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New form of Laguerre fractional differential equation and applications

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Abstract: Laguerre differential equation is a well known equation that appears in the quantum mechanical description of the hydrogen atom. In this paper, we aim to develop a new form of Laguerre Fractional Differential Equation (LFDE) of order 2α and we investigate the solutions and their properties. For a positive real number α, we prove that the equation has solutions of the form $L_{n,\alpha}(x) = \sum_{k=0}^{n} a_k x^k$, where the coefficients of the polynomials are computed explicitly. For integer case $\alpha = 1$ we show that these polynomials are identical to classical Laguerre polynomials. Finally, we solve some fractional differential equations by defining a suitable integral transform.

Key words: Fractional Laguerre equation, Fractional Sturm-Liouville operator, Riemann-Liouville and Caputo derivatives

1. Introduction

Developing classical integer order differential equations to the fractional order is one of those mathematical topics that received much attention. It has been shown for many years that the use of this emerging tool in modeling and design helps to improve the efficiency of various sciences. In recent years, it has been proved that in some applications modeling by fractional derivatives generates more accurate solutions than modeling by integer order derivatives [5, 9, 16, 18]. The recent applications of fractional differential equations in science and engineering are significant motivation for researchers to study and develop their research in this subject, for more details see [7, 10–13]. It is well known that polynomials such as Legendre, Chebyshev and Laguerre play a fundamental role in studying ordinary and partial differential equation [4, 6, 8, 22]. Laguerre differential equation is a well-known equation that appears in the quantum mechanical description of the hydrogen atom [20]. M. Klimek and O.P. Agrawal considered the fractional Legendre equation as a singular fractional Sturm-Liouville problem and they presented some results on the applications of Legendre polynomials in ordinary and partial differential equations in a finite interval $[a, b]$ [15]. The research of fractional differential equations in semiinfinite interval $(0, \infty)$ is also of great interest. Some authors tried to obtain analytical and numerical solutions for fractional order differential equations on an unbounded interval, for more details see [1–3, 17, 21]. T. Aboelenen et al. [1] considered a generalized Laguerre fractional differential equation in half-line and investigated spectral data subject to a homogeneous Dirichlet and homogeneous fractional integro-differential boundary conditions. In this paper, we aim to develop a new type of LFDE in half-line domain which is different from [1]. Our method is constructive in finding solutions of LFDE. The classical Laguerre differential equation is a second-order ordinary

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differential equation, where the solutions are Laguerre polynomials. It is well-known that the classical Laguerre differential equation has the following form [20, 22]

$$\left[Dxe^{-x}D + ne^{-x}\right] L_n(x) = 0, \quad 0 < x < \infty, \tag{1.1}$$

where $L_n(x)$ are Laguerre polynomials of degree n. In this paper, we define a fractional order Laguerre differential equation of the following form

$$\left[-D_-^\alpha x^\alpha e^{-\alpha x} D_{0+}^\alpha + \lambda e^{-\alpha x}\right] y(x) = 0, \tag{1.2}$$

where D_-^α and D_{0+}^α are left and right Riemann-Liouville fractional derivatives, respectively. We find values of parameter λ for which the solutions are polynomials $L_{n,\alpha}$. This paper is organised in the following manner. In section 2 some preliminary materials on fractional calculus and Laguerre polynomials are given. We find the solutions of (1.2) in section 3. We prove that the solutions are polynomials of the form $L_{n,\alpha}(x) = \sum_{k=0}^n a_k x^k$, where the coefficients of the polynomials are computed by solving a system of algebraic equations by a backward recursive formula with $a_n = \frac{(-1)^n \alpha^n}{\Gamma(n+1)}$. Orthogonality of the polynomials $L_{n,\alpha}(x)$ with weight function $e^{-\alpha x}$ are proved. Moreover, we show that for $\alpha = 1$ the polynomials $L_{n,\alpha}(x)$ are indeed the classical Laguerre polynomials $L_n(x)$ and we show that $L_{n,\alpha}(\frac{x}{\alpha}) = L_n(x)$. In section 4 we give some applications in solving fractional differential equations.

2. Preliminaries

In this section, we give some preliminary materials of fractional calculus [14, 18, 19] and Laguerre polynomials [22]. It is well-known that the Sturm-Liouville form of the classical Laguerre equation is of the form (1.1), where the Laguerre polynomial $L_n(x)$ are given by

$$L_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{k!} x^k. \tag{2.1}$$

Moreover, the following orthogonality property holds

$$\int_0^{+\infty} e^{-x} L_m(x) L_n(x) dx = \delta_{m,n}. \tag{2.2}$$

The Laguerre polynomials satisfy the following recursive relation

$$L_{n+1}(x) = \frac{(2n+1-x)L_n(x) - nL_{n-1}(x)}{n+1}, \qquad L_0(x) = 1,\ L_1(x) = 1 - x. \tag{2.3}$$

Let α be a positive real number. Left and right Riemann-Liouville integrals of order α are defined by

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(s)}{(x-s)^{1-\alpha}} ds, \qquad x > a, \tag{2.4}$$

$$I_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(s)}{(s-x)^{1-\alpha}} ds, \qquad x < b. \tag{2.5}$$

If we consider half line then left and right Riemann-Liouville integrals of order α are defined by

$$I_{+}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(s)}{(x-s)^{1-\alpha}} ds, \tag{2.6}$$

$$I_{-}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} \frac{f(s)}{(s-x)^{1-\alpha}} ds. \tag{2.7}$$

Let $m-1<\alpha<m$ where $m \in \mathbb{N}$. Then left and right Riemann-Liouville derivatives are defined by

$$(D_{a+}^{\alpha} f)(x)=D^{m}(I_{a+}^{m-\alpha} f)(x),$$

$$(D_{b-}^{\alpha} f)(x)=(-D)^{m}(I_{b-}^{m-\alpha} f)(x).$$

If we consider half line then left and right Riemann-Liouville derivative of order α is defined by

$$(D_{+}^{\alpha} f)(x)=D^{m}(I_{+}^{m-\alpha} f)(x), \tag{2.8}$$

$$(D_{-}^{\alpha} f)(x)=(-D)^{m}(I_{-}^{m-\alpha} f)(x). \tag{2.9}$$

If $m-1<\alpha<m$ and $f \in C[a,b]$ then it can be proved that the following properties are satisfied

$$D_{a+}^{\alpha} I_{a+}^{\alpha} f(x)=f(x), \quad D_{b-}^{\alpha} I_{b-}^{\alpha} f(x)=f(x),$$

$$D_{a+}^{\alpha} I_{a+}^{\beta} f(x)=I_{a+}^{\beta-\alpha} f(x), \quad D_{b-}^{\alpha} I_{b-}^{\beta} f(x)=I_{b-}^{\beta-\alpha} f(x),$$

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x)=I_{a+}^{\alpha+\beta} f(x), \quad I_{b-}^{\alpha} I_{b-}^{\beta} f(x)=I_{b-}^{\alpha+\beta} f(x).$$

Moreover if $\alpha>0$ and $\beta>\alpha$ then we have

$$I_{a+}^{\alpha}(x-a)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1},$$

$$D_{a+}^{\alpha}(x-a)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1},$$

$$I_{b-}^{\alpha}(b-x)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1},$$

$$D_{b-}^{\alpha}(b-x)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1}.$$

Theorem 2.1 *[19] Let $\alpha>0, p \geq 1, q \geq 1, \frac{1}{p}+\frac{1}{q} \leq 1+\alpha, f(x) \in I_{0+}^{\alpha}(L_p), g(x) \in I_{-}^{\alpha}(L_q)$. Then*

$$\int_{0}^{\infty} g(x) D_{0+}^{\alpha} f(x) dx=\int_{0}^{\infty} f(x) D_{-}^{\alpha} g(x) dx. \tag{2.10}$$

3. Laguerre fractional differential equation

In this section, we define Laguerre fractional differential equation and obtain the main results of this paper. We introduced the classical Laguerre differential equation by (1.1). Now, we define the fractional form of the Laguerre differential equation as follows

$$\left[-D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}+\lambda_{n,\alpha}e^{-\alpha x}\right]L_{n,\alpha}(x)=0, \tag{3.1}$$

where

$$\lambda_{n,\alpha}=\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}\alpha^{\alpha}. \tag{3.2}$$

We find an explicit formula for $L_{n,\alpha}(x)$ in Theorem 3.2. In the following Lemma we prove the self-adjoint property of fractional operator $\ell_{\alpha}:=-D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}$.

Lemma 3.1 *If f and g satisfy in the conditions of Theorem 2.1, then we have $\langle \ell_{\alpha}f,g\rangle=\langle f,\ell_{\alpha}g\rangle$, i.e.*

$$\int_{0}^{+\infty}\left[f(x)D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}g(x)-g(x)D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}f(x)\right]dx=0. \tag{3.3}$$

Proof Using the equation (2.10) two times implies

$$\begin{aligned}\int_{0}^{+\infty}f(x)D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}g(x)dx&=\int_{0}^{+\infty}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}g(x).D_{0+}^{\alpha}f(x)dx\\&=\int_{0}^{+\infty}g(x)D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}f(x)dx.\end{aligned}$$

□

Now we prove that the solutions of the LFDE for $\lambda=\lambda_{n,\alpha}$ are orthogonal polynomials and we find a procedure to construct the coefficients of these polynomials.

Theorem 3.2 *The solutions of the LFDE (3.1) are orthogonal polynomials of the form*

$$L_{n,\alpha}(x)=\sum_{k=0}^{n}a_{k}x^{k}, \tag{3.4}$$

where a_k are computed by a constructive backward recursive formula of the form

$$a_{k}=-\frac{1}{\Gamma(1-\alpha)(\lambda_{\alpha,n}-\lambda_{\alpha,k})}\sum_{j=k+1}^{n}\left[\lambda_{\alpha,j}\binom{j}{k}\Gamma(j-k-\alpha)\alpha^{k-j+1}a_{j}\right], \qquad k=n-1,n-2,\dots,1,0, \tag{3.5}$$

and a_n is an arbitrary nonzero constant.

Proof We assume that the solutions of the LFDE are polynomials of the form (3.5) and then we construct the coefficients a_k. First, we compute the first term of the left-hand side of the equation (3.1).

$$\begin{aligned}
\left[-D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}\right]L_{n,\alpha}(x) &= DI_{-}^{1-\alpha}x^{\alpha}e^{-\alpha x}\sum_{k=0}^{n}a_k\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha}\\
&= D\sum_{k=0}^{n}a_k\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}.\frac{1}{\Gamma(1-\alpha)}\int_x^{+\infty}(t-x)^{-\alpha}t^k e^{-\alpha t}dt\\
&= D\sum_{k=0}^{n}a_k\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}.\frac{1}{\Gamma(1-\alpha)}\int_0^{+\infty}u^{-\alpha}(u+x)^k e^{-\alpha(u+x)}du\\
&= D\sum_{k=0}^{n}a_k\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}.\frac{1}{\Gamma(1-\alpha)}e^{-\alpha x}\int_0^{+\infty}e^{-\alpha u}u^{-\alpha}\left(\sum_{i=0}^{k}\binom{k}{i}u^i x^{k-i}\right)du\\
&= D\sum_{k=0}^{n}\sum_{i=0}^{k}a_k\binom{k}{i}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}.\frac{1}{\Gamma(1-\alpha)}e^{-\alpha x}x^{k-i}\left(\int_0^{+\infty}e^{-\alpha u}u^{i-\alpha}du\right)\\
&= D\sum_{k=0}^{n}\sum_{i=0}^{k}a_k\binom{k}{i}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}.\frac{1}{\Gamma(1-\alpha)}e^{-\alpha x}x^{k-i}\frac{\Gamma(i-\alpha+1)}{\alpha^{i-\alpha+1}}\\
&= -\sum_{k=0}^{n}\sum_{i=0}^{k}a_k\binom{k}{i}\frac{\Gamma(k+1)\Gamma(i-\alpha+1)}{\Gamma(k-\alpha+1)\Gamma(1-\alpha)}\alpha^{\alpha-i}e^{-\alpha x}x^{k-i}\\
&+ \sum_{k=0}^{n}\sum_{i=0}^{k}a_k\binom{k}{i}\frac{\Gamma(k+1)\Gamma(i-\alpha+1)}{\Gamma(k-\alpha+1)\Gamma(1-\alpha)}(k-i)\alpha^{\alpha-i-1}e^{-\alpha x}x^{k-i-1}.
\end{aligned}$$

Now by substituting the resulting expression above in (3.1) and deleting the factor $e^{-\alpha x}$, we find

$$\begin{aligned}
&-\textstyle\sum_{k=0}^{n}\sum_{i=0}^{k}a_k\binom{k}{i}\dfrac{\Gamma(k+1)\Gamma(i-\alpha+1)}{\Gamma(k-\alpha+1)\Gamma(1-\alpha)}\alpha^{\alpha-i}x^{k-i}\\
&+\textstyle\sum_{k=0}^{n}\sum_{i=0}^{k}a_k\binom{k}{i}\dfrac{\Gamma(k+1)\Gamma(i-\alpha+1)}{\Gamma(k-\alpha+1)\Gamma(1-\alpha)}(k-i)\alpha^{\alpha-i-1}x^{k-i-1}+\lambda_{n,\alpha}\textstyle\sum_{k=0}^{n}a_kx^k=0.
\end{aligned}\tag{3.6}$$

Equating the coefficients of x^k to zero we find an algebraic system that computes a_k. Equating the coefficient of x^n to zero we find

$$-a_n\frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n-\alpha+1)\Gamma(1-\alpha)}\alpha^{\alpha}x^n+\lambda_{n,\alpha}a_nx^n=0,$$

thus we obtain $\lambda_{n,\alpha}=\dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}\alpha^{\alpha}$. It is clear that a_n can be any arbitrary nonzero real number. Now we find a recursive formula to compute the coefficients a_k for $k<n$. We can rewrite the equation (3.6) in the following form

$$\sum_{k=0}^{n}\sum_{i=0}^{k}a_k\binom{k}{i}\frac{\Gamma(k+1)\Gamma(i-\alpha+1)}{\Gamma(k-\alpha+1)\Gamma(1-\alpha)}\alpha^{\alpha-i}x^{k-i}((k-i)x^{-1}\alpha^{-1}-1)+\lambda_{n,\alpha}\sum_{k=0}^{n}a_kx^k=0.\tag{3.7}$$

Expanding the sums, choosing a_n an arbitrary real number and equating the coefficient of x^{n-1} to zero implies that

$$a_{n-1} = \frac{-1}{\lambda_{n,\alpha} - \dfrac{\Gamma(n)}{\Gamma(n-\alpha)}\alpha^{\alpha}} \frac{n\Gamma(n+1)}{\Gamma(n+1-\alpha)}\alpha^{\alpha} a_n,$$

Similarly, we find

$$\begin{aligned} a_{n-2} = -\frac{1}{\Gamma(1-\alpha)[\lambda_{n,\alpha} - \lambda_{n-2,\alpha}]}[\lambda_{n-1,\alpha}\binom{n-1}{n-2}\Gamma(1-\alpha)a_{n-1} \\ +\lambda_{n,\alpha}\binom{n}{n-2}\Gamma(2-\alpha)\alpha^{-1}a_n]. \end{aligned}$$

Using induction and simple calculations, we obtain the final formula (3.5). Now we show the orthogonality of the polynomials $L_{n,\alpha}(x)$. Suppose that $m \neq n$. We have

$$\left[-D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha} + \lambda_{n,\alpha}e^{-\alpha x}\right] L_{n,\alpha}(x) = 0,$$
$$\left[-D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha} + \lambda_{m,\alpha}e^{-\alpha x}\right] L_{m,\alpha}(x) = 0.$$

Multiplying the first equation by $L_{m,\alpha}(x)$ and the second equation by $L_{n,\alpha}(x)$ then subtracting implies that

$$\begin{aligned} -L_{m,\alpha}(x)D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}L_{n,\alpha}(x) + L_{n,\alpha}(x)D_{-}^{\alpha}x^{\alpha}e^{-\alpha x}D_{0+}^{\alpha}L_{m,\alpha}(x) \\ = (\lambda_{m,\alpha} - \lambda_{n,\alpha})e^{-\alpha x}L_{n,\alpha}(x)L_{m,\alpha}(x). \end{aligned}$$

Integrating over $[0,\infty)$ and using Lemma 3.1 implies that

$$(\lambda_{m,\alpha} - \lambda_{n,\alpha})\int_0^{+\infty} e^{-\alpha x}L_{n,\alpha}(x)L_{m,\alpha}(x)dx = 0. \tag{3.8}$$

Since m and n are distinct, this completes the orthogonality of the polynomials.

□

Corollary 3.3 *For $\alpha = 1$ the polynomials $L_{n,1}$ are identical to the classical Laguerre polynomials.*

Proof We may write the equation (3.5) for $k = n-1, n-2, \ldots, 1, 0$ as follows

$$\begin{aligned} a_k = & - \frac{1}{(\lambda_{n,\alpha} - \lambda_{k,\alpha})}\lambda_{k+1,\alpha}\binom{k+1}{k}a_{k+1} \\ & - \frac{1}{(\lambda_{n,\alpha} - \lambda_{k,\alpha})}\sum_{j=k+2}^{n}\left[\lambda_{j,\alpha}\binom{j}{k}(j-k-\alpha-1)(j-k-\alpha-2)\ldots(1-\alpha)\alpha^{k-j+1}a_j\right]. \end{aligned}$$

The second sum for $\alpha = 1$ is zero and $\lambda_{n,1} = n$, thus we have

$$a_k = -\frac{(k+1)^2}{(n-k)}a_{k+1}, \qquad k = n-1, n-2, \ldots, 1, 0. \tag{3.9}$$

Considering $a_n = \frac{(-1)^n}{n!}$ and using induction we find

$$a_j = \frac{(-1)^j}{j!}\binom{n}{j}, \qquad j = 1, 2, \ldots, n. \tag{3.10}$$

According to (2.1) a_j given by (3.10) is the same as coefficients of the classical Laguerre polynomials. □

Now we give an interesting relation between the polynomials $L_{n,\alpha}(x)$ and the classical Laguerre polynomials L_n in the following theorem.

Theorem 3.4 *If n is a nonnegative integer and $x \in [0, \infty)$ then for $a_n = \frac{(-1)^n}{n!}\alpha^n$, we have*

$$L_{n,\alpha}(\frac{x}{\alpha}) = L_n(x). \tag{3.11}$$

Proof We use induction to prove this result. For $n = 0$ the statement is true since we have

$$L_{0,\alpha}(\frac{x}{\alpha}) = 1 = L_0(x).$$

Suppose that the statement is true for all $j < n$, i.e.

$$L_{j,\alpha}(\frac{x}{\alpha}) = L_j(x).$$

We may write $L_{n,\alpha}(\frac{x}{\alpha})$ as a linear combination as follows

$$L_{n,\alpha}(\frac{x}{\alpha}) = \sum_{k=0}^{n} A_k^n L_k(x) = A_n^n L_n(x) + \sum_{k=0}^{n-1} A_k^n L_{k,\alpha}(\frac{x}{\alpha}). \tag{3.12}$$

Using (3.8) for two different and arbitrary indices n, m and changing variables $\alpha x = u$ imply that

$$\int_0^{+\infty} e^{-u} L_{n,\alpha}(\frac{u}{\alpha}) L_{j,\alpha}(\frac{u}{\alpha}) du = 0.$$

Multiplying both sides of (3.12) by $e^{-x} L_{j,\alpha}(\frac{x}{\alpha})$ for $j < n$ and integrating over $[0, \infty)$ and using orthogonality of the classical Laguerre polynomials given by (2.2) implies that $A_j^n = 0$, for $j = 0, 1, 2, \cdots, n-1$. Thus by using (3.12) we find

$$L_{n,\alpha}(\frac{x}{\alpha}) = A_n^n L_n(x).$$

On the other hand we have $a_n = \frac{(-1)^n \alpha^n}{n!}$. Since the leading coefficients of $L_{n,\alpha}(\frac{x}{\alpha})$ and $L_n(x)$ are both $\frac{(-1)^n}{n!}$. Thus, we conclude $A_n^n = 1$ that completes the proof. □

Corollary 3.5 *Theorem 3.4 implies that the coefficients of polynomial $L_{n,\alpha}(x)$ are $a_k = \dfrac{(-1)^k}{k!}\binom{n}{k}\alpha^k$.*

Corollary 3.6 *Orthogonality of the polynomials $L_{n,\alpha}(x)$ implies that*

$$\int_0^{+\infty} e^{-\alpha x} L_{m,\alpha}(x) L_{n,\alpha}(x) dx = \frac{1}{\alpha}\delta_{m,n}. \tag{3.13}$$

Proof If $m = n$ and $u = \alpha x$ then we have

$$\int_0^{+\infty} e^{-\alpha x} L_{n,\alpha}^2(x)dx = \frac{1}{\alpha}\int_0^{+\infty} e^{-u} L_{n,\alpha}^2(\frac{u}{\alpha})du.$$

Using (3.11) and (2.2) implies (3.13). □

Definition 3.7 *We define the Laguerre norm of the function* $f(x)$ *as follows*

$$\|f\|_L = (\int_0^{+\infty} e^{-\alpha x} f^2(x)dx)^{\frac{1}{2}}. \tag{3.14}$$

Corollary 3.8 *Orthogonal property in Corollary 3.6 implies that*

$$\|L_{n,\alpha}\|_L^2 = \frac{1}{\alpha}. \tag{3.15}$$

Now we introduce a recursive formula to compute fractional Laguerre polynomials in the following theorem.

Theorem 3.9 *The fractional Laguerre polynomials* $L_{k,\alpha}(x)$ *satisfy the following recursive formula*

$$\begin{aligned} &L_{n+1,\alpha}(x) = \frac{1}{n+1}\left[(2n+1-\alpha x)L_{n,\alpha}(x) - nL_{n-1,\alpha}(x)\right], \quad n \geq 1, \\ &L_{0,\alpha}(x) = 1, \quad L_{1,\alpha}(x) = 1 - \alpha x. \end{aligned} \tag{3.16}$$

Proof It is well known that the classical Laguerre polynomials $L_n(x)$ satisfy the following recursive relation [22]

$$L_{n+1}(x) = \frac{1}{n+1}\left[(2n+1-x)L_n(x) - nL_{n-1}(x)\right], \quad n \geq 1, \quad L_0(x) = 1, L_1(x) = 1 - x.$$

Now using (3.11) implies that

$$\begin{aligned} &L_{n+1,\alpha}(\frac{x}{\alpha}) = \frac{1}{n+1}\left[(2n+1-x)L_{n,\alpha}(\frac{x}{\alpha}) - nL_{n-1,\alpha}(\frac{x}{\alpha})\right], \quad n \geq 1, \\ &L_{0,\alpha}(\frac{x}{\alpha}) = 1, L_{1,\alpha}(\frac{x}{\alpha}) = 1 - x, \end{aligned}$$

Changing variable $u = \frac{x}{\alpha}$ in the last equations implies the result. □

4. Integral transform and applications

In this section, we define integral transform corresponding to fractional Laguerre polynomials $L_{n,\alpha}(x)$ and introduce inverse transform to solve some fractional differential equations.

Integral transform of a function $f \in L^2(0, +\infty)$ corresponding to Laguerre polynomials $L_{n,\alpha}(x)$ is defined by

$$F(n) = T[f](n) = \int_0^{+\infty} e^{-\alpha x} f(x) L_{n,\alpha}(x)dx. \tag{4.1}$$

The corresponding inverse transform is defined by

$$T^{-1}[F(n)] = \sum_{n=0}^{+\infty} \alpha F(n) L_{n,\alpha}(x). \tag{4.2}$$

Lemma 4.1 *Considering the Laguerre operator* $\ell_\alpha = D_-^\alpha x^\alpha e^{-\alpha x} D_{0^+}^\alpha$, *we have*

$$T[e^{\alpha x} \ell_\alpha f(x)] = \lambda_{n,\alpha} F(n). \tag{4.3}$$

Proof Using Lemma 3.1 we have the following equality

$$\begin{aligned} T[e^{\alpha x} \ell_\alpha f(x)] &= \int_0^{+\infty} \ell_\alpha f(x).L_{n,\alpha}(x)dx = \int_0^{+\infty} f(x).\ell_\alpha L_{n,\alpha}(x)dx \\ &= \lambda_{n,\alpha} \int_0^{+\infty} e^{-\alpha x} f(x) L_{n,\alpha}(x)dx = \lambda_{n,\alpha} F(n). \end{aligned}$$

□

Now we give some applications in finding the solutions of some special fractional differential equations. We use $L_{n,\alpha}$ to solve typical fractional differential equations. Indeed, we generate a particular solution in the form of series in terms of $L_{n,\alpha}$ using the integral transform (4.1).

Theorem 4.2 *Suppose* $g \in L^2(0, \infty]$ *and* $\lambda \neq \lambda_{n,\alpha}$. *If the integral transform of* g *satisfies the following inequality*

$$|G(n)| \leq k_\alpha n^\beta, \tag{4.4}$$

then the solution of the fractional differential equation

$$[e^{\alpha x} \ell_\alpha - \lambda] f(x) = g(x), \tag{4.5}$$

for $\alpha > \beta + 1$ *is given by the following infinite series*

$$f(x) = \sum_{n=0}^{\infty} \alpha \frac{G(n)}{\lambda_{n,\alpha} - \lambda} L_{n,\alpha}(x). \tag{4.6}$$

Proof Taking integral transform of (4.5) and using (4.3) implies that

$$[\lambda_{n,\alpha} - \lambda] F(n) = G(n) \Longrightarrow F(n) = \frac{G(n)}{\lambda_{n,\alpha} - \lambda}.$$

Applying inverse transform (4.2), the function f could be expressed in the form (4.6). For $n > n_0$ using Corollary 3.8 we have

$$\|\alpha \frac{G(n)}{\lambda_{n,\alpha} - \lambda} L_{n,\alpha}\|_L \leqslant \frac{\sqrt{\alpha}}{|\lambda_{n,\alpha} - \lambda|} k_\alpha n^\beta.$$

Using the asymptotic property of the eigenvalues [14] we find

$$\lambda_{n,\alpha} \cong \alpha^{\alpha}(n+1)^{\alpha}, \quad n \longrightarrow \infty. \tag{4.7}$$

Thus we have

$$\sqrt{\alpha}k_{\alpha}n^{\beta}\frac{1}{|\lambda_{n,\alpha}-\lambda|} \cong \frac{\sqrt{\alpha}k_{\alpha}}{n^{\alpha-\beta}}, \quad n \longrightarrow \infty.$$

The assumption $\alpha-\beta>1$ implies uniform convergence of (4.6) in $[0,\infty)$. □

Corollary 4.3 *If we truncate the series (4.6) as* $f_N(x)=\sum_{n=0}^{N}\alpha\dfrac{G(n)}{\lambda_{n,\alpha}-\lambda}L_{n,\alpha}(x)$*, then we have* $\|f-f_N\|_L = O(\frac{1}{n^{\alpha-\beta}})$*.*

Example 4.4 *For a fix* $m\in\mathbb{N}$*, we consider the following nonhomogeneous equation in* $[0,+\infty)$

$$[e^{\alpha x}\ell_{\alpha}-\lambda]\, f(x) = L_{m,\alpha}(x). \tag{4.8}$$

Taking $L_{n,\alpha}$ *integral transform of (4.4) implies*

$$F(m)=\frac{1}{\alpha(\lambda_{m,\alpha}-\lambda)},$$

and $F(n)=0$ *for* $n\neq m$*. Using relation (4.6), the particular solution of nonhomogeneous fractional equation (4.8) is obtained as follows*

$$f(x)=\frac{1}{\lambda_{m,\alpha}-\lambda}L_{\alpha,m}(x).$$

Example 4.5 *We consider the following nonhomogeneous equation in* $[0,+\infty)$

$$[e^{\alpha x}J_{\alpha}-\lambda]\, f(x) = e^{-x}. \tag{4.9}$$

The $L_{n,\alpha}$ *integral transform of the above equation gives,*

$$\begin{aligned}(\lambda_{n,\alpha}-\lambda)F(n) &= \int_0^{+\infty} e^{-x(\alpha+1)}L_{n,\alpha}(x)dx \\ &= \sum_{k=0}^{n} a_k\Big(\int_0^{+\infty} e^{-x(\alpha+1)}x^k dx\Big) = \sum_{k=0}^{n} a_k\frac{\Gamma(k+1)}{(1+\alpha)^{k+1}}.\end{aligned}$$

Substituting the values of a_k *from Corollary 3.5, we obtain*

$$F(n)=\frac{1}{(1+\alpha)^{n+1}(\lambda_{n,\alpha}-\lambda)}.$$

Using relation (4.2), the particular solution of nonhomogeneous fractional equation (4.9) is obtained as follows:

$$f(x)=\sum_{n=0}^{\infty}\alpha\frac{L_{n,\alpha}(x)}{(1+\alpha)^{n+1}(\lambda_{n,\alpha}-\lambda)}.$$

We truncate the series above and approximate the solution $f(x)$ with $f_N(x)$. For $\alpha = 1.5, 0.9$ and different values of N, the graphs of $f_N(x)$ are plotted in Figures 1 and 2. The Infinity norm and Laguerre norm (Definition 3.7) of $(f_{N+2} - f_N)$ for different values of N are computed in Tables 1 and 2.

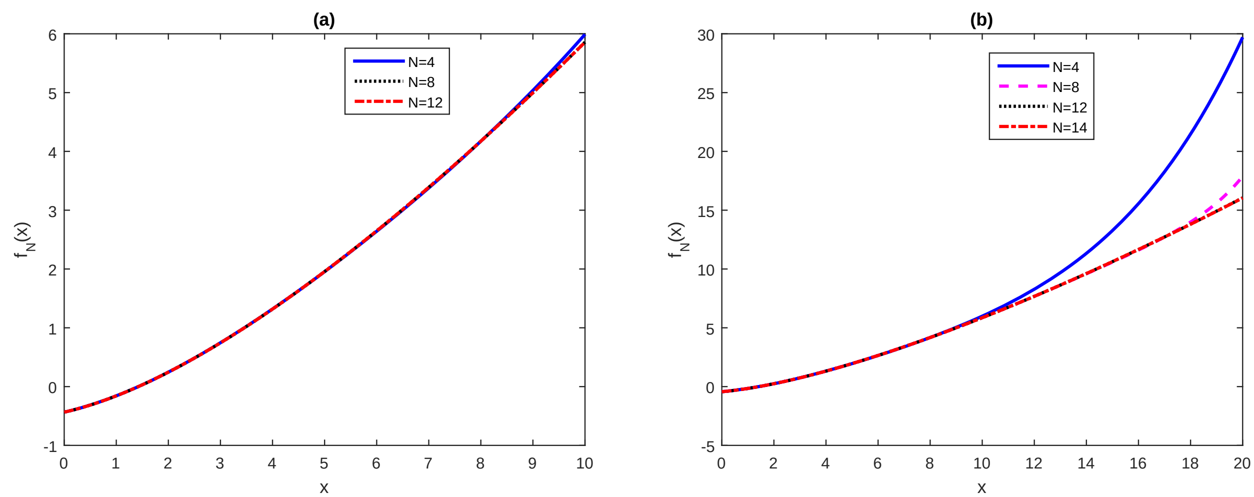

Figure 1. Graphs of truncated solutions $f_N(x)$ for Example 4.5 with $\alpha = 1.5$, $\lambda = 2$. (a) for interval $[0, 10]$ and (b) for interval $[0, 20]$.

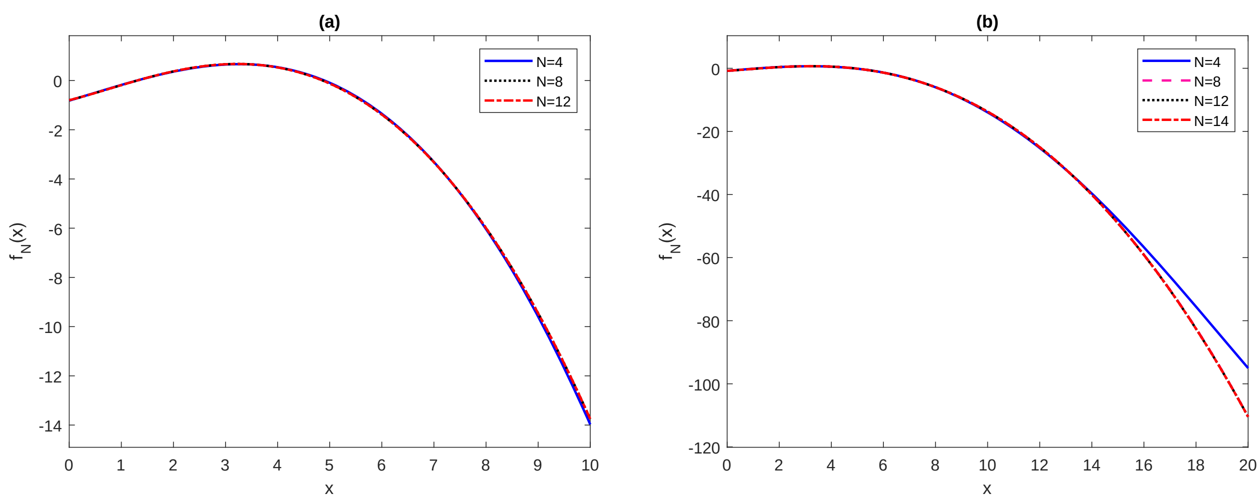

Figure 2. Graphs of truncated solutions $f_N(x)$ for Example 4.5 with $\alpha = 0.9$, $\lambda = 2$. (a) for interval $[0, 10]$ and (b) for interval $[0, 20]$.

5. Conclusions

In this paper, we introduce a fractional Laguerre differential equation which leads to a family of orthogonal polynomials $L_{n,\alpha}(x)$, where α is a positive real number. For $\alpha = 1$, we prove that $L_{n,1}(x)$ are identical to the classical Laguerre polynomials. We find the relation between $L_{n,\alpha}(x)$ and the classical Laguerre polynomials. Moreover, by defining an integral transform corresponding to $L_{n,\alpha}(x)$, we find the exact series solution of

Table 1. Results of Example 4.5 for $\alpha = 1.5$ and $\lambda = 2$.

N	$\|f_{N+2} - f_N\|_\infty$ in $[0,10]$	$\|f_{N+2} - f_N\|_\infty$ in $[0,20]$	$\|f_{N+2} - f_N\|_L$
4	0.1436	4.2109	3.08×10^{-4}
6	0.0129	8.1319	2.79×10^{-5}
8	0.0012	1.9839	2.98×10^{-6}
10	9.24×10^{-5}	0.1624	3.47×10^{-7}
12	8.98×10^{-6}	0.0126	4.28×10^{-8}
14	2.06×10^{-6}	0.0015	5.48×10^{-9}
16	2.46×10^{-7}	9.29×10^{-5}	7.23×10^{-10}

Table 2. Results of Example 4.5 for $\alpha = 0.9$ and $\lambda = 2$.

N	$\|f_{N+2} - f_N\|_\infty$ in $[0,10]$	$\|f_{N+2} - f_N\|_\infty$ in $[0,20]$	$\|f_{N+2} - f_N\|_L$
4	0.2504	15.8528	0.0113
6	0.0317	0.9845	0.0019
8	0.0039	0.1994	3.70×10^{-4}
10	0.0015	0.0582	8.04×10^{-5}
12	1.66×10^{-4}	0.0201	1.84×10^{-5}
14	6.99×10^{-5}	0.0057	4.34×10^{-6}
16	9.93×10^{-6}	6.13×10^{-4}	1.05×10^{-6}

some fractional differential equations. Finally, we introduced two numerical examples showing truncated approximation $f_N(x)$ of the solutions for different values of N and α, shown in Figures 1 and 2 with intervals $[0,10]$ and $[0,20]$, respectively. The norm $\|f_{N+2} - f_N\|$ is decreasing sharply in both examples with increasing N. The results show that using this method we can obtain efficient solutions on large intervals.

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