Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2977 – 2997
© TÜBİTAK
doi:10.55730/1300-0098.3313

Classical solutions for 1-dimensional and 2-dimensional Boussinesq equations

Svetlin GEORGIEV[1], **Aissa BOUKAROU**[2,*], **Khaled ZENNIR**[3]
[1]Department of Mathematics, Faculty of Mathematics and Informatics, University of Sofia, Bulgaria
[2]Department of Mathematics, Faculty of Mathematics and Informatics, University of Ghardaia, Algeria
[3]Department of Mathematics, College of Sciences and Arts, Qassim University, Ar-Rass, Saudi Arabia

Received: 09.01.2022 • **Accepted/Published Online:** 03.08.2022 • **Final Version:** 05.09.2022

Abstract: In this article we investigate the IVPs for 1-dimensional and 2-dimensional Boussinesq equations. A new topological approach is applied to prove the existence of at least one classical solution and at least two nonnegative classical solutions for the considered IVPs. The arguments are based upon recent theoretical results.

Key words: Boussinesq equation, existence, classical solution

1. Introduction

In this paper, we investigate the IVPs for 1-dimensional and 2-dimensional Boussinesq equation

$$\begin{aligned} u_{tt} &= u_{xx} + \beta u_{xxxx} + \left(u^2\right)_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) &= u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} u_{tt} + \left(u_{xx} + u^2 - u\right)_{xx} - u_{yy} &= 0, \quad t > 0, \quad (x,y) \in \mathbb{R}^2, \\ u(0,x,y) = v_0(x,y), \quad u_t(0,x,y) &= v_1(x,y), \quad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{1.2}$$

respectively, where

(H1) $u_0, u_1 \in \mathcal{C}^4(\mathbb{R})$, $0 \leq u_0, u_1 \leq B$ on $\mathbb{R}$, $\beta = \pm 1$,

(G1) $v_0, v_1 \in \mathcal{C}^4(\mathbb{R}^2)$, $0 \leq v_0, v_1 \leq B$ on $\mathbb{R}^2$,

for some positive constant B.

The local well-posedness for dispersive equations with quadratic nonlinearities has been extensively studied in Sobolev spaces with negative indices. The proof of these results is based on the Fourier restriction norm approach introduced by Bourgain [2, 3].

In [7], Luiz Farah proved that the good Boussinesq equation with data in $H^s(\mathbb{R})$, $s > -1/4$, is well-posed. Esfahani and Farah [6] proved local well-posedness in $H^s(\mathbb{R})$ with $s > -1/2$ for the sixth-order Boussinesq

*Correspondence: boukarouaissa@gmail.com
2010 *AMS Mathematics Subject Classification:* 37C25, 47H10, 58J20

equation

$$\begin{cases} u_{tt} = u_{xx} + \beta u_{xxxx} + u_{xxxxxx} + \left(u^2\right)_{xx}, & x \in \mathbb{R}, t \geq 0 \\ u(0,x) = \varphi(x); \quad u_t(0,x) = \psi_x(x). \end{cases} \tag{1.3}$$

The well-posedness of IVP (1.3) on a periodic domain is shown in [13] for $s > -\frac{1}{2}$.

The main aim of this paper is to investigate the IVPs (1.1) and (1.2) for the existence of at least one classical solution and the existence of at least two classical solutions. We propose a new approach for investigating for the existence of classical solutions. This approach can be applied to other classes IVPs for ordinary and partial differential equations.

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3 we prove the existence of at least one classical solution and the existence of at least two nonnegative classical solutions for the IVP (1.1). In Section 4 we prove the existence of at least one classical solution and the existence of at least two nonnegative classical solutions for the IVP (1.2).

2. Preliminary results

To prove our existing result we will use the following fixed point theorem. Its proof can be found in [8].

Theorem 2.1 *Let $\epsilon > 0$, $B > 0$, E be a Banach space and $X = \{x \in E : \|x\| \leq B\}$. Let also, $Tx = -\epsilon x$, $x \in X$, $S : X \to E$ is a continuous, $(I - S)(X)$ resides in a compact subset of E and*

$$\{x \in E : x = \lambda(I - S)x, \quad \|x\| = B\} = \emptyset \tag{2.1}$$

for any $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then there exists $x^* \in X$ so that

$$Tx^* + Sx^* = 0.$$

Let X be a real Banach space.

Definition 2.2 *A mapping $K : X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for l-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.3 *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0, \infty)$ is defined by*

$$\alpha(Y) = \inf \left\{ \delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad and \quad diam(Y_j) \leq \delta, \quad j \in \{1, \ldots, m\} \right\},$$

where $diam(Y_j) = \sup\{\|x - y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1, \ldots, m\}$.

For the main properties of the measure of noncompactness, we refer the reader to [1].

Definition 2.4 *A mapping $K : X \to X$ is said to be l-set contraction if it is continuous, bounded, and there exists a constant $l \geq 0$ such that*

$$\alpha(K(Y)) \leq l\alpha(Y),$$

for any bounded set $Y \subset X$. The mapping K is said to be a strict set contraction if $l < 1$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction (see [5], pp. 264).

Definition 2.5 *Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that*

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X,$$

for any $x, y \in X$.

Definition 2.6 *A closed, convex set $\mathcal{P}$ in X is said to be cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies that $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$.

Lemma 2.7 *Let X be a closed convex subset of a Banach space E, $\mathcal{P}$ be a cone in E and $U \subset X$ a bounded open subset with $0 \in U$. Assume that there exists $\varepsilon > 0$ small enough and that $K : \overline{U} \to X$ is a strict k-set contraction that satisfies the boundary condition:*

$$Kx \notin \{x,\ \lambda x\} \text{ for all } x \in \partial U \text{ and } \lambda \geq 1 + \varepsilon.$$

Then the fixed point index $i\,(K, U, X) = 1$.

Proof Consider the homotopic deformation $H : [0, 1] \times \overline{U} \to X$ defined by

$$H(t, x) = \frac{1}{\varepsilon + 1}\, tKx.$$

The operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t, .)$ is a strict set contraction for each $t \in [0, 1]$. In addition, $H(t, .)$ has no fixed point on ∂U. On the contrary,

• If $t = 0$, there exists some $x_0 \in \partial U$ such that $x_0 = 0$, contradicting $x_0 \in U$.

• If $t \in (0, 1]$, there exists some $x_0 \in \mathcal{P} \cap \partial U$ such that $\frac{1}{\varepsilon+1}\, tKx_0 = x_0$; then $Kx_0 = \frac{1+\varepsilon}{t} x_0$ with $\frac{1+\varepsilon}{t} \geq 1 + \varepsilon$, contradicting the assumption. From the invariance under homotopy and the normalization properties of the index, we deduce

$$i\,(\frac{1}{\varepsilon + 1}\, K, U, X) = i\,(0, U, X) = 1.$$

Now, we show that

$$i\,(K, U, X) = i\,(\frac{1}{\varepsilon + 1}\, K, U, X).$$

We have

$$\frac{1}{\varepsilon + 1} Kx \neq x,\ \forall\, x \in \partial U. \tag{2.2}$$

Then there exists $\gamma > 0$ such that

$$\|x - \frac{1}{\varepsilon + 1} Kx\| \geq \gamma, \ \forall x \in \partial U.$$

On the other hand, we have $\frac{1}{\epsilon+1}Kx \to Kx$ as $\epsilon \to 0$, for $x \in \overline{U}$. So, for ε small enough, we have

$$\|Kx - \frac{1}{\varepsilon + 1} Kx\| < \frac{\gamma}{2}, \ \forall x \in \partial U.$$

Define the convex deformation $G : [0,1] \times \overline{U} \to X$ by

$$G(t,x) = tKx + (1-t)\frac{1}{\varepsilon + 1} Kx.$$

The operator G is continuous and uniformly continuous in t for each x, and the mapping $G(t,.)$ is a strict set contraction for each $t \in [0,1]$ (since $t + \frac{1}{\varepsilon+1}(1-t) < t + 1 - t = 1$). In addition, $G(t,.)$ has no fixed point on ∂U. In fact, for all $x \in \partial U$, we have

$$\begin{array}{rcl} \|x - G(t,x)\| & = & \|x - tKx - (1-t)\frac{1}{\varepsilon+1}Kx\| \\ & \geq & \|x - \frac{1}{\varepsilon+1}Kx\| - t\|Kx - \frac{1}{\varepsilon+1}Kx\| \\ & > & \gamma - \frac{\gamma}{2} > \frac{\gamma}{2}. \end{array}$$

Then our claim follows from the invariance property by homotopy of the index.

□

Proposition 2.8 *Let $\mathcal{P}$ be a cone in a Banach space E. Let also, U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $S : \overline{U} \to E$ is a l-set contraction with $0 \leq l < h - 1$, and $S(\overline{U}) \subset (I - T)(\Omega)$. If there exists $\varepsilon \geq 0$ such that*

$$Sx \notin \{(I-T)(x), \quad (I-T)(\lambda x)\} \ \textit{for all } x \in \partial U \cap \Omega \ \textit{ and } \ \lambda \geq 1 + \varepsilon,$$

then the fixed point index $i_* (T + S, U \cap \Omega, \mathcal{P}) = 1$.

Proof The mapping $(I-T)^{-1}S : \overline{U} \to \mathcal{P}$ is a strict set contraction and it is readily seen that the following condition is satisfied

$$(I-T)^{-1}Sx \notin \{x, \lambda x\} \quad \text{for} \quad \text{all} \quad x \in \partial U \quad \text{and} \quad \lambda \geq 1 + \epsilon.$$

Our claim then follows from the definition of i_* and Lemma 2.7. □

The following result will be used to prove our main result.

Theorem 2.9 *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a k-set contraction with $0 \leq k < h - 1$ and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx \neq (I-T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exists $\epsilon \geq 0$ such that $Sx \neq (I-T)(\lambda x)$, for all $\lambda \geq 1 + \epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii) $Sx \neq (I-T)(x-\lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T+S$ *has at least two nonzero fixed points* $x_1, x_2 \in \mathcal{P}$ *such that*

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega,$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof If $Sx = (I-T)x$ for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_2 \cap \Omega$ of the operator $T+S$. Suppose that $Sx \neq (I-T)x$ for any $x \in \partial U_2 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_1 \cap \Omega$ and $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$, otherwise the conclusion has been proved. By [4, Proposition 2.16] and Proposition 2.8, we have

$$i_*\left(T+S, U_1 \cap \Omega, \mathcal{P}\right) = i_*\left(T+S, U_3 \cap \Omega, \mathcal{P}\right) = 0 \text{ and } i_*\left(T+S, U_2 \cap \Omega, \mathcal{P}\right) = 1.$$

The additivity property of the index yields

$$i_*\left(T+S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{P}\right) = 1 \text{ and } i_*\left(T+S, (U_3 \setminus \overline{U}_2) \cap \Omega, \mathcal{P}\right) = -1.$$

Consequently, by the existence property of the index, $T+S$ has at least two fixed points $x_1 \in (U_2 \setminus U_1) \cap \Omega$ and $x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$. □

3. The 1-dimensional Boussinesq equation

In this section, we will investigate the IVP (1.1).

3.1. Auxiliary results

Let $X = \mathcal{C}^2([0,\infty), \mathcal{C}^4(\mathbb{R}))$ be endowed with the norm

$$\begin{aligned} \|u\| &= \max\Bigg\{ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_t(t,x)|, \\ &\quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_{tt}(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_x(t,x)|, \\ &\quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_{xx}(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_{xxx}(t,x)|, \\ &\quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_{xxxx}(t,x)| \Bigg\}, \end{aligned}$$

provided it exists. For $u \in X$, define the operator

$$\begin{aligned} S_1 u(t,x) &= u(t,x) - u_0(x) - t u_1(x) \\ &\quad - \int_0^t (t-t_1)\Big(u_{xx}(t_1,x) + \beta u_{xxxx}(t_1,x) \\ &\quad + 2\left(u_x(t_1,x)\right)^2 + 2u(t_1,x)u_{xx}(t_1,x) \Big) dt_1, \end{aligned}$$

$(t,x) \in [0,\infty) \times \mathbb{R}$.

Lemma 3.1 *Suppose that* $(H1)$ *holds. Let* $u \in X$ *satisfies the equation*

$$S_1 u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{3.1}$$

Then u *is a solution to the IVP* (1.1).

Proof We have

$$\begin{aligned} 0 &= u(t,x) - u_0(x) - tu_1(x) \\ &\quad - \int_0^t (t-t_1)\Big(u_{xx}(t_1,x) + \beta u_{xxxx}(t_1,x) \\ &\quad + 2\left(u_x(t_1,x)\right)^2 + 2u(t_1,x)u_{xx}(t_1,x) \Big) dt_1, \end{aligned} \tag{3.2}$$

$(t,x) \in [0,\infty) \times \mathbb{R}$, which we differentiate with respect to t and we get

$$\begin{aligned} 0 &= u_t(t,x) - u_1(x) \\ &\quad - \int_0^t \Big(u_{xx}(t_1,x) + \beta u_{xxxx}(t_1,x) \\ &\quad + 2\left(u_x(t_1,x)\right)^2 + 2u(t_1,x)u_{xx}(t_1,x) \Big) dt_1, \end{aligned} \tag{3.3}$$

$(t,x) \in [0,\infty) \times \mathbb{R}$. We differentiate (3.3) with respect to t and we find

$$\begin{aligned} 0 &= u_{tt}(t,x) - u_{xx}(t,x) - \beta u_{xxxx}(t,x) \\ &\quad - 2\left(u_x(t,x)\right)^2 - 2u(t,x)u_{xx}(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

i.e. u satisfies the first equation of (1.1). Now, we put $t = 0$ into (3.2) and (3.3) and we arrive at

$$0 = u(0,x) - u_0(x), \quad 0 = u_t(0,x) - u_1(x), \quad x \in \mathbb{R}.$$

This completes the proof. □

Let $B_1 = 4B^2$.

Lemma 3.2 *Suppose that* $(H1)$ *holds. If* $u \in X$, $\|u\| \leq B$, *then*

$$|S_1 u(t,x)| \leq (1 + t + t^2) B_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Proof We have

$$\begin{aligned}
|S_1u(t,x)| &= \Bigg| u(t,x) - u_0(x) - tu_1(x) \\
&\quad - \int_0^t (t-t_1)\Bigg(u_{xx}(t_1,x) + \beta u_{xxxx}(t_1,x) \\
&\quad + 2\left(u_x(t_1,x)\right)^2 + 2u(t_1,x)u_{xx}(t_1,x)\Bigg)dt_1\Bigg| \\
&\le |u(t,x)| + |u_0(x)| + t|u_1(x)| \\
&\quad + \int_0^t (t-t_1)\Bigg(|u_{xx}(t_1,x)| + |u_{xxxx}(t_1,x)| \\
&\quad + 2\left|u_x(t_1,x)\right|^2 + 2|u(t_1,x)||u_{xx}(t_1,x)|\Bigg)dt_1 \\
&\le 2B + tB + \int_0^t (t-t_1)(2B+4B^2)dt_1 \\
&\le B_1(1+t+t^2), \quad (t,x)\in[0,\infty)\times\mathbb{R}.
\end{aligned}$$

This completes the proof. □

Below, suppose that

(H2) there exists a function $g \in \mathcal{C}([0,\infty)\times\mathbb{R})$ so that $g > 0$ on $(0,\infty)\times(\mathbb{R}\backslash\{0\})$, $g(0,x) = g(t,0) = 0$, $(t,x)\in[0,\infty)\times\mathbb{R}$, and a positive constant A such that

$$4!2^8(1+t+t^2+t^3+t^4)(1+|x|+x^2+|x|^3+x^4)$$

$$\times \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \le A.$$

In the end of this section we will give an example for such function g and such constant A. For $u \in X$, define the operator

$$S_2u(t,x) = \int_0^t \int_0^x (t-t_1)^2(x-x_1)^4 g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Lemma 3.3 *Suppose that* $(H1)$ *and* $(H2)$ *hold. If* $u \in X$ *satisfies the equation*

$$S_2u(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}, \tag{3.4}$$

then u *is a solution to the IVP* (1.1).

Proof We differentiate three times with respect to t and five times with respect to x and we find

$$g(t,x)S_1u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Hence,

$$S_1u(t,x) = 0, \quad (t,x) \in (0,\infty) \times (\mathbb{R}\backslash\{0\}).$$

Since $S_1u(\cdot,\cdot)$ is a continuous function on $[0,\infty) \times \mathbb{R}$, we get

$$\begin{aligned} 0 &= \lim_{t\to 0} S_1u(t,x) = S_1u(0,x) \\ &= \lim_{x\to 0} S_1u(t,x) = S_1u(t,0), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

Thus,

$$S_1u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Now, applying Lemma 3.1, we get the desired result. □

Lemma 3.4 *Suppose that* $(H1)$ *and* $(H2)$ *hold. If* $u \in X$, $\|u\| \leq B$, *then*

$$\|S_2u\| \leq AB_1.$$

Proof We will use the inequality $(x+y)^p \leq 2^p(x^p + y^p)$, $o > 0, x > 0, y > 0$. We have

$$\begin{aligned} |S_2u(t,x)| &= \left| \int_0^t \int_0^x (t-t_1)^2(x-x_1)^4 g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1 \right| \\ &\leq \int_0^t \left| \int_0^x (t-t_1)^2(x-x_1)^4 g(t_1,x_1)|S_1u(t_1,x_1)|dx_1 \right| dt_1 \\ &\leq B_1 2^8 |x|^4 t^2 \int_0^t \left| \int_0^x g(t_1,x_1)(1+t_1+t_1^2)dx_1 \right| dt_1 \\ &\leq B_1 2^8 |x|^4 (1+t+t^2+t^3+t^4) \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \\ &\leq AB_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned}
|(S_2u)_t(t,x)| &= 2\left|\int_0^t\int_0^x (t-t_1)(x-x_1)^4 g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\right| \\
&\leq 2\int_0^t\left|\int_0^x (t-t_1)(x-x_1)^4 g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\right|dt_1 \\
&\leq B_1 2^8|x|^4 t\int_0^t\left|\int_0^x g(t_1,x_1)(1+t_1+t_1^2)dx_1\right|dt_1 \\
&\leq B_1 2^8|x|^4(1+t+t^2+t^3)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \\
&\leq AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
|(S_2u)_{tt}(t,x)| &= 2\left|\int_0^t\int_0^x (x-x_1)^4 g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\right| \\
&\leq 2\int_0^t\left|\int_0^x (x-x_1)^4 g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\right|dt_1 \\
&\leq B_1 2^6|x|^4\int_0^t\left|\int_0^x g(t_1,x_1)(1+t_1+t_1^2)dx_1\right|dt_1 \\
&\leq B_1 2^6|x|^4(1+t+t^2)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \\
&\leq AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
|(S_2u)_x(t,x)| &= 4\left|\int_0^t\int_0^x (t-t_1)^2(x-x_1)^3 g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\right| \\
&\leq 4\int_0^t\left|\int_0^x (t-t_1)^2(x-x_1)^3 g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\right|dt_1 \\
&\leq B_1 2^8|x|^3 t^2\int_0^t\left|\int_0^x g(t_1,x_1)(1+t_1+t_1^2)dx_1\right|dt_1 \\
&\leq B_1 2^8|x|^3(1+t+t^2+t^3+t^4)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \\
&\leq AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
|(S_2u)_{xx}(t,x)| &= 12\left|\int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\right| \\
&\le 12\int_0^t\left|\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\right|dt_1 \\
&\le 12B_1 2^6 x^2 t^2\int_0^t\left|\int_0^x g(t_1,x_1)(1+t_1+t_1^2)dx_1\right|dt_1 \\
&\le 12B_1 2^6 x^2(1+t+t^2+t^3+t^4)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
|(S_2u)_{xxx}(t,x)| &= 24\left|\int_0^t\int_0^x (t-t_1)^2(x-x_1) g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\right| \\
&\le 24\int_0^t\left|\int_0^x (t-t_1)^2(x-x_1) g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\right|dt_1 \\
&\le 24B_1 2^8 |x| t^2\int_0^t\left|\int_0^x g(t_1,x_1)(1+t_1+t_1^2)dx_1\right|dt_1 \\
&\le 24B_1 2^8 |x|(1+t+t^2+t^3+t^4)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
|(S_2u)_{xxxx}(t,x)| &= 24\left|\int_0^t\int_0^x (t-t_1)^2 g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\right| \\
&\le 24\int_0^t\left|\int_0^x (t-t_1)^2 g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\right|dt_1 \\
&\le 24B_1 2^8 t^2\int_0^t\left|\int_0^x g(t_1,x_1)(1+t_1+t_1^2)dx_1\right|dt_1 \\
&\le 24B_1 2^8(1+t+t^2+t^3+t^4)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}.
\end{aligned}$$

Consequently,

$$\|S_2 u\| \leq AB_1.$$

This completes the proof. □

3.2. Main results

Our first main result is read as follows.

Theorem 3.5 *Suppose that* $(H1)$ *and* $(H2)$ *hold. Then the IVP* (1.1) *has at least one solution in* $\mathcal{C}^2([0,\infty),\mathcal{C}^4(\mathbb{R}))$.

Proof Below, suppose

(H3) $\epsilon \in (0,1)$, A and B satisfy the inequalities $\epsilon B_1(1+A) < B$ and $AB_1 < B$.

Let $\widetilde{\widetilde{\widetilde{Y}}}$ denote the set of all equicontinuous families in X with respect to the norm $\|\cdot\|$. Let also, $\widetilde{\widetilde{Y}} = \overline{\widetilde{\widetilde{\widetilde{Y}}}}$ be the closure of $\widetilde{\widetilde{\widetilde{Y}}}$, $\widetilde{Y} = \widetilde{\widetilde{Y}} \cup \{u_0, u_1\}$,

$$Y = \{u \in \widetilde{Y} : \|u\| \leq B\}.$$

Note that Y is a compact set in X. For $u \in X$, define the operators

$$\begin{aligned} Tu(t,x) &= -\epsilon u(t,x), \\ Su(t,x) &= u(t,x) + \epsilon u(t,x) + \epsilon S_2 u(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

For $u \in Y$, using Lemma 3.4, we have

$$\begin{aligned} \|(I-S)u\| &= \|\epsilon u - \epsilon S_2 u\| \\ &\leq \epsilon\|u\| + \epsilon\|S_2 u\| \\ &\leq \epsilon B_1 + \epsilon A B_1 \\ &= \epsilon B_1(1+A) \\ &< B. \end{aligned}$$

Thus, $S : Y \to E$ is continuous and $(I-S)(Y)$ resides in a compact subset of E. Now, suppose that there is a $u \in E$ so that $\|u\| = B$ and

$$u = \lambda(I-S)u,$$

or

$$\frac{1}{\lambda}u = (I-S)u = -\epsilon u - \epsilon S_2 u,$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right) u = -\epsilon S_2 u,$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2u\| \le AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right) B = \left(\frac{1}{\lambda} + \epsilon\right) \|u\| = \epsilon \|S_2 u\| < \epsilon B,$$

which is a contradiction. Hence and Theorem 2.1, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned} u^*(t,x) &= Tu^*(t,x) + Su^*(t,x) \\ &= -\epsilon u^*(t,x) + u^*(t,x) + \epsilon u^*(t,x) + \epsilon S_2 u^*(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

whereupon

$$0 = S_2 u^*(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

From here and from Lemma 3.3, it follows that u is a solution to the IVP (1.1). This completes the proof. □

Our next result is as follows.

Theorem 3.6 *Suppose that* $(H1)$ *and* $(H2)$ *hold. Then the IVP* (1.1) *has at least two nonnegative solutions in* $\mathcal{C}^2([0,\infty), \mathcal{C}^4(\mathbb{R}))$.

Proof Suppose

(H4) Let $m > 0$ be large enough and A, B, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L,$$

$$AB_1 < \frac{L}{5}.$$

Let

$$\widetilde{P} = \{u \in X : u \ge 0 \quad \text{on} \quad [0,\infty) \times \mathbb{R}\}.$$

With P we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators

$$\begin{aligned} T_1 v(t,x) &= (1 + m\epsilon) v(t,x) - \epsilon \frac{L}{10}, \\ S_3 v(t,x) &= -\epsilon S_2 v(t,x) - m\epsilon v(t,x) - \epsilon \frac{L}{10}, \end{aligned}$$

$t \in [0, \infty)$, $x \in \mathbb{R}$. Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1.1). Define

$$\begin{aligned} U_1 &= P_r = \{v \in P : \|v\| < r\}, \\ U_2 &= P_L = \{v \in P : \|v\| < L\}, \\ U_3 &= P_{R_1} = \{v \in P : \|v\| < R_1\}, \\ R_2 &= R_1 + \frac{A}{m} B_1 + \frac{L}{5m}, \\ \Omega &= \overline{P}_{R_2} = \{v \in P : \|v\| \le R_2\}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have

$$\|T_1 v_1 - T_1 v_2\| = (1 + m\varepsilon)\|v_1 - v_2\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

2. For $v \in \overline{P}_{R_1}$, we get

$$\begin{aligned} \|S_3 v\| &\le \varepsilon\|S_2 v\| + m\varepsilon\|v\| + \varepsilon\frac{L}{10} \\ &\le \varepsilon\left(AB_1 + mR_1 + \frac{L}{10}\right). \end{aligned}$$

Therefore $S_3(\overline{P}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{P}_{R_1} \to X$ is continuous, we have that $S_3(\overline{P}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{P}_{R_1} \to X$ is a 0-set contraction.

3. Let $v_1 \in \overline{P}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m} S_2 v_1 + \frac{L}{5m}.$$

Note that $S_2 v_1 + \frac{L}{5} \ge 0$ on $[0, \infty) \times \mathbb{R}^2$. We have $v_2 \ge 0$ on $[0, \infty) \times \mathbb{R}^2$ and

$$\begin{aligned} \|v_2\| &\le \|v_1\| + \frac{1}{m}\|S_2 v_1\| + \frac{L}{5m} \\ &\le R_1 + \frac{A}{m} B_1 + \frac{L}{5m} \\ &= R_2. \end{aligned}$$

Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2 v_1 - \varepsilon\frac{L}{10} - \varepsilon\frac{L}{10}$$

or

$$\begin{aligned}(I-T_1)v_2 &= -\varepsilon m v_2 + \varepsilon\frac{L}{10}\\ &= S_3 v_1.\end{aligned}$$

Consequently $S_3(\overline{P}_{R_1}) \subset (I-T_1)(\Omega)$.

4. Assume that for any $v_0 \in P^*$ there exist $\lambda > 0$ and $z \in \partial P_r \cap (\Omega + \lambda v_0)$ or $z \in \partial P_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$S_3 z = (I-T_1)(z - \lambda v_0).$$

Then

$$-\epsilon S_2 z - m\epsilon z - \epsilon\frac{L}{10} = -m\epsilon(z-\lambda v_0) + \epsilon\frac{L}{10},$$

or

$$-S_2 z = \lambda m v_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 z\| = \left\|\lambda m v_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

This is a contradiction.

5. Suppose that for any $\epsilon_1 \geq 0$ small enough there exist a $x_1 \in \partial P_L$ and $\lambda_1 \geq 1+\epsilon_1$ such that $\lambda_1 x_1 \in \overline{P}_{R_1}$ and

$$S_3 x_1 = (I-T_1)(\lambda_1 x_1). \tag{3.5}$$

In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $x_1 \in \partial P_L$, $\lambda_1 x_1 \in \overline{P}_{R_1}$, $\lambda_1 \geq 1+\epsilon_1$ and (3.5) holds. Since $x_1 \in \partial P_L$ and $\lambda_1 x_1 \in \overline{P}_{R_1}$, it follows that

$$\left(\frac{2}{5m}+1\right)L < \lambda_1 L = \lambda_1\|x_1\| \leq R_1.$$

Moreover,

$$-\epsilon S_2 x_1 - m\epsilon x_1 - \epsilon\frac{L}{10} = -\lambda_1 m\epsilon x_1 + \epsilon\frac{L}{10},$$

or

$$S_2 x_1 + \frac{L}{5} = (\lambda_1 - 1)m x_1.$$

From here,

$$2\frac{L}{5} \geq \left\|S_2 x_1 + \frac{L}{5}\right\| = (\lambda_1-1)m\|x_1\| = (\lambda_1-1)mL,$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.9 hold. Hence, the IVP (1.1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1,$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

□

Example 3.7 *Below, we will illustrate our main results in this section. Let* $B = 1$ *and*

$$R_1 = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \frac{1}{5B_1}, \quad \epsilon = \frac{1}{5B_1(1+A)}.$$

Then $B_1 = 4$ *and*

$$AB_1 = \frac{1}{5} < B, \quad \epsilon B_1(1+A) < B,$$

i.e. $(H3)$ *holds. Next,*

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L, \quad AB_1 < \frac{L}{5}.$$

i.e. $(H4)$ *holds. Take*

$$h(s) = \log \frac{1 + s^4\sqrt{2} + s^8}{1 - s^4\sqrt{2} + s^8}, \quad l(s) = \arctan \frac{s^4\sqrt{2}}{1 - s^8}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$\begin{aligned} h'(s) &= \frac{8\sqrt{2}s^3(1-s^8)}{(1-s^4\sqrt{2}+s^8)(1-s^4\sqrt{2}+s^8)}, \\ l'(s) &= \frac{4\sqrt{2}s^3(1+s^8)}{1+s^{16}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{s\to\pm\infty} \sum_{r=0}^{4} s^r h(s) &= \lim_{s\to\pm\infty} \frac{h(s)}{\frac{1}{\sum_{r=0}^4 s^r}} \\ &= \lim_{s\to\pm\infty} \frac{h'(s)}{-\frac{\sum_{r=0}^3 (r+1)s^r}{\left(\sum_{r=0}^4 s^r\right)^2}} \\ &= -\lim_{s\to\pm\infty} \frac{8\sqrt{2}s^3(1-s^8)\left(\sum_{r=0}^4 s^r\right)^2}{\left(\sum_{r=0}^3 (r+1)s^r\right)(1-s^4\sqrt{2}+s^8)(1-s^4\sqrt{2}+s^8)} \\ &\neq \pm\infty \end{aligned}$$

and

$$\lim_{s\to\pm\infty}\sum_{r=0}^{4}s^r l(s) = \lim_{s\to\pm\infty}\frac{l(s)}{\frac{1}{\sum_{r=0}^{4}s^r}}$$

$$= \lim_{s\to\pm\infty}\frac{l'(s)}{-\frac{\sum_{r=0}^{3}(r+1)s^r}{\left(\sum_{r=0}^{4}s^r\right)^2}}$$

$$= -\lim_{s\to\pm\infty}\frac{4\sqrt{2}s^3(1+s^8)\left(\sum_{r=0}^{4}s^r\right)^2}{(1+s^{16})\left(\sum_{r=0}^{3}(r+1)s^r\right)}$$

$$\neq \pm\infty.$$

Consequently,

$$-\infty < \lim_{s\to\pm\infty}\left(\sum_{r=0}^{4}s^r\right)h(s) < \infty,$$

$$-\infty < \lim_{s\to\pm\infty}\left(\sum_{r=0}^{4}s^r\right)l(s) < \infty.$$

Hence, there exists a positive constant C_1 *so that*

$$\sum_{r=0}^{4}|s|^r\left(\frac{1}{16\sqrt{2}}\log\frac{1+s^4\sqrt{2}+s^8}{1-s^4\sqrt{2}+s^8}+\frac{1}{8\sqrt{2}}\arctan\frac{s^4\sqrt{2}}{1-s^8}\right) \leq C_1,$$

$s\in\mathbb{R}$. *Note that* $\lim_{s\to\pm 1} l(s) = \frac{\pi}{2}$ *and by [12] (pp. 707, Integral 79), we have*

$$\int\frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^3}{(1+s^{16})}, \quad s\in\mathbb{R},$$

and

$$g_1(t,x) = Q(t)Q(x), \quad t\in[0,\infty), \quad x\in\mathbb{R}.$$

Then there exists a constant $C>0$ *such that*

$$2^8 4!\left(\sum_{r=0}^{4}t^r\right)\left(\sum_{r=0}^{4}|x|^r\right)$$

$$\times\int_0^t\left|\int_0^x g_1(t_1,x_1)dx_1\right|dt_1 \leq C, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Let

$$g(t,x)=\frac{A}{C}g_1(t,x),\quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Then

$$2^8 4!\left(\sum_{r=0}^{4}t^r\right)\left(\sum_{r=0}^{4}|x|^r\right)$$

$$\times\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\le A,\quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

i.e. $(H2)$ *holds. Therefore for the IVP*

$$\begin{aligned} u_{tt} &= u_{xx}+u_{xxxx}+\left(u^2\right)_{xx}, \quad t>0,\quad x\in\mathbb{R},\\ u(0,x) &= \tfrac{1}{(1+x^2)(1+x^4)},\quad x\in\mathbb{R},\\ u_t(0,x) &= \tfrac{1}{(1+x^4)(1+x^6)},\quad x\in\mathbb{R},\end{aligned}$$

are fulfilled all conditions of Theorem 3.5 and Theorem 3.6.

4. The 2-dimensional Boussinesq equation

In this section, we will investigate the IVP (1.2). Let $X=\mathcal{C}^2([0,\infty),\mathcal{C}^4(\mathbb{R},\mathcal{C}^2(\mathbb{R})))$ be endowed with the norm

$$\begin{aligned} \|u\| &= \max\Bigg\{\sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2}|u(t,x,y)|,\quad \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2}|u_t(t,x,y)|,\\ &\quad \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2}|u_{tt}(t,x,y)|,\quad \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2}|u_x(t,x,y)|,\\ &\quad \sup_{(t,x,y)\in[0,\infty)\times 0,\infty)\times\mathbb{R}^2}|u_{xx}(t,x,y)|,\quad \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2}|u_{xxx}(t,x,y)|,\\ &\quad \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2}|u_{xxxx}(t,x,y)|,\quad \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2}|u_y(t,x,y)|,\\ &\quad \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2}|u_{yy}(t,x,y)|\Bigg\},\end{aligned}$$

provided it exists. For $u\in X$, define the operator

$$\begin{aligned} S_1u(t,x,y) &= u(t,x,y)-v_0(x,y)-tv_1(x,y)\\ &\quad +\int_0^t(t-t_1)\Big(-u_{xx}(t_1,x,y)+u_{xxxx}(t_1,x,y)-u_{yy}(t_1,x,y)\\ &\quad +2\left(u_x(t_1,x,y)\right)^2+2u(t_1,x,y)u_{xx}(t_1,x,y)\Big)dt_1,\end{aligned}$$

$(t,x,y) \in [0,\infty) \times \mathbb{R}^2$.

Lemma 4.1 *Suppose that* $(G1)$ *holds. Let* $u \in X$ *satisfies the equation*

$$S_1 u(t,x,y) = 0, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2. \tag{4.1}$$

Then u *is a solution to the IVP* (1.2).

Proof We have

$$\begin{aligned} 0 &= u(t,x,y) - v_0(x,y) - tv_1(x,y) \\ &+ \int_0^t (t-t_1)\Big(- u_{xx}(t_1,x,y) + u_{xxxx}(t_1,x,y) - u_{yy}(t_1,x,y) \\ &+2\left(u_x(t_1,x,y)\right)^2 + 2u(t_1,x,y)u_{xx}(t_1,x,y)\Big) dt_1, \end{aligned} \tag{4.2}$$

$(t,x,y) \in [0,\infty) \times \mathbb{R}^2$, which we differentiate with respect to t and we get

$$\begin{aligned} 0 &= u_t(t,x,y) - v_1(x,y) \\ &+ \int_0^t \Big(- u_{xx}(t_1,x,y) + u_{xxxx}(t_1,x,y) - u_{yy}(t_1,x,y) \\ &+2\left(u_x(t_1,x,y)\right)^2 + 2u(t_1,x,y)u_{xx}(t_1,x,y)\Big) dt_1, \end{aligned} \tag{4.3}$$

$(t,x,y) \in [0,\infty) \times \mathbb{R}^2$. We differentiate (4.3) with respect to t and we find

$$\begin{aligned} 0 &= u_{tt}(t,x,y) - u_{xx}(t,x,y) + u_{xxxx}(t,x,y) - u_{yy}(t,x,y) \\ &+2\left(u_x(t,x,y)\right)^2 + 2u(t,x,y)u_{xx}(t,x,y), \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2, \end{aligned}$$

i.e. u satisfies the first equation of (1.2). Now, we put $t=0$ into (4.2) and (4.3) and we arrive at

$$0 = u(0,x,y) - v_0(x,y), \quad 0 = u_t(0,x,y) - v_1(x,y), \quad (x,y) \in \mathbb{R}^2.$$

This completes the proof. □

Let $B_1 = 4B^2$.

Lemma 4.2 *Suppose that* $(G1)$ *holds. If* $u \in X$, $\|u\| \le B$, *then*

$$|S_1 u(t,x,y)| \le (1+t+t^2)B_1, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2.$$

Proof We have

$$\begin{aligned}
|S_1u(t,x,y)| &= \Bigg| u(t,x,y) - v_0(x,y) - tv_1(x,y) \\
&\quad + \int_0^t (t-t_1)\Big(- u_{xx}(t_1,x,y) + u_{xxxx}(t_1,x,y) - u_{yy}(t_1,x,y) \\
&\quad +2\left(u_x(t_1,x,y)\right)^2 + 2u(t_1,x)u_{xx}(t_1,x,y)\Big)dt_1\Bigg| \\
&\leq |u(t,x,y)| + |u_0(x,y)| + t|u_1(x,y)| \\
&\quad + \int_0^t (t-t_1)\Big(|u_{xx}(t_1,x,y)| + |u_{xxxx}(t_1,x)| + |u_{yy}(t_1,x,y)| \\
&\quad +2\left|u_x(t_1,x)\right|^2 + 2|u(t_1,x)||u_{xx}(t_1,x)|\Big)dt_1 \\
&\leq 2B + tB + \int_0^t (t-t_1)(3B+4B^2)dt_1 \\
&\leq B_1(1+t+t^2), \quad (t,x,y) \in [0,\infty)\times\mathbb{R}^2.
\end{aligned}$$

This completes the proof. □

Below, suppose that

(G2) there exists a function $g \in \mathcal{C}([0,\infty)\times\mathbb{R}^2)$ so that $g > 0$ on $(0,\infty)\times(\mathbb{R}^2\backslash(\{x=0\}\cup\{y=0\}))$, $g(0,x,y) = g(t,x,0) = g(t,0,y) = 0$, $(t,x,y) \in [0,\infty)\times\mathbb{R}^2$, and a positive constant A such that

$$\begin{aligned}
&4!2^8(1+t+t^2+t^3+t^4)(1+|x|+x^2+|x|^3+x^4)(1+|y|+y^2) \\
&\times \int_0^t \left| \int_0^x \int_0^y g(t_1,x_1,y_1)dx_1dy_1 \right| dt_1 \leq A.
\end{aligned}$$

In the end of this section we will give an example for such function g and such constant A. For $u \in X$, define the operator

$$S_2u(t,x,y) = \int_0^t \int_0^x (t-t_1)^2(x-x_1)^4(y-y_1)^2 g(t_1,x_1,y_1)S_1u(t_1,x_1,y_1)dx_1dy_1dt_1,$$

$(t,x,y) \in [0,\infty)\times\mathbb{R}^2$.

Lemma 4.3 *Suppose that $(G1)$ and $(G2)$ hold. If $u \in X$ satisfies the equation*

$$S_2u(t,x,y) = 0, \quad (t,x,y) \in [0,\infty)\times\mathbb{R}^2, \tag{4.4}$$

then u is a solution to the IVP (1.2).

Proof We differentiate three times with respect to t and y and five times with respect to x and we find

$$g(t,x,y)S_1u(t,x,y) = 0, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2.$$

Hence,

$$S_1u(t,x,y) = 0, \quad (t,x,y) \in (0,\infty) \times (\mathbb{R}\backslash(\{x=0\} \cup \{y=0\})).$$

Since $S_1u(\cdot,\cdot,\cdot)$ is a continuous function on $[0,\infty) \times \mathbb{R}^2$, we get

$$\begin{aligned} 0 &= \lim_{t\to 0} S_1u(t,x,y) = S_1u(0,x,y) \\ &= \lim_{x\to 0} S_1u(t,x,y) = S_1u(t,0,y) \\ &= \lim_{y\to 0} S_1u(t,x,y) = S_1u(t,x,0), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

Thus,

$$S_1u(t,x,y) = 0, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2.$$

Now, applying Lemma 4.1, we get the desired result. □

As we have proved Lemma 3.4, one can obtain the following result

Lemma 4.4 *Suppose that* $(G1)$ *and* $(G2)$ *hold. If* $u \in X$, $\|u\| \leq B$, *then*

$$\|S_2u\| \leq AB_1.$$

As we have proved Theorem 3.5 and Theorem 3.6, one can obtain the following results.

Theorem 4.5 *Suppose that* $(G1)$ *and* $(G2)$ *hold. Then the IVP (1.2) has at least one solution in* $\mathcal{C}^2([0,\infty),\mathcal{C}^4(\mathbb{R},\mathcal{C}^2(\mathbb{R}^2)))$.

Theorem 4.6 *Suppose* $(G1)$ *and* $(G2)$. *Then the IVP (1.2) has at least two nonnegative solutions in* $\mathcal{C}^2([0,\infty),\mathcal{C}^4(\mathbb{R},\mathcal{C}^2(\mathbb{R})))$.

Example 4.7 *Let* A, B, R_1, L, r, m *and* ϵ *be as in Example 3.7. Then* $B_1 = 4$ *and* $(H3)$ *and* $(H4)$ *hold. Let also,* Q *be the same function as in Example 3.7. Take*

$$g_1(t,x,y) = Q(t)Q(x)Q(y), \quad t \in [0,\infty), \quad x \in \mathbb{R}.$$

Then there exists a constant $C > 0$ *such that*

$$\begin{aligned} &2^8 4! \left(\sum_{r=0}^{4} t^r\right)\left(\sum_{r=0}^{4} |x|^r\right)(1+|y|+y^2) \\ &\times \int_0^t \left| \int_0^x \int_0^y g_1(t_1,x_1,y_1)dx_1dy_1 \right| dt_1 \leq C, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2. \end{aligned}$$

Let

$$g(t,x,y) = \frac{A}{C} g_1(t,x,y), \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2.$$

Then

$$2^8 4! \left(\sum_{r=0}^{4} t^r\right)\left(\sum_{r=0}^{4} |x|^r\right)(1+|y|+y^2)$$

$$\times \int_0^t \left| \int_0^x g(t_1,x_1,y_1)dx_1dy_1 \right| dt_1 \leq A, \quad (t,x,y)\in[0,\infty)\times\mathbb{R}^2.$$

i.e. $(G2)$ *holds. Therefore for the IVP*

$$\begin{aligned} u_{tt} &= u_{xx}+u_{yy}-u_{xxxx}-\left(u^2\right)_{xx}, \quad t>0, \quad (x,y)\in\mathbb{R}^2,\\ u(0,x,y) &= \tfrac{1}{(1+3x^2)(4+9y^4)}, \quad (x,y)\in\mathbb{R}^2,\\ u_t(0,x,y) &= \tfrac{1}{(1+11x^6)(1+12y^8)}, \quad (x,y)\in\mathbb{R}^2, \end{aligned}$$

are fulfilled all conditions of Theorem 4.5 and Theorem 4.6.

References

[1] Banas J, Goebel K.Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, 60. Marcel Dekker, New York, 1980.

[2] Bourgain J. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geometric and Functional Analysis 1993; 3: 107-156.

[3] Bourgain J. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation, Geometric and Functional Analysis 1993; 3: 209-262.

[4] Djebali S, Mebarki K.Fixed point index theory for perturbation of expansive mappings by k-set contractions, Topological Methods in Nonlinear Analysis 2019; 54 (2): 613-640.

[5] Drabek P, Milota J. Methods in Nonlinear Analysis, Applications to Differential Equations, Birkhäuser, 2007.

[6] Esfahani A, Farah L. Local well-posedness for the sixth-order Boussinesq equation. Journal of Mathematical Analysis and Applications 2012; 385: 230-242.

[7] Farah L. Local solutions in Sobolev spaces with negative indices for the Good Boussinesq equation, Comm. Partial Differential Equations 2009; 34: 52-73.

[8] Georgiev S, Zennir K. Existence of solutions for a class of nonlinear impulsive wave equations, Ricerchat (2021), https://doi.org/10.1007/s11587-021-00649-2.

[9] Kadomtsev B, Petviashvili V. On the stability of solitary waves in weakly dispersive media, Soviet Physics Doklady 1970; 15: 539-541 .

[10] Manakov S, Zakharov V. Twodimensional solitons of the Kadomtsev–Petviashvili equation and their interaction, Physics Letters A 1977;63 (3): 205-206.

[11] Manakov S, Santini P. On the solutions of the dKP equation: the nonlinear Riemann-Hilbert problem, longtime behaviour, implicit solutions and wave breaking. Mathematical and Theoretical 2008; 41:055204.

[12] Polyanin A, Manzhirov A. Handbook of integral equations, Chemical Rubber Company Press, 1998.

[13] Wang H, Esfahani A. Well-posedness for the Cauchy problem associated to a periodic Boussinesq equation. Nonlinear Analysis: Theory, Methods & Applications 2013; 89: 267-275.

[14] Yan W, Li YS, Huang JH, Duan JQ. The Cauchy problem for a two-dimensional generalized Kadomtsev-Petviashvili I equation in anisotropic Sobolev spaces, Analysis and Applications 2020; 18: 469-522.