, if*

$$\max \left\{ \frac{qMb^\gamma}{\Gamma(\gamma+1)}, \frac{Mb^\beta(\lambda + L_\sigma)}{\Gamma(\beta+1)}, \frac{Mb^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} \left[p + qM \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \right] + \frac{\kappa Mb^\beta}{\Gamma(\beta+1)} \right\} < 1.$$

Proof Let $B(r)$ be the nonempty, closed, and convex subset of $C(J, X)$ such that

$$B(r) := \{u \in C(J, X) : \|u\| \leq r, r > 0\}.$$

Let $W : C(J, X) \rightarrow C(J, X)$ be the operator given by

$$Wu(t) = Ku(t) + Qu(t)$$

where

$$Ku(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s)) ds dz, \tag{3.22}$$

and

$$Qu(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) H(\sigma(u), u) dz. \tag{3.23}$$

We apply Krasnoselskii's fixed point theorem (Theorem 2.5) in four steps.

Step 1. ($Ku + Qv \in B(r)$ whenever $u, v \in B(r)$)

Let $u, v \in B(r)$. Using (3.22),(3.23) with applying (H_4) and Lemma 3.5, we have

$$\begin{aligned} \|Ku(t) + Qv(t)\| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} \|R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s))\| ds dz \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \|S_\gamma(z) H(\sigma(v), v)\| dz \\ &\leq \frac{\gamma M}{\Gamma(\beta)\Gamma(\gamma+1)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} \|F(s, u(s), {}^R D^\beta u(s))\| ds dz \\ &\quad + \frac{M}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \|H(\sigma(v), v)\| dz \\ &\leq \frac{Mr}{\Gamma(\beta)\Gamma(\gamma+1)} \left[p + qM \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \right] \int_0^t z^\gamma (t-z)^{\beta-1} dz \\ &\quad + \frac{\kappa Mr}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} dz. \end{aligned}$$

One can prove that

$$\int_0^t z^\gamma (t-z)^{\beta-1} dz = \frac{\Gamma(\beta)\Gamma(\gamma+1)}{\Gamma(\gamma+\beta+1)} t^{\gamma+\beta}.$$

Thus, we get

$$\|Ku + Qv\| \leq \frac{Mb^{\gamma+\beta}r}{\Gamma(\gamma+\beta+1)} \left[p + qM \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \right] + \frac{\kappa Mb^\beta r}{\Gamma(\beta+1)} < r.$$

Thus, $Ku + Qv \in B(r)$ whenever $u, v \in B(r)$.

Step 2. (K is continuous)

Let $v_0 \in \mathcal{D}(K)$ and $\|u - v_0\| < \delta$ for all $u \in \mathcal{D}(K)$.

Using (3.22), (3.21), (H_2) , and Lemma 3.4 (part 1), we have

$$\begin{aligned} &\|Ku(t) - Kv_0(t)\| \\ &\leq \int_0^t \frac{(t-z)^{\beta-1}}{\Gamma(\beta)} \int_0^z (z-s)^{\gamma-1} \|R_\gamma(z-s) [F(s, u(s), {}^R D^\beta u(s)) - F(s, v_0(s), {}^R D^\beta v_0(s))]\| ds dz \\ &\leq \frac{\gamma M}{\Gamma(\beta)\Gamma(\gamma+1)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} [p\|u(s) - v_0(s)\| + q\|{}^R D^\beta (u(s) - v_0(s))\|] ds dz \\ &\leq \frac{M}{\Gamma(\beta)\Gamma(\gamma+1)} \left[p\|u - v_0\| + qM \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \|u - v_0\| \right] \int_0^t z^\gamma (t-z)^{\beta-1} dz. \end{aligned}$$

Hence,

$$\|Ku - Kv_0\| \leq \frac{Mb^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} \left[p + qM \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \right] \|u - v_0\|.$$

Then for every $\epsilon > 0$, there is a $\delta > 0$ such that $\|Ku - Kv_0\| < \epsilon$ for all $u \in \mathcal{D}(K)$ satisfying $\|u - v_0\| < \delta$. Thus, K is continuous at $v_0 \in \mathcal{D}(K)$. Since it is valid at every $v \in \mathcal{D}(K)$, K is a continuous operator.

Step 3. (K is compact)

Let $u \in B(r)$. From (3.22), Lemma 3.4 (part 1) and, Lemma 3.5, we have

$$\begin{aligned} \|Ku(t)\| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} \|R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s))\| ds dz \\ &\leq \frac{\gamma M}{\Gamma(\beta) \Gamma(\gamma+1)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} \|F(s, u(s), {}^R D^\beta u(s))\| ds dz \\ &\leq \frac{Mr}{\Gamma(\beta) \Gamma(\gamma+1)} \left[p + qM \left(\frac{\kappa \Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \right] \int_0^t z^\gamma (t-z)^{\beta-1} dz. \end{aligned}$$

Then,

$$\|Ku\| \leq \frac{Mb^{\gamma+\beta}r}{\Gamma(\gamma+\beta+1)} \left[p + qM \left(\frac{\kappa \Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \right].$$

Therefore, the class of functions $\{Ku(t)\}$ is uniformly bounded in $B(r)$.

Now, let $0 \leq t_1 \leq t_2 \leq b$,

$$\begin{aligned} Ku(t_2) - Ku(t_1) &= \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_2-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s)) ds dz \\ &\quad + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s)) ds dz \\ &\quad - \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s)) ds dz; \end{aligned}$$

hence,

$$\begin{aligned} &\|Ku(t_2) - Ku(t_1)\| \\ &\leq \frac{\gamma M}{\Gamma(\beta) \Gamma(\gamma+1)} \int_0^{t_1} [(t_2-z)^{\beta-1} - (t_1-z)^{\beta-1}] \int_0^z (z-s)^{\gamma-1} \|F(s, u(s), {}^R D^\beta u(s))\| ds dz \\ &\quad + \frac{\gamma M}{\Gamma(\beta) \Gamma(\gamma+1)} \int_{t_1}^{t_2} (t_2-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} \|F(s, u(s), {}^R D^\beta u(s))\| ds dz \\ &\leq \frac{Mr}{\Gamma(\beta) \Gamma(\gamma+1)} \left[p + qM \left(\frac{\kappa \Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \right) \right] \times \\ &\quad \left[\int_0^{t_1} z^\gamma [(t_2-z)^{\beta-1} - (t_1-z)^{\beta-1}] dz + \int_{t_1}^{t_2} z^\gamma (t_2-z)^{\beta-1} dz \right]. \end{aligned} \tag{3.24}$$

Since

$$\int_0^{t_1} z^\gamma [(t_2-z)^{\beta-1} - (t_1-z)^{\beta-1}] dz = t_2^{\gamma+\beta} \int_0^{t_1/t_2} x^\gamma (1-x)^{\beta-1} dx - t_1^{\gamma+\beta} \int_0^1 x^\gamma (1-x)^{\beta-1} dx \tag{3.25}$$

and

$$\int_{t_1}^{t_2} z^\gamma (t_2-z)^{\beta-1} dz = t_2^{\gamma+\beta} \int_{t_1/t_2}^1 x^\gamma (1-x)^{\beta-1} dx. \tag{3.26}$$

Then $\|Ku(t_2) - Ku(t_1)\| \rightarrow 0$ as $t_2 \rightarrow t_1$ and the set of functions $\{Ku(t), u \in B(r)\}$ is equicontinuous. By Arzela-Ascoli theorem, $\{Ku(t), u \in B(r)\}$ is relatively compact and K is a compact operator.

Step 4. (Q is a contraction)

From (H_4) ,

$$\|H(\sigma(u), u) - H(\sigma(v), v)\| \leq (\lambda + L_\sigma)\|u - v\|. \tag{3.27}$$

Using (3.23), (3.27) with applying Lemma 3.4 (part 1), we have

$$\begin{aligned} \|Qu(t) - Qv(t)\| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \|S_\gamma(z) [H(\sigma(u), u) - H(\sigma(v), v)]\| dz \\ &\leq \frac{M}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \|H(\sigma(u), u) - H(\sigma(v), v)\| dz \\ &\leq \frac{M(\lambda + L_\sigma)}{\Gamma(\beta)} \|u - v\| \int_0^t (t-z)^{\beta-1} dz. \end{aligned}$$

Then,

$$\|Qu - Qv\| \leq \frac{M(\lambda + L_\sigma)b^\beta}{\Gamma(\beta + 1)} \|u - v\|.$$

Since $\frac{M(\lambda + L_\sigma)T^\beta}{\Gamma(\beta + 1)} < 1$, Q is a contraction operator.

As a consequence of Krasonskii's fixed point theorem, the operator W has at least one fixed point. Therefore, Problem (1.1)-(1.2) has at least one mild solution $u \in B(r)$, and the proof is completed. \square

Theorem 3.7 *Let (H_1) - (H_4) be satisfied. Then Problem (1)-(2) has a unique mild solution $u \in C(J, X)$, if $\frac{qMb^\gamma}{\Gamma(\gamma+1)} < 1$, and there exists $\rho \in (0, 1)$ such that*

$$\rho = \frac{Mb^{\gamma+\beta}}{\Gamma(\gamma + \beta + 1)} \left[p + qM \left(\frac{\kappa\Gamma(\gamma + 1) + pb^\gamma}{\Gamma(\gamma + 1) - qMb^\gamma} \right) \right] \quad \text{and} \quad \max \left\{ \frac{\kappa Mb^\beta}{\Gamma(\beta + 1)}, \frac{M(\lambda + L_\sigma)b^\beta}{\Gamma(\beta + 1)} \right\} < 1 - \rho.$$

Proof Consider the operator $W : C(J, X) \rightarrow C(J, X)$ such that

$$\begin{aligned} Wu(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s)) ds dz \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) H(\sigma(u), u) dz. \end{aligned} \tag{3.28}$$

Let $u, v \in B(r)$. In the view of the contraction mapping principle, the proof will be given in two steps.

Step 1. (W maps $B(r)$ into itself)

As in proving Step 1 in the previous theorem, we get

$$\|Ku\| \leq \frac{Mb^{\gamma+\beta}r}{\Gamma(\gamma + \beta + 1)} \left[p + qM \left(\frac{\kappa\Gamma(\gamma + 1) + pT^\gamma}{\Gamma(\gamma + 1) - qMT^\gamma} \right) \right] + \frac{\kappa Mb^\beta r}{\Gamma(\beta + 1)} < r.$$

Therefore, $WB(r) \subseteq B(r)$.

Step 2. (W is a contraction)

Using (3.28) and Lemma 3.4 (part 1),

$$\begin{aligned} & \|Wu(t) - Wv(t)\| \\ & \leq \int_0^t \frac{(t-z)^{\beta-1}}{\Gamma(\beta)} \int_0^z (z-s)^{\gamma-1} \|R_\gamma(z-s) [F(s, u(s), {}^R D^\beta u(s)) - F(s, v(s), {}^R D^\beta v(s))]\| ds dz \\ & \quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \|S_\gamma(z) [H(\sigma(u), u) - H(\sigma(v), v)]\| dz \\ & \leq \frac{\gamma M}{\Gamma(\beta) \Gamma(\gamma+1)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} \|F(s, u(s), {}^R D^\beta u(s)) - F(s, v(s), {}^R D^\beta v(s))\| ds dz \\ & \quad + \frac{M}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \|H(\sigma(u), u) - H(\sigma(v), v)\| dz. \end{aligned}$$

Using (H_2) , (3.27) and (3.21), we get

$$\begin{aligned} & \|Wu(t) - Wv(t)\| \\ & \leq \frac{\gamma M}{\Gamma(\beta) \Gamma(\gamma+1)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} [p\|u(s) - v(s)\| + q\|{}^R D^\beta (u(s) - v(s))\|] ds dz \\ & \quad + \frac{M(\lambda + L_\sigma)b^\beta}{\Gamma(\beta+1)} \|u - v\| \\ & \leq \frac{M}{\Gamma(\beta) \Gamma(\gamma+1)} \left[p\|u - v\| + qM \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\alpha} \right) \|u - v\| \right] \int_0^t z^\gamma (t-z)^{\beta-1} dz \\ & \quad + \frac{M(\lambda + L_\sigma)b^\beta}{\Gamma(\beta+1)} \|u - v\|. \end{aligned}$$

Then,

$$\|Ku - Kv\| \leq \left\{ \frac{Mb^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} \left[p + qM \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\alpha} \right) \right] + \frac{M(\lambda + L_\sigma)b^\beta}{\Gamma(\beta+1)} \right\} \|u - v\|.$$

Since $\frac{Mb^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} \left[p + qM \left(\frac{\kappa\Gamma(\gamma+1) + pb^\gamma}{\Gamma(\gamma+1) - qMb^\alpha} \right) \right] + \frac{M(\lambda + L_\sigma)b^\beta}{\Gamma(\beta+1)} < 1$, W is a contraction operator and it has a unique fixed point $u \in B(r)$ which is the unique mild solution of Problem (1.1)-(1.2). Therefore, we get the required. \square

4. Some estimates

Here, we derive an estimate on the mild solution, then we discuss the continuous dependence of solutions on initial conditions, and local ϵ -approximate mild solutions.

Theorem 4.1 *Let (H_1) , (H_2) be satisfied, and*

$$\max \left\{ \frac{qMb^\gamma}{\Gamma(\gamma+1)}, \frac{pMb^{\gamma+\beta} \Gamma(\gamma+1)}{\Gamma(\gamma+\beta+1) [\Gamma(\gamma+1) - qMb^\gamma]} \right\} < 1.$$

If $u(t)$ is a mild solution of Problem (1.1) subject to ${}^R D^\beta u(t)|_{t=0} = u_0$ and $A(I^{1-\beta}u(t)|_{t=0}) = 0$, then

$$\|u\| \leq \frac{M^2 b^{\gamma+\beta} \|u_0\| \Gamma(\beta+1) \Gamma(\gamma+1) + M \|u_0\| b^\beta \Gamma(\gamma+\beta+1) [\Gamma(\gamma+1) - qMb^\gamma]}{\Gamma(\beta+1) [(\Gamma(\gamma+1) - qMb^\gamma) \Gamma(\gamma+\beta+1) - pMb^{\gamma+\beta} \Gamma(\gamma+1)]}. \tag{4.1}$$

Proof Similar to Lemma 3.5 and its proof, if $\frac{qMb^\gamma}{\Gamma(\gamma+1)} < 1$, we get

$$\|{}^R D^\beta u\| \leq M \left(\frac{\|u_0\| \Gamma(\gamma+1) + pb^\gamma \|u\|}{\Gamma(\gamma+1) - qMb^\gamma} \right), \tag{4.2}$$

and

$$\|f(\cdot, u(\cdot), {}^R D^\beta u(\cdot))\| \leq \frac{M \|u_0\| \Gamma(\gamma+1)}{\Gamma(\gamma+1) - qMb^\gamma} + p \left(1 + \frac{qb^\gamma M}{\Gamma(\gamma+1) - qMb^\gamma} \right) \|u\|.$$

Using (3.16),

$$\begin{aligned} \|u(t)\| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} \|R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s))\| ds dz \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \|S_\gamma(z) u_0\| dz \\ &\leq \frac{\gamma M}{\Gamma(\beta) \Gamma(\gamma+1)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} \|F(s, u(s), {}^R D^\beta u(s))\| ds dz \\ &\quad + \frac{M \|u_0\|}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} dz \\ &\leq \frac{M}{\Gamma(\beta) \Gamma(\gamma+1)} \left[\frac{M \|u_0\| \Gamma(\gamma+1)}{\Gamma(\gamma+1) - qMb^\gamma} + p \left(1 + \frac{qb^\gamma M}{\Gamma(\gamma+1) - qMb^\gamma} \right) \|u\| \right] \int_0^t z^\gamma (t-z)^{\beta-1} dz \\ &\quad + \frac{Mb^\beta \|u_0\|}{\Gamma(\beta+1)} \\ &\leq \frac{Mb^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} \left[\frac{M \|u_0\| \Gamma(\gamma+1)}{\Gamma(\gamma+1) - qMb^\gamma} + p \left(1 + \frac{qb^\gamma M}{\Gamma(\gamma+1) - qMb^\gamma} \right) \|u\| \right] + \frac{Mb^\beta \|u_0\|}{\Gamma(\beta+1)}. \end{aligned}$$

Simple calculations give the required inequality (4.1). □

Theorem 4.2 Let the assumptions (H_1) and (H_2) be satisfied, and

$$\max \left\{ \frac{qMb^\gamma}{\Gamma(\gamma+1)}, \frac{pMb^{\gamma+\beta}}{\Gamma(1+\gamma+\beta)} \left[1 + \frac{qMb^\gamma}{\Gamma(1+\gamma) - qMb^\gamma} \right] \right\} < 1. \tag{4.3}$$

Suppose that $u_1(t)$ and $u_2(t)$ are two solutions for Problem (1.1) corresponding to ${}^R D^\beta u_1(t)|_{t=0} = u_1^0$, $A(I^{1-\beta}u_1(t)|_{t=0}) = 0$, and ${}^R D^\beta u_2(t)|_{t=0} = u_2^0$, $A(I^{1-\beta}u_2(t)|_{t=0}) = 0$, respectively. Then

$$\|u_1 - u_2\| \leq \frac{\frac{Mb^\beta}{\Gamma(1+\beta)} \left[1 + \frac{qMb^\gamma \Gamma(\gamma+1)}{(\gamma+\beta) \Gamma(\gamma+1) - qMb^\gamma} \right]}{1 - \frac{pMb^{\gamma+\beta}}{\Gamma(1+\gamma+\beta)} \left[1 + \frac{qMb^\gamma}{\Gamma(1+\gamma) - qMb^\gamma} \right]} \|u_1^0 - u_2^0\|. \tag{4.4}$$

Proof Let $u_1(t)$ and $u_2(t)$ be two solutions of Problem (1.1). Then

$$\begin{aligned}
 {}^C D^\alpha {}^R D^\beta u_1(t) &= Au_1(t) + I^\beta F(t, u_1(t), {}^R D^\beta u_1(t)), \quad t \in J; \\
 {}^R D^\beta u_1(t)|_{t=0} &= u_1^0, \quad A(I^{1-\beta} u_1(t)|_{t=0}) = 0,
 \end{aligned}$$

and

$$\begin{aligned}
 {}^C D^\alpha {}^R D^\beta u_2(t) &= Au_2(t) + I^\beta F(t, u_2(t), {}^R D^\beta u_2(t)), \quad t \in J; \\
 {}^R D^\beta u_2(t)|_{t=0} &= u_2^0, \quad A(I^{1-\beta} u_2(t)|_{t=0}) = 0.
 \end{aligned}$$

This implies

$$\begin{aligned}
 u_1(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) u_1^0 dz \\
 &+ \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u_1(s), {}^R D^\beta u_1(s)) ds dz.
 \end{aligned}$$

and

$$\begin{aligned}
 u_2(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) u_2^0 dz \\
 &+ \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u_2(s), {}^R D^\beta u_2(s)) ds dz.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \|u_1(t) - u_2(t)\| &\leq \int_0^t \frac{(t-z)^{\beta-1}}{\Gamma(\beta)} \|S_\gamma(z) (u_1^0 - u_2^0)\| dz \\
 &+ \int_0^t \frac{(t-z)^{\beta-1}}{\Gamma(\beta)} \int_0^z (z-s)^{\gamma-1} \|R_\gamma(z-s) [F(s, u_1(s), {}^R D^\beta u_1(s)) - F(s, u_2(s), {}^R D^\beta u_2(s))]\| ds dz.
 \end{aligned}$$

Applying part 1 of Lemma 3.4 and (H_2) ,

$$\begin{aligned}
 \|u_1(t) - u_2(t)\| &\leq \frac{Mb^\beta}{\Gamma(1+\beta)} \|u_1^0 - u_2^0\| \\
 &+ \frac{\gamma M}{\Gamma(\beta)\Gamma(1+\gamma)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} [p\|u_1(s) - u_2(s)\| + q\|{}^R D^\beta (u_1 - u_2)(s)\|] ds dz. \tag{4.5}
 \end{aligned}$$

From (4.2), if $\frac{qMb^\gamma}{\Gamma(\gamma+1)} < 1$,

$$\|{}^R D^\beta (u_1 - u_2)\| \leq \frac{M\Gamma(1+\gamma)}{\Gamma(\gamma+1) - qMb^\gamma} \|u_1^0 - u_2^0\| + \frac{pMb^\gamma}{\Gamma(\gamma+1) - qMb^\gamma} \|u_1 - u_2\| \tag{4.6}$$

From (4.6) and (4.5),

$$\begin{aligned}
 \|u_1 - u_2\| &\leq \frac{Mb^\beta}{\Gamma(1+\beta)} \left[1 + \frac{qMb^\gamma\Gamma(\gamma+1)}{(\gamma+\beta)[\Gamma(\gamma+1) - qMb^\gamma]} \right] \|u_1^0 - u_2^0\| \\
 &+ \frac{pMb^{\gamma+\beta}}{\Gamma(1+\gamma+\beta)} \left[1 + \frac{qMb^\gamma}{\Gamma(1+\gamma) - qMb^\gamma} \right] \|u_1 - u_2\|.
 \end{aligned}$$

Therefore, we can easily get Inequality (4.4). □

Remark 4.3 Inequality (4.4) shows continuous dependence of solutions of Problem (1.1) on initial conditions as well as it gives the uniqueness which follows by putting $u_1^0 = u_2^0$.

Definition 4.4 A solution of the integrodifferential inequality

$$\left\| u(t) - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) ({}^R D^\beta u(t)|_{t=0}) dz - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u(s), {}^R D^\beta u(s)) ds dz \right\| \leq \epsilon \quad (4.7)$$

is called an ϵ -approximate mild solution of Problem (1.1).

Remark 4.5 From (4.7), if $\epsilon = 0$, then $u(t)$ is a solution of the integrodifferential equation

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) ({}^R D^\beta u(t)|_{t=0}) dz + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\alpha(z-s) F(s, u(s), {}^R D^\beta u(s)) ds dz,$$

which is a mild solution of (1.1).

Theorem 4.6 Let the assumptions (H_1) , (H_2) , and (4.3) be satisfied.

Suppose that $u_1(t)$ and $u_2(t)$ are two ϵ -approximate mild solutions for Problem (1.1) corresponding to ${}^R D^\beta u_1(t)|_{t=0} = u_1^0$, $A(I^{1-\beta} u_1(t)|_{t=0}) = 0$, and ${}^R D^\beta u_2(t)|_{t=0} = u_2^0$, $A(I^{1-\beta} u_2(t)|_{t=0}) = 0$, respectively. Then

$$\|u_1 - u_2\| \leq \frac{\epsilon_1 + \epsilon_2 + \frac{Mb^\beta}{\Gamma(1+\beta)} \left[1 + \frac{qMb^\gamma \Gamma(\gamma+1)}{(\gamma+\beta)[\Gamma(\gamma+1)-qMb^\gamma]} \right] \|u_1^0 - u_2^0\|}{1 - \frac{pMb^{\gamma+\beta}}{\Gamma(1+\gamma+\beta)} \left[1 + \frac{qMb^\gamma}{\Gamma(1+\gamma)-qMb^\gamma} \right]}. \quad (4.8)$$

Proof Let $u_1(t)$ and $u_2(t)$ be two ϵ -approximate mild solutions of Problem (1.1). Then,

$$\left\| u_1(t) - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) u_1^0 dz - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u_1(s), {}^R D^\beta u_1(s)) ds dz \right\| \leq \epsilon_1,$$

and

$$\left\| u_2(t) - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) u_2^0 dz - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u_2(s), {}^R D^\beta u_2(s)) ds dz \right\| \leq \epsilon_2.$$

We know that for all $x_1, x_2 \in X$, we have $\|x_1\| - \|x_2\| \leq \|x_1 - x_2\| \leq \|x_1\| + \|x_2\|$.

Thus, let $x_1 = u_1(t) - u_2(t)$, and

$$x_2 = \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) (u_1^0 - u_2^0) dz + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) [F(s, u_1(s), {}^R D^\beta u_1(s)) - F(s, u_2(s), {}^R D^\beta u_2(s))] ds dz.$$

Then,

$$\begin{aligned} & \|u_1(t) - u_2(t)\| - \left\| \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) (u_1^0 - u_2^0) dz + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) [F(s, u_1(s), {}^R D^\beta u_1(s)) - F(s, u_2(s), {}^R D^\beta u_2(s))] ds dz \right\| \\ & \leq \left\| \left[u_1(t) - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) u_1^0 dz - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u_1(s), {}^R D^\beta u_1(s)) ds dz \right] \right. \\ & \quad \left. - \left[u_2(t) - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) u_2^0 dz - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u_2(s), {}^R D^\beta u_2(s)) ds dz \right] \right\| \\ & \leq \left\| u_1(t) - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) u_1^0 dz - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u_1(s), {}^R D^\beta u_1(s)) ds dz \right\| \\ & \quad + \left\| u_2(t) - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) u_2^0 dz - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) F(s, u_2(s), {}^R D^\beta u_2(s)) ds dz \right\| \\ & \leq \epsilon_1 + \epsilon_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u_1(t) - u_2(t)\| & \leq \epsilon_1 + \epsilon_2 + \left\| \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} S_\gamma(z) (u_1^0 - u_2^0) dz \right\| \\ & \quad + \left\| \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} \int_0^z (z-s)^{\gamma-1} R_\gamma(z-s) [F(s, u_1(s), {}^R D^\beta u_1(s)) - F(s, u_2(s), {}^R D^\beta u_2(t))] ds dz \right\| \end{aligned}$$

As in proving Theorem 4.2, One can get (4.8). □

Remark 4.7 If $\epsilon_1 = \epsilon_2 = 0$, we deduce that $u_1(t)$ and $u_2(t)$ are mild solutions for Problem (1.1), and then inequality (4.8) reduced to inequality (4.4) which gives continuous dependence of mild solutions on initial conditions for Problem (1.1).

Remark 4.8 *Inequality(4.8) proves the uniqueness of the mild solution of Problem (1.1), if $\epsilon_1 = \epsilon_2 = 0$ and $u_1^0 = u_2^0$.*

5. Conclusion

The existence of at least one continuous mild solution to a class (involves both Caputo and Riemann–Liouville fractional derivatives) of semilinear fractional integrodifferential equations with nonlocal conditions, was investigated. Continuous dependence of solutions on initial conditions and local ϵ -approximate mild solution of the considered problem were discussed.

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