Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2572 – 2609
© TÜBİTAK
doi:10.55730/1300-0098.3289

Bernstein–Walsh-type inequalities for derivatives of algebraic polynomials on the regions of complex plane

Naciye Pelin ÖZKARTEPE[1], **Cevahir Doğanay GÜN**[2], **Fahreddin ABDULLAYEV**[3,*]
[1]Department of Mathematics, Faculty of Arts and Science, Gaziantep University, Gaziantep, Turkey
[2]Department of Mathematics and Science Education, Nizip Faculty of Education, Gaziantep University, Gaziantep, Turkey
[3]Department of Mathematics, Faculty of Arts and Science, Kyrgyz-Turkish Manas University, Bishkek, Kyrgyz Republic Mersin University, Turkey

Received: 27.12.2021 • **Accepted/Published Online:** 02.06.2022 • **Final Version:** 05.09.2022

Abstract: In this paper, we study Bernstein–Walsh-type estimates for the derivatives of an arbitrary algebraic polynomial on some general regions of the complex plane.

Key words: Algebraic polynomial, quasiconformal mapping, smooth curve, quasismooth curve, asymptotic conformal curve

1. Introduction

Let $\mathbb{C}$ be a complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$ such that $0 \in G$; $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$; $\Delta := \Delta(0,1) := \{w : |w| > 1\}$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto Δ such that $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$; $\Psi := \Phi^{-1}$. For $R > 1$, we take $L_R := \{z : |\Phi(z)| = R\}$, $G_R := intL_R$ and $\Omega_R := extL_R$.

Let $\wp_n$ denote the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $\{z_j\}_{j=1}^{l}$ be the fixed system of distinct points on curve L. For some fixed R_0, $1 < R_0 < \infty$, and $z \in \overline{G}_{R_0}$, consider generalized Jacobi weight function $h(z)$, which is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^{l} |z - z_j|^{\gamma_j}, \tag{1.1}$$

where $\gamma_j > -2$, for all $j = 1, 2, ..., l$, and h_0 is uniformly separated from zero in G_{R_0}, i.e. there exists a constant $c_1(G) > 0$ such that for all $z \in G_{R_0}$, $h_0(z) \geq c_1(G) > 0$.

*Correspondence: fahreddin.abdullayev@manas.edu.kg; fabdul@mersin.edu.tr
2010 *AMS Mathematics Subject Classification:* 30C30, 30E10, 30C70

Let $0<p\leq\infty$ and σ be the two-dimensional Lebesgue measure. For the Jordan region G, we introduce:

$$\begin{aligned}
\|P_n\|_p \quad &: \quad =\|P_n\|_{A_p(h,G)}:=\left(\iint\limits_G h(z)\left|P_n(z)\right|^p d\sigma_z\right)^{1/p},\ 0<p<\infty, \qquad (1.2)\\
\|P_n\|_\infty \quad &: \quad =\|P_n\|_{A_\infty(1,G)}:=\max_{z\in\overline{G}}\left|P_n(z)\right|,\ p=\infty,\\
A_p(1,G) \quad &= \quad :A_p(G),
\end{aligned}$$

and when L is rectifiable:

$$\begin{aligned}
\|P_n\|_{\mathcal{L}_p(h,L)} \quad &: \quad =\left(\int\limits_L h(z)\left|P_n(z)\right|^p |dz|\right)^{1/p}<\infty, 0<p<\infty, \qquad (1.3)\\
\|P_n\|_{\mathcal{L}_\infty(1,L)} \quad &: \quad =\max_{z\in L}\left|P_n(z)\right|,\ p=\infty,\\
\mathcal{L}_p(1,L) \quad &= \quad :\mathcal{L}_p(L).
\end{aligned}$$

Well known Bernstein–Walsh inequality [49] shows that for any $P_n(z)\in\wp_n$

$$\|P_n\|_{C(\overline{G}_R)}\leq\left|\Phi(z)\right|^n\|P_n\|_{C(\overline{G})} \qquad (1.4)$$

holds. Thus, for the points $z\in\overline{G}_R$, $R=1+\frac{1}{n}$, the $\|P_n\|_C$ −norm in $\overline{G}_R$ and $\overline{G}$ have the same order of growth. Also, in [49] some similar estimates for various norms were given for the right hand side of (1.3).

For the weight function $h(z)\equiv 1$ and arbitrary polynomial $P_n(z)\in\wp_n$ in [33] it was shown:

$$\|P_n\|_{\mathcal{L}_p(L_R)}\leq\left|\Phi(z)\right|^{n+\frac{1}{p}}\|P_n\|_{\mathcal{L}_p(L)},\quad p>0. \qquad (1.5)$$

The estimate (1.5) has been generalized in [12, Lemma 2.4] for the weight function $h(z)\neq 1$, defined as in (1.1) for the $\gamma_j>-1$, $j=1,2,...,l$, and it was obtained as follows:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)}\leq R^{n+\frac{1+\gamma^*}{p}}\|P_n\|_{\mathcal{L}_p(h,L)},\ \gamma^*=\max\left\{0;\gamma_j:1\leq j\leq\ l\right\}. \qquad (1.6)$$

To give a similar estimation to (1.6) for the $A_p(h,G)-$norm, first of all, we will give the following definition.

Let the function φ maps G conformally and univalently onto $B:=B(0,1):=\{w:|w|<1\}$ which is normalized by $\varphi(0)=0$ and $\varphi'(0)>0$ and let $\psi:=\varphi^{-1}$.

Following [41, p. 286], a bounded Jordan region G is called a κ-quasidisk, $0\leq\kappa<1$ if any conformal mapping ψ can be extended to a K-quasiconformal homeomorphism of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$ with $K=\frac{1+\kappa}{1-\kappa}$. In that case the curve $L:=\partial G$ is called a κ-quasicircle. The region G (curve L) is called a quasidisk (quasicircle), if it is κ-quasidisk (κ-quasicircle) with some $0\leq\kappa<1$.

A Jordan curve L is called a quasicircle or quasiconformal curve if it is the image of the unit circle under a quasiconformal mapping of $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ ([34]). On the other hand, some geometric criteria of quasiconformality of the curves was given in [15, p. 81] (also, see [42, p. 107]). It is well known that quasicircles can be nonrectifiable (see, for example, [24], [34, p. 104]).

In [8] (also, see [6]) it was given an analog of the estimates (1.4) and (1.6) for the quasidisks in the norm (1.2), with the weight function $h(z)$ which is defined by (1.1), as follows:

$$\|P_n\|_{A_p(h,G_R)} \leq c_1 \; {R^*}^{n+\frac{1}{p}} \, \|P_n\|_{A_p(h,G)} \, , \; p>0,$$

where $R^* := 1 + c_2(R-1)$ and $c_2>0,\; c_1 := c_1(G,p,c_2) > 0$ are constants independent of n and R.

Further, for an arbitrary Jordan region G, any $P_n \in \wp_n,\; R_1 = 1+\frac{1}{n}$, in [9, Theorem1.1] it was obtained that

$$\|P_n\|_{A_p(G_R)} \leq cR^{n+\frac{2}{p}} \, \|P_n\|_{A_p(G_{R_1})} \, , \; p>0,$$

is true for arbitrary $R > R_1 = 1+\frac{1}{n}$, where $c = \left(\frac{2}{e^p-1}\right)^{\frac{1}{p}} \left[1+O(\frac{1}{n})\right], \; n \to \infty.$

N. Stylianopoulos in [45] replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (1.4) and found a new version of the Bernstein–Walsh Lemma: Assume that L is quasiconformal and rectifiable. Then there exists a constant $c = c(L) > 0$ depending only on L such that

$$|P_n(z)| \leq c\frac{\sqrt{n}}{d(z,L)} \, \|P_n\|_{A_2(G)} \, |\Phi(z)|^{n+1} \, , \quad z \in \Omega,$$

where $d(z,L) := \inf\{|\zeta - z| : \; \zeta \in L\}$, holds for every $P_n \in \wp_n$.

In this paper, we continue the study of the problem on pointwise estimates of the derivatives $\left|P_n^{(m)}(z)\right|$ for $m \geq 1$, in unbounded regions of the complex plane and we obtained the estimates as the following type:

$$\left|P_n^{(m)}(z)\right| \leq \eta_n(G,h,p,m,z) \, \|P_n\|_p \, , z \in \Omega, \tag{1.7}$$

where $\eta_n(.) \to \infty$ as $n \to \infty$, depending on the properties of the $G,\; h$.

Analogous results of (1.7)-type for $m=0$, a different weight funtion h, an unbounded region and some norms were obtained in [28, p. 418–428], [36]–[40] and the others.

To get an estimate for the $\left|P_n^{(m)}(z)\right|$ on the whole complex plane, we will need an estimate for the $\left|P_n^{(m)}(z)\right|$ in the bounded region G. To do this, we will use Bernstein–Markov–Nikolsky type estimates for $\left|P_n^{(m)}(z)\right|$, $z \in \overline{G}$, as the following type:

$$\|P_n^{(m)}\|_\infty \leq \lambda_n(G,h,p)\|P_n\|_p, m = 0,1,2,..., \tag{1.8}$$

where $\lambda_n := \lambda_n(G,h,p,m) > 0,\; \lambda_n \to \infty$ as $n \to \infty$, is a constant depending on the properties of the region G and the weight function h.

Estimates of (1.8)- type were studied since the beginning of the 20th century([22], [23], [46]). In recent years, (1.8)- type inequalities for various spaces have been studied in [5], [6], [7], [14] (see also the references cited therein), [18], [26], [27], [28, pp. 418–428], [36], [37, subsection 5.3], [38], [39, pp. 122–133], [44] and the others. For $m = 0$ the similar estimates were continued to be studied in [3]-[8], [10], [13], [20], [21] and the others.

Therefore, combining the estimates (1.7) and (1.8), we will obtain the estimate of $\left|P_n^{(m)}(z)\right|$ for any $m = 1, 2, ...,$ in whole complex plane $\mathbb{C} = \overline{G}_R \cup \Omega_R$, $R > 1$:

$$\left|P_n^{(m)}(z)\right| \leq c_4 \left\|P_n\right\|_p \begin{cases} \lambda_n(G, h, p), & z \in \overline{G}_R \\ \eta_n(G, h, p, d(z, L) \left|\Phi(z)\right|^{n+1}), & z \in \Omega_R, \end{cases} \tag{1.9}$$

where $c_4 = c_4(G, p) > 0$ is a constant independent of n, h, P_n, and $\lambda_n(G, h, p) \to \infty$, $\eta_n(.) \to \infty$ as $n \to \infty$, depending on the properties of the G, h.

2. Definitions and main results

Throughout this paper, $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which in general, depends on G and parameters that inessential for the argument. Otherwise, the dependence will be explicitly stated. For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k + 1, ..., m$.

Let z_1, z_2 be arbitrary points on L and $L(z_1, z_2)$ denotes the subarc of L of shorter diameter with endpoints z_1 and z_2. The curve L is a quasicircle if and only if the quantity

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|}$$

is bounded for all $z_1, z_2 \in L$ and $z \in L(z_1, z_2)$ (three point property). A curve L is called "$c-$quasiconformal" by Lesley (see [35, p. 341]), if there exists the positive constant c, independent of points z_1, z_2 and z such that

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \leq c.$$

The Jordan curve L is called asymptotically conformal [25], [42], if

$$\max_{z \in L(z_1, z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \to 1, \qquad |z_1 - z_2| \to 0.$$

Some various properties of the asymptotically conformal curves have been studied in [16], [29], [30], [31], [32], [43], and references provided therein. According to the geometric criteria of quasiconformality of the curves ([15, p. 81], [42, p. 107]) every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. The asymptotically conformal curves can be nonrectifiable.

As stated in [41, p. 163], we say that a bounded Jordan curve L is $\lambda-$quasismooth or Lavrentiev curve if for every pair z_1, $z_2 \in L$, where $l(z_1, z_2)$ denote the shorter subarc of L, joining $z_1 \in L$ and $z_2 \in L$ and $|l(z_1, z_2)|$ is the linear measure (length) of $l(z_1, z_2)$, there exists a constant $\lambda := \lambda(L) \geq 1$, such that

$$|l(z_1, z_2)| \leq \lambda \left|z_1 - z_2\right|, \; z_1, \; z_2 \in L.$$

In this case, the inner region $intL$ of a Lavrentiev curve L is called a Lavrentiev region.

Let S be a rectifiable Jordan curve or an arc and let $z=z(s),\ s\in[0,\ |S|],\ |S|:=mes\,S$, be the natural parametrization of S.

A Jordan curve or an arc $S\in C_{\theta}$, if S has a continuous tangent $\theta(z):=\theta(z(s))$ at every point $z(s)$. We will write $G\in C_{\theta}$, if $\partial G\in C_{\theta}$.

Following [41, p. 48](see also [19, p. 32]), we say that a Jordan curve S called Dini-smooth, if it has a parametrization $z=z(s),\ 0\leq s\leq |S|:=mes\ S$ such that $z^{'}(s)\neq 0,\ 0\leq s\leq |S|$ and $\left|z^{'}(s_2)-z^{'}(s_1)\right|<g(s_2-s_1),\ s_1<s_2$, where g is an increasing function for which

$$\int\limits_0^1\frac{g(x)}{x}dx<\infty.$$

A Jordan region G has a piecewise Dini-smooth boundary, if $L:=\partial G$ consists of the union of finite Dini-smooth arcs $L_j,\ j=\overline{1,m}$, such that they have exterior (with respect to $\overline{G}$) angles $\lambda_j\pi,\ 0<\lambda_j<2$, at the corner points $\{z_j\}$, $j=\overline{1,m}$, where two arcs meet.

According to the "three-point" criterion [34, p. 100], every piecewise $C_{\theta}-$curve and Dini-smooth curve (without cusps) is quasiconformal.

Now, we give the definitions of quasidisks with some general functional conditions.

Definition 2.1 *We say that $G\in Q_{\alpha}$ if G is a quasicircle and $\Phi\in H^{\alpha}(\overline{\Omega})$ for some $0<\alpha\leq 1$.*

We note that the class Q_{α} is sufficiently large. A detailed account on it and the related topics are contained in [35], [42], [47] (see also the references cited therein). We consider only some cases:

a) If L is a Dini-smooth curve [42], then $G\in Q_1$.

b) If L is a piecewise Dini-smooth curve and largest exterior angle on L has opening $\alpha\pi,\ 0<\alpha<1$, [42, p.52] , then $G\in Q_{\alpha}$.

c) If $L=:\partial G$ is a smooth curve having continuous tangent line, then $G\in Q_{\alpha}$ for all $0<\alpha<1$.

d) If G is "L-shaped" region, then $\Phi\in Lip\frac{2}{3}$.

e) If L is quasismooth (in the sense of Lavrentiev), then $\Phi\in Lip\ \alpha$ for $\alpha=\frac{1}{2}(1-\frac{1}{\pi}\arcsin\frac{1}{c})^{-1}$ [47], [48].

f) If L is "c-quasiconformal" (see, for example, [35]), then $\Phi\in Lip\ \alpha$ for $\alpha=\frac{\pi}{2(\pi-\arcsin\frac{1}{c})}$.

g) If L is an asymptotic conformal curve, then $\Phi\in Lip\ \alpha$ for all $0<\alpha<1$ [35].

Now, we can state the corresponding results for the class of regions $G\in Q_{\alpha}$. First, we give one more theorem that we will use in this case, and after that we give estimate for $\left|P_n^{(m)}(z)\right|,\ z\in\overline{G}$, for each $m\geq 0$.

Theorem A [14, Th. 1] Let $0<p\leq\infty$; $L\in Q_{\alpha}$ for some $0<\alpha\leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n\in\wp_n,\ n\in\mathbb{N}$, and every $m=0,1,2,\ldots$

$$\|P_n^{(m)}\|_{\infty}\leq c_4\|P_n\|_p\begin{cases}n^{\frac{1}{\alpha}\left(\frac{\gamma^*+2}{p}+m\right)}, & \frac{1}{2}\leq\alpha\leq 1,\\ n^{\delta\left(\frac{\gamma^*+2}{p}+m\right)}, & 0<\alpha<\frac{1}{2},\end{cases}\tag{2.1}$$

where here and throughout this paper

$$\gamma^{*} := \max\left\{0; \gamma_j,\ j=\overline{1,l}\right\}, \tag{2.2}$$

and $\delta=\delta(G),\ 1\le\delta\le 2$, is a certain number depending on the specified region.

Recall that for any $m\ge 1$ and for $m=0$, the sharpness of estimate (2.1) was given in [13, Th. 2.1].

For $0<\delta_j<\delta_0:=\frac{1}{4}\min\{|z_i-z_j|: i,j=1,2,...,l,\ i\neq j\}$, let $\Omega(z_j,\ \delta_j):=\Omega\cap\{z:|z-z_j|\le\delta_j\}$; $\delta:=\min\limits_{1\le j\le l}\delta_j$; for $L=\partial G$ we set: $U_\infty(L,\delta):=\bigcup\limits_{\zeta\in L}U(\zeta,\ \delta)-$infinite open cover of the curve L; $U_N(L,\delta):=\bigcup\limits_{j=1}^{N}U_j(L,\delta)\subset U_\infty(L,\delta)-$finite open cover of the curve L; $\Omega(\delta):=\Omega(L,\delta):=\Omega\cap U_N(L,\delta)$, $\widehat{\Omega}:=\Omega\setminus\Omega(\delta)$; $\Omega_R(\delta):=\Omega(L_R,\delta):=\Omega_R\cap U_N(L_R,\delta)$, $\widehat{\Omega}_R:=\Omega_R\setminus\Omega_R(\delta)$.

Now, we start to formulate the new results.

2.1. The general estimate (recurrence formula)

First, we present a general estimate of $\left|P_n^{(m)}(z)\right|$ for which it will be possible to obtain estimates for the derivative for each order $m=1,2,...$

Theorem 2.2 *Let $p\ge 2$; $G\in Q_\alpha$ for some $\frac{1}{2}\le\alpha\le 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n\in\wp_n$, $n\in\mathbb{N}$, and every $m=1,2,...,$ we have:*

$$\left|P_n^{(m)}(z)\right|\le c_1\ \left|\Phi^{n+1}(z)\right|\left\{\frac{\|P_n\|_p}{d(z,L)}A^1_{n,p}(z,m)+\sum_{j=1}^{m}C_m^jB^1_{n,j}(z)\left|P_n^{(m-j)}(z)\right|\right\}, \tag{2.3}$$

where $c_1=c_1(G,\gamma,m,p)>0$ is a constant independent of n and z;

$$A^1_{n,p}(z,m)\ :\ =\begin{cases} n^{\left(\frac{\gamma^*+2}{p}+m-1\right)\frac{1}{\alpha}}, & p\ge 2, \quad m\ge 2, \quad \gamma>-2,\\ n^{\frac{\gamma^*+2}{p\alpha}}, & 2\le p<1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad \gamma>-2+\alpha,\\ (n\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad \gamma>-2+\alpha,\\ n^{1-\frac{1}{p}}, & p>1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad \gamma>-2+\alpha,\\ n^{1-\frac{1}{p}}, & p\ge 2, \quad m=1, \quad -2<\gamma\le -2+\alpha,\end{cases}$$

$$B^1_{n,j}(z)\ :\ =n^{\frac{j}{\alpha}}, j=1,2,...,m,$$

if $z\in\Omega(\delta)$, and

$$A^1_{n,p}(z,m)\ :\ =\begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & 2\le p<\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, \quad \gamma>\alpha,\\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, \quad \gamma>\alpha,\\ n^{1-\frac{1}{p}}, & p>\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, \quad \gamma>\alpha,\\ n^{1-\frac{1}{p}}, & p\ge 2, \quad -2<\gamma\le\alpha,\end{cases}$$

$$B^1_{n,j}(z)\ :\ =n, j=\overline{1,m},$$

if $z\in\widehat{\Omega}(\delta)$.

Theorem 2.3 *Let* $1 < p < 2$; $G \in Q_\alpha$ *for some* $\frac{1}{2} \leq \alpha \leq 1$, *and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and every* $m = 1, 2, ...,$ *we have:*

$$\left|P_n^{(m)}(z)\right| \leq c_2 \ \left|\Phi^{n+1}(z)\right| \left\{ \frac{\|P_n\|_p}{d(z,L)} A^2_{n,p}(z,m) + \sum_{j=1}^{m} C_m^j B^1_{n,j}(z) \left|P_n^{(m-j)}(z)\right| \right\}, \tag{2.4}$$

where $c_2 = c_2(G, \gamma, m, p) > 0$ *is a constant independent of* n *and* z;

$$A^2_{n,p}(z,m) := n^{(\frac{\gamma^*+2}{p}+m-1)\frac{1}{\alpha}},$$

if $z \in \Omega(\delta)$,

$$A^2_{n,p}(z,m) := \begin{cases} n^{(\frac{\gamma+2}{p}+m-1)\frac{1}{\alpha}}, & 1<p<2, \quad \gamma \geq \alpha, \\ n^{(\frac{\gamma+2}{p}+m-1)\frac{1}{\alpha}}, & p<1+\frac{\gamma}{\alpha}, \quad 0<\gamma<\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}}(n\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma}{\alpha}, \quad 0<\gamma<\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}, & p>1+\frac{\gamma}{\alpha}, \quad 0<\gamma<\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1<p<2, \quad -2<\gamma\leq 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$ *and* $B^1_{n,j}(z)$ *is defined as in Theorem 2.2.*

In order to get the final estimates for $\left|P_n^{(m)}(z)\right|$, $m \geq 2$, we need an estimate for the $\left|P_n^{(m-j)}(z)\right|$ for each $j = \overline{1,m}$. Consider the case $m = 2$ and $j = 1, 2$. Cases of $m \geq 3$ are carried out sequentially by applying the estimates obtained from (2.3) and (2.4).

2.2. Estimate for $|P_n(z)|$

Theorem 2.4 *Let* $p > 1$; $G \in Q_\alpha$ *for some* $\frac{1}{2} \leq \alpha \leq 1$, *and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and* $z \in \Omega_R$, *we have:*

$$|P_n(z)| \leq c_3 \frac{\left|\Phi^{n+1}(z)\right|}{d(z,L)} A^3_{n,p} \|P_n\|_p, \tag{2.5}$$

where $c_3 = c_3(G, \gamma, p) > 0$ *is a constant independent of* n *and* z;

$$A^3_{n,p} := \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, \quad \gamma \geq \alpha, \\ n^{1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, \quad \gamma \geq \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, \quad \gamma \geq \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, \quad 0 \leq \gamma < \alpha, \\ n^{\left(1-\frac{2}{p}\right)\frac{1}{\alpha}+\frac{3}{p}-1}, & p \geq 2, \quad -2 < \gamma < 0, \end{cases}$$

for any $p \geq 2$ *and*

$$A^3_{n,p} := \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}}, & 1<p<2, \quad \gamma \geq \alpha, \\ n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}}, & 1<p<1+\frac{\gamma}{\alpha}, \quad 0<\gamma<\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}}(n\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma}{\alpha}, \quad 0<\gamma<\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1+\frac{\gamma}{\alpha}<p<2, \quad 0\leq\gamma<\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1<p<2, \quad -2<\gamma<0, \end{cases}$$

for $1 < \ p < 2.$

The following remark shows the sharpness of Theorem 2.4.

Remark 2.5 *For any* $n \in \mathbb{N}$ *there exists* $Q_n \in \wp_n$ *and* $G^* \subset \mathbb{C}$ *such that*

$$|Q_n(z)| \geq c\frac{\sqrt{n}}{d(z,L)} \left\|Q_n\right\|_{A_2(G^*)} \ |\Phi(z)|^{n+1}, \quad z \in F \Subset \Omega^* := C\overline{G}^*,$$

where $c = c(G^*) > 0.$

We note that the estimate for $|P_n(z)|$ was previously obtained in our paper [13, Th. 3] for $p > 0$. But this result is better for $p \geq 2$ and coincides with that of [13, Th. 3] for $1 < p < 2$.

2.3. Estimate for $|P_n'(z)|$

Now, using Theorems 2.2, 2.3 and 2.4, we can give an estimate for $|P_n'(z)|$ for $z \in \Omega_R$.

Theorem 2.6 *Let* $p \geq 2$; $G \in Q_\alpha$ *for some* $\frac{1}{2} \leq \alpha \leq 1$, *and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have:*

$$|P_n'(z)| \leq c_4\frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} \left\|P_n\right\|_p \ A^4_{n,p}(z), \tag{2.6}$$

where $c_4 = c_4(G,\gamma,p) > 0$ *is a constant independent of* n *and* z;

$$A^4_{n,p}(z) := \begin{cases} n^{\frac{\gamma+2}{p\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{\frac{1}{\alpha}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{\frac{1}{\alpha}+1-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{\frac{1}{\alpha}+1-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma < \alpha, \\ n^{\frac{3}{p}+\left(2-\frac{2}{p}\right)\frac{1}{\alpha}-1}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, *and*

$$A^4_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}+1}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{2-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{2-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{2-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma < \alpha, \\ n^{\left(1-\frac{2}{p}\right)\frac{1}{\alpha}+\frac{3}{p}}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

Theorem 2.7 *Let* $1 < p < 2$; $G \in Q_\alpha$ *for some* $\frac{1}{2} \leq \alpha \leq 1$, *and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have:*

$$|P_n'(z)| \leq c_5\frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} \left\|P_n\right\|_p \ A^5_{n,p}(z), \tag{2.7}$$

where $c_5 = c_5(G, \gamma, p) > 0$ *is a constant independent of* n *and* z*;*

$$A^5_{n,p}(z) := \begin{cases} n^{\frac{\gamma+2}{p\alpha}}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{\frac{\gamma+2}{p\alpha}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{\frac{2}{p\alpha}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{\frac{2}{p\alpha}+1-\frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{\frac{2}{p\alpha}+1-\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$*, and*

$$A^5_{n,p}(z) = \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}+1}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}+1}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1} (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+2-\frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+2-\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

2.4. Estimate for $|P''_n(z)|$

Considering the estimates obtained in Theorems 2.6, 2.7 for $|P'_n(z)|$ and Theorem 2.4 for $|P_n(z)|$ in the Theorems 2.2, 2.3 respectively, we can give the following:

Theorem 2.8 *Let* $p \geq 2$*;* $G \in Q_\alpha$ *for some* $\frac{1}{2} \leq \alpha \leq 1$*, and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$*,* $n \in \mathbb{N}$*, we have:*

$$|P''_n(z)| \leq c_6 \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L)} \|P_n\|_p A^6_{n,p}(z), \tag{2.8}$$

where $c_6 = c_6(G, \gamma, p) > 0$ *is a constant independent of* n *and* z*;*

$$A^6_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & 0 \leq \gamma < \alpha, \\ n^{\left(3-\frac{2}{p}\right)\frac{1}{\alpha}+\frac{3}{p}-1}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$*, and*

$$A^6_{n,p}(z) := \begin{cases} n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{3-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{3-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{3-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma < \alpha \\ n^{(1-\frac{2}{p})\frac{1}{\alpha}+\frac{3}{p}+1}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

Theorem 2.9 *Let* $1 < p < 2$; $G \in Q_\alpha$ *for some* $\frac{1}{2} \le \alpha \le 1$, *and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have:*

$$|P_n''(z)| \le c_7 \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L)} \|P_n\|_p A^7_{n,p}(z), \tag{2.9}$$

where $c_7 = c_7(G,\gamma,p) > 0$ *is a constant independent of* n *and* z;

$$A^7_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1 < p < 2, \quad \gamma \ge \alpha, \\ n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1 < p < 1+\frac{\gamma}{\alpha}, \quad 0 < \gamma < \alpha, \\ n^{\left(\frac{2}{p}+1\right)\frac{1}{\alpha}} (n \ln n)^{1-\frac{1}{p}}, & p = 1+\frac{\gamma}{\alpha}, \quad 0 < \gamma < \alpha, \\ n^{\left(\frac{2}{p}+1\right)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1+\frac{\gamma}{\alpha} < p < 2, \quad 0 \le \gamma < \alpha, \\ n^{\left(\frac{2}{p}+1\right)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1 < p < 2, \quad -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, *and*

$$A^7_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1 < p < 2, \quad \gamma \ge \alpha, \\ n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1 < p < 1+\frac{\gamma}{\alpha}, \quad 0 < \gamma < \alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+3-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1+\frac{\gamma}{\alpha}, \quad 0 < \gamma < \alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+3-\frac{1}{p}}, & 1+\frac{\gamma}{\alpha} < p < 2, \quad 0 \le \gamma < \alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+3-\frac{1}{p}}, & 1 < p < 2, \quad -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

2.5. Estimates for $|P_n'(z)|$ and $|P_n''(z)|$ in whole plane

According to (1.4) (applied to the polynomial $Q_{n-1}(z) := P_n'(z)$), the estimation (2.1) is also true for the points $z \in \overline{G}_R$, $R = 1 + \varepsilon_0 n^{-1}$, with a different constant. Therefore, combining the estimates (2.1) (for the $z \in \overline{G}_R$) with (2.6), (2.7), (2.8), (2.9), we will obtain the estimate on the growth of $|P_n'(z)|$ and $|P_n''(z)|$ respectively, in the whole complex plane:

Theorem 2.10 *Let* $p \ge 2$; $G \in Q_\alpha$ *for some* $\frac{1}{2} \le \alpha \le 1$, *and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have:*

$$|P_n'(z)| \le c_8 \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+1\right)\frac{1}{\alpha}}, & z \in \overline{G}_R, \\ \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} A^4_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_8 = c_8(G,\gamma,p) > 0$ *is a constant independent of* n *and* z; $A^4_{n,p}(z)$ *is defined as in Theorem 2.6 for all* $z \in \Omega_R$.

Theorem 2.11 *Let* $1 < p < 2$; $G \in Q_\alpha$ *for some* $\frac{1}{2} \le \alpha \le 1$, *and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have:*

$$|P_n'(z)| \le c_9 \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+1\right)\frac{1}{\alpha}}, & z \in \overline{G}_R, \\ \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} A^5_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_9 = c_9(G,\gamma,p) > 0$ *is a constant independent of* n *and* z; $A^5_{n,p}(z)$ *is defined as in Theorem 2.7 for all* $z \in \Omega_R$.

Theorem 2.12 *Let* $p \geq 2$; $G \in Q_\alpha$ *for some* $\frac{1}{2} \leq \alpha \leq 1$, *and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have:*

$$|P''_n(z)| \leq c_{10} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+2\right)\frac{1}{\alpha}}, & z \in \overline{G}_R, \\ \frac{|\Phi^{3(n+1)}(z)|}{d(z,L)} A^6_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_{10} = c_{10}(G,\gamma,p) > 0$ *is a constant independent of* n *and* z; $A^6_{n,p}(z)$ *is defined as in Theorem 2.8 for all* $z \in \Omega_R$.

Theorem 2.13 *Let* $1 < p < 2$; $G \in Q_\alpha$ *for some* $\frac{1}{2} \leq \alpha \leq 1$, *and* $h(z)$ *be defined by (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have:*

$$|P''_n(z)| \leq c_{11} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+2\right)\frac{1}{\alpha}}, & z \in \overline{G}_R, \\ \frac{|\Phi^{3(n+1)}(z)|}{d(z,L)} A^7_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_{11} = c_{11}(G,\gamma,p) > 0$ *is a constant independent of* n *and* z; $A^7_{n,p}(z)$ *is defined as in Theorem 2.9 for all* $z \in \Omega_R$.

Thus, using Theorems 2.2, 2.3 and estimating the $\left|P^{(m)}_n(z)\right|$ sequentially for each $m \geq 3$, and combining the obtained estimates with Theorem A, we can obtain estimates for the $\left|P^{(m)}_n(z)\right|$ on the whole complex plane.

3. Some auxiliary results

Throughout this paper we denote "$a \preceq b$" and "$a \asymp b$" are equivalent to $a \leq cb$ and $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 respectively.

Lemma 3.1 *[1]Let* G *be a quasidisk,* $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1,2,3$. *Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ *and* $|w_1 - w_2| \preceq |w_1 - w_3|$ *are equivalent. So are* $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$.

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_1} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_2},$$

where $0 < r_0 < 1$ *is a constant depending on* G.

Corollary 3.2 *Under the conditions of Lemma 3.1, we have:*

$$|w_1 - w_2|^{c_1} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{\varepsilon},$$

where $\varepsilon = \varepsilon(G) < 1$.

Lemma 3.3 *Let $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$. Then, for all $w_1, w_2 \in \overline{\Omega}'$, we have:*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{\frac{1}{\alpha}}.$$

This fact follows from of an appropriate result for the mapping $f \in \sum(\kappa)$ [41, p. 287] and estimation for the Ψ' [17, Th. 2.8]:

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{3.1}$$

Lemma 3.4 *[11] Let L be a $K-$quasiconformal curve; $R = 1 + \frac{c}{n}$. Then, for any fixed $\varepsilon \in (0,1)$ there exists a level curve $L_{1+\varepsilon(R-1)}$ such that the following holds for any polynomial $P_n(z) \in \wp_n$,*

$$\|P_n\|_{\mathcal{L}_p\left(\frac{h}{|\Phi'|},\, L_{1+\varepsilon(R-1)}\right)} \preceq n^{\frac{1}{p}} \|P_n\|_p, \ p > 0.$$

Let $\{z_j\}_{j=1}^{l}$ be a fixed system of the points on L and the weight function $h(z)$ be defined as in (1.1). The following result is the integral analog of the familiar lemma of Bernstein–Walsh [49, p. 101] for the $A_p(h, G)$ -norm.

Lemma 3.5 *[5] Let G be a quasidisk and $P_n(z)$, $\deg P_n \leq n, n = 1, 2, ...,$ is arbitrary polynomial and weight function $h(z)$ satisfied the condition(1.1). Then for any $R > 1$, $p > 0$ and $n = 1, 2, ...$*

$$\|P_n\|_{A_p(h, G_R)} \leq c_3 \left(1 + c(R-1)\right)^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}$$

where c, c_3 are independent of n and G.

4. Proof of Theorems

Proof [of Theorems 2.2 and 2.3]. The proofs of Theorem 2.2 and 2.3 will be simultaneously. Let $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$ and let $R = 1 + \frac{1}{n}$, $R_1 := 1 + \frac{R-1}{2}$. For $z \in \Omega$ assume that:

$$H_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}.$$

We represent the $m-$th derivative of $H_{n,p}(z)$ as follows:

$$\begin{aligned} H_n^{(m)}(z) &= \left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)} \\ &= \sum_{j=0}^{m} C_m^j \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} P_n^{(m-j)}(z) \\ &= \frac{P_n^{(m)}(z)}{\Phi^{n+1}(z)} + \sum_{j=1}^{m} C_m^j \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} P_n^{(m-j)}(z), \end{aligned}$$

where $C_m^j := \frac{m(m-1)...(m-j+1)}{j!}$. Therefore,

$$P_n^{(m)}(z) = \Phi^{n+1}(z)\left\{\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)} - \sum_{j=1}^{m} C_m^j \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} P_n^{(m-j)}(z)\right\}, z \in \Omega.$$

Then,

$$\left|P_n^{(m)}(z)\right| \leq \left|\Phi^{n+1}(z)\right|\left\{\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right| + \sum_{j=1}^{m} C_m^j \left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right| \left|P_n^{(m-j)}(z)\right|\right\}. \tag{4.1}$$

As can be seen from (4.1), the following two statements on the right side need to be evaluated for $z \in \Omega$ in order to obtain the evaluation for $\left|P_n^{(m)}(z)\right|$:

$$\text{A)} \left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right|; \ \text{B)} \left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right|.$$

Now, we start to evaluate them.

A) Since the function $H_n(z)$, $H_n(\infty) = 0$, is analytic in Ω, continuous on $\overline{\Omega}$, then Cauchy integral representation for the derivatives gives:

$$H_n^{(m)}(z) = -\frac{1}{2\pi i}\int_{L_{R_1}} H_n(\zeta)\frac{d\zeta}{(\zeta - z)^{m+1}}, \ z \in \Omega_R, \ m \geq 1.$$

Then,

$$\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right| \leq \frac{1}{2\pi}\int_{L_{R_1}}\left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta - z|^{m+1}} \leq \frac{1}{2\pi d(z, L_{R_1})}\int_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta - z|^{m}}. \tag{4.2}$$

Denote by

$$A_n(z) := \int_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta - z|^{m}}, \tag{4.3}$$

and estimate these integrals separately.

For this we give some notations.

Let $w_j := \Phi(z_j), \varphi_j := \arg w_j$. Without loss of generality, we will assume that $\varphi_l < 2\pi$. For $\eta := \min\left\{\eta_j, j = \overline{1, l}\right\}$, where $\eta_j = \min\limits_{t \in \partial\Phi(\Omega(z_j,\ \delta_j))}|t - w_j| > 0$, we set that:

$$\begin{aligned}
\Delta_j(\eta_j) \ &: = \{t : |t - w_j| \leq \eta_j\} \subset \Phi(\Omega(z_j,\ \delta_j)),\\
\Delta(\eta) \ &: = \bigcup_{j=1}^{l}\Delta_j(\eta), \ \widehat{\Delta}_j = \Delta\backslash\Delta(\eta_j); \widehat{\Delta}(\eta) := \Delta\backslash\Delta(\eta); \ \Delta_1^{'} := \Delta_1^{'}(1),\\
\Delta_1^{'}(\rho) \ &: = \left\{t = Re^{i\theta} : R \geq \rho > 1, \ \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\},\\
\Delta_j^{'} \ &: = \Delta_j^{'}(1), \Delta_j^{'}(\rho) := \left\{t = Re^{i\theta} : R \geq \rho > 1, \ \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_0}{2}\right\}, \ j = 2, 3..., \ l,
\end{aligned}$$

where $\varphi_0 = 2\pi - \varphi_l;\ \Omega_j := \Psi(\Delta_j^{'}), L_{R_1}^j := L_{R_1} \cap \Omega_j;\ \Omega = \bigcup_{j=1}^{l} \Omega_j$.

For the simplicity of the calculations, we can limit ourselves to only one point on the boundary where the weight function has singularities, i.e. let $h(z)$ be defined as in (1.1) for $l = 1$ and we put: $\gamma_1 =: \gamma$. To estimate $A_n(z)$, first of all replacing the variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the integrand by $|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}\ |\Psi'(\tau)|^{\frac{2}{p}}$ and applying the Hölder inequality, we obtain:

$$\begin{aligned} A_n(z) &= \int\limits_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^m} = \sum_{i=1}^{2} \int\limits_{F_{R_1}^i} \frac{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma_j}{p}} \left| P_n(\Psi(\tau))(\Psi'(\tau))^{\frac{2}{p}} \right| |\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}} |\Psi(\tau) - \Psi(w)|^m} |d\tau| \\ &\leq \sum_{i=1}^{2} \left(\int\limits_{F_{R_1}^i} |\Psi(\tau) - \Psi(w_1)|^{\gamma} |P_n(\Psi(\tau))|^p |\Psi'(\tau)|^2 |d\tau| \right)^{\frac{1}{p}} \\ &\times \left(\int\limits_{F_{R_1}^i} \left(\frac{|\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}} |\Psi(\tau) - \Psi(w)|^m} \right)^q |d\tau| \right)^{\frac{1}{q}} =: \sum_{i=1}^{2} A_n^i(z), \end{aligned}$$

where $F_{R_1}^1 := \Phi(L_{R_1}^1) = \Delta_1^{'} \cap \{\tau : |\tau| = R_1\}$, $F_{R_1}^2 := \Phi(L_{R_1}) \backslash F_{R_1}^1$ and

$$\begin{aligned} A_n^i(z) &:= \left(\int\limits_{F_{R_1}^i} |f_{n,p}(\tau)|^p |d\tau| \right)^{\frac{1}{p}} \left(\int\limits_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}} |d\tau| \right)^{\frac{1}{q}} \\ &=: J_{n,1}^i \cdot J_{n,2}^i(z), \\ f_{n,p}(\tau) &:= (\Psi(\tau) - \Psi(w_1))^{\frac{\gamma}{p}} P_n(\Psi(\tau))(\Psi'(\tau))^{\frac{2}{p}},\ |\tau| = R_1. \end{aligned}$$

Applying to Lemma 3.4, we get:

$$J_{n,1}^i \preceq n^{\frac{1}{p}} \|P_n\|_p,\ i = 1, 2.$$

For the estimation of the integral

$$\left(J_{n,2}^i(z)\right)^q = \int\limits_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}} |d\tau| \tag{4.4}$$

for $i = 1, 2$, we set:

$$\begin{aligned} E_{R_1}^{11} &:= \left\{\tau : \tau \in F_{R_1}^1,\ |\tau - w_1| < c_1(R_1 - 1)\right\}, \\ E_{R_1}^{12} &:= \left\{\tau : \tau \in F_{R_1}^1,\ c_1(R_1 - 1) \leq |\tau - w_1| < \eta\right\}, \\ E_{R_1}^{13} &:= \{\tau : \tau \in \Phi(L_{R_1}), \eta \leq |\tau - w_1| < \eta^*\}, \eta^* = \eta^*(G), \end{aligned} \tag{4.5}$$

where $0 < c_1 < \eta$ is chosen so that $\{\tau : |\tau - w_1| < c_1(R_1 - 1)\} \cap \Delta \neq \varnothing$ and $\Phi(L_{R_1}) = \bigcup_{k=1}^{3} E_{R_1}^{1k}$. Taking into consideration these notations, from (4.4) we have:

$$J_{n,2}^1(z) + J_{n,2}^2(z) =: J_2(z) = J_2(E_{R_1}^{11}) + J_2(E_{R_1}^{12}) + J_2(E_{R_1}^{13}) =: J_2^1(z) + J_2^2(z) + J_2^3(z)$$

and, consequently,

$$\begin{aligned} A_n(z) &= A_n^1(z) + A_n^2(z) \preceq n^{\frac{1}{p}} \|P_n\|_p \cdot \left[J_2^1(z) + J_2^2(z) + J_2^3(z) + J_2^4(z)\right] \\ &= : A_{n,1}(z) + A_{n,2}(z) + A_{n,3}(z), \end{aligned} \tag{4.6}$$

where

$$A_{n,k}(z) := n^{\frac{1}{p}} \|P_n\|_p \cdot J_2^k(z), \; k = 1,2,3,$$

$$\left(J_2^k(z)\right)^q := \int\limits_{E_{R_1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}}, \; k = 1,2,3.$$

For any $k = 1,2,$ denote by

$$E_{R_1,1}^{1k} := \left\{\tau \in E_{R_1}^{1k} : |\Psi(\tau) - \Psi(w_1)| \geq |\Psi(\tau) - \Psi(w)|\right\}, \; E_{R_1,2}^{1k} := E_{R_1}^{1k} \backslash E_{R_1,1}^{1k},$$

$$\begin{aligned} \left(I(E_{R_1,1}^{1k})\right)^q &: = \begin{cases} \int\limits_{E_{R_1,1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}}, & if \; \gamma \geq 0, \\ \int\limits_{E_{R_1,1}^{1k}} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}}, & if \; \gamma < 0, \end{cases} \\ \left(I(E_{R_1,2}^{1k})\right)^q &: = \int\limits_{E_{R_1,2}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+qm}}, \; k = 1,2, \end{aligned} \tag{4.7}$$

and estimate the last integrals.

Given the possible values of q ($q > 2$ and $q \leq 2$) and γ ($-2 < \gamma < 0$ and $\gamma \geq 0$), we will consider the cases separately.

Case 1. Let $1 < q \leq 2$ ($p \geq 2$). Then,

$$\left(I(E_{R_1,1}^{1k})\right)^q = \int\limits_{E_{R_1,1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}}.$$

1.1. Let $\gamma \geq 0$. If $z \in \Omega(\delta)$, applying Lemma 3.3 to (3.1), we get:

$$\left(I(E_{R_1,1}^{11})\right)^q \leq \int\limits_{E_{R_1,1}^{11}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}} \tag{4.8}$$

$$\begin{aligned}
&\asymp \int\limits_{E^{11}_{R_1,1}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \preceq n^{2-q} \int\limits_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha}}} \\
&\preceq \quad n^{2-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}} mesE^{11}_{R_1,1} \preceq n^{1-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}};
\end{aligned}$$

$$I(E^{11}_{R_1,1}) \preceq n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}.$$

$$\begin{aligned}
\left(I(E^{11}_{R_1,2})\right)^q &\preceq \int\limits_{E^{11}_{R_1,2}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \\
&\preceq \quad n^{2-q} \int\limits_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha}}} \preceq n^{2-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}} mesE^{11}_{R_1,2} \\
&\preceq \quad n^{1-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}}; \\
I(E^{11}_{R_1,2}) &\preceq \quad n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned}$$

$$\left(I(E^{12}_{R_1,1})\right)^q \preceq \int\limits_{E^{12}_{R_1,1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \tag{4.9}$$

$$\begin{aligned}
&\preceq \int\limits_{E^{12}_{R_1,1}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \preceq n^{2-q} \int\limits_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha}}} \\
&\preceq n^{2-q} \begin{cases} n^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha}-1}, & \gamma(q-1)+qm-(2-q)>\alpha, \\ \ln n, & \gamma(q-1)+qm-(2-q)=\alpha, \\ 1, & \gamma(q-1)+qm-(2-q)<\alpha; \end{cases}
\end{aligned}$$

$$I(E^{12}_{R_1,1}) \preceq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma>-2, \quad m\ge 2, \\ n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p<1+\frac{\gamma+2}{\alpha}, \quad m=1, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, \quad m=1, \\ n^{1-\frac{2}{p}}, & p>1+\frac{\gamma+2}{\alpha}, \quad m=1. \end{cases}$$

$$\left(I(E^{12}_{R_1,2})\right)^q \le \int\limits_{E^{12}_{R_1,2}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \preceq n^{2-q} \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{\gamma(q-1)+qm-(2-q)}} \tag{4.10}$$

$$\preceq n^{2-q} \begin{cases} n^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha}-1}, & \gamma(q-1)+qm-(2-q)>\alpha, \\ \ln n, & \gamma(q-1)+qm-(2-q)=\alpha, \\ 1, & \gamma(q-1)+qm-(2-q)<\alpha; \end{cases}$$

$$I(E^{12}_{R_1,2}) \preceq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma>-2, \quad m\ge 2, \\ n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p<1+\frac{\gamma+2}{\alpha}, \quad m=1, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, \quad m=1, \\ n^{1-\frac{2}{p}}, & p>1+\frac{\gamma+2}{\alpha}, \quad m=1, \end{cases}$$

$$I(E^{12}_{R_1,1})+I(E^{12}_{R_1,2})\preceq\begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma>-2, \quad m\geq 2,\\ n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p<1+\frac{\gamma+2}{\alpha}, \quad m=1,\\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, \quad m=1,\\ n^{1-\frac{2}{p}}, & p>1+\frac{\gamma+2}{\alpha}, \quad m=1.\end{cases}$$

For $\tau\in E^{13}_{R_1}$ we see that $\eta<|\tau-w_1|<2\pi\dot{R}_1$, $|\tau-w|\geq\eta-c_1$. Therefore, $|\Psi(\tau)-\Psi(w_1)|\succeq 1$, from Lemma 3.1 and for $|\tau-w_1|\geq\eta$, $|\Psi(\tau)-\Psi(w)|\succeq|\tau-w|^{\frac{1}{\alpha}}$, from Lemma 3.3. Then, for $w\in\Delta(w_1,\eta)$, applying (3.1), we get:

$$\begin{aligned}\left(J^3_2(z)\right)^q &\preceq \int\limits_{E^{13}_{R_1}}\frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}}\preceq\int\limits_{E^{13}_{R_1}}\left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}}\\ &\preceq n^{2-q}\int\limits_{E^{11}_{R_1,1}}\frac{|d\tau|}{|\tau-w|^{\frac{qm-(2-q)}{\alpha}}}\\ &\preceq n^{2-q}\begin{cases} n^{\frac{qm-(2-q)}{\alpha}-1}, & qm-(2-q)>\alpha, \quad m\geq 2,\\ n^{\frac{qm-(2-q)}{\alpha}-1}, & qm-(2-q)>\alpha, \quad m=1,\\ \ln n, & qm-(2-q)=\alpha, \quad m=1,\\ 1, & qm-(2-q)<\alpha, \quad m=1;\end{cases}\end{aligned}$$

$$\begin{aligned} J^3_2(z) &\preceq \begin{cases} n^{\left(\frac{2}{p}+m-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & p\geq 2, \quad m\geq 2,\\ n^{\frac{2}{p\alpha}-\frac{1}{p}}, & p<1+\frac{2}{\alpha}, \quad m=1,\\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=1+\frac{2}{\alpha}, \quad m=1,\\ n^{1-\frac{2}{p}}, & p>1+\frac{2}{\alpha}, \quad m=1,\end{cases} \quad z\in\Omega(\delta), \\ J^3_2(z) &\preceq n^{\left(\frac{1}{\alpha}-1\right)\left(1-\frac{2}{p}\right)}, \quad z\in\widehat{\Omega}(\delta).\end{aligned} \tag{4.11}$$

Combining (4.8–4.11), for $p\geq 2,\gamma\geq 0$ and $z\in\Omega(\delta)$, we get:

$$\sum_{k=1}^{3}J^k_2(z)\preceq\begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma>-2, \quad m\geq 2,\\ n^{\left(\frac{\gamma}{p}+1\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & 2\leq p<1+\frac{\gamma+2}{\alpha}, \quad m=1,\\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, \quad m=1,\\ n^{1-\frac{2}{p}}, & p>1+\frac{\gamma+2}{\alpha}, \quad m=1.\end{cases} \tag{4.12}$$

If $z\in\widehat{\Omega}(\delta)$, then

$$\begin{aligned}\left(J^1_2(z)\right)^q &\preceq \int\limits_{E^{11}_{R_1}}\frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}}\preceq\int\limits_{E^{11}_{R_1}}\left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}\frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)}{\alpha}}}\\ &\preceq n^{2-q}\int\limits_{E^{11}_{R_1}}\frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)-(2-q)}{\alpha}}}\preceq n^{2-q+\frac{\gamma(q-1)-(2-q}{\alpha}}mesE^{11}_{R_1}\\ &\preceq n^{1-q+\frac{\gamma(q-1)-(2-q)}{\alpha}};\\ J^1_2(z) &\preceq n^{\frac{\gamma}{p\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}.\end{aligned} \tag{4.13}$$

$$\left(J_2^2(z)\right)^q \preceq \int_{E_{R_1}^{12}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preceq \int_{E_{R_1}^{12}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)}{\alpha}}}$$

$$\preceq n^{2-q} \int_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)-(2-q)}{\alpha}}}$$

$$\preceq n^{2-q} \begin{cases} n^{\frac{\gamma(q-1)-(2-q)}{\alpha}-1}, & \gamma(q-1)-(2-q)>\alpha, \\ \ln n, & \gamma(q-1)-(2-q)=\alpha, \\ 1, & \gamma(q-1)-(2-q)<\alpha; \end{cases}$$

$$J_2^2(z) \preceq \begin{cases} n^{\frac{\gamma}{p\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & 2\le p<\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, & \gamma>\alpha, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{1-\frac{1}{p}}, & p=\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, & \gamma>\alpha, \\ n^{1-\frac{2}{p}}, & p>\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, & \gamma>\alpha, \\ n^{1-\frac{2}{p}}, & p\ge 2, & -2<\gamma\le\alpha. \end{cases}$$

$$\left(J_2^3(z)\right)^q \preceq \int_{E_{R_1}^{13}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preceq \int_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}|d\tau| \preceq n^{\left(\frac{1}{\alpha}-1\right)(2-q)};$$

$$J_2^3(z) \preceq n^{\left(\frac{1}{\alpha}-1\right)\left(1-\frac{2}{p}\right)},$$

and, from (4.6)–(4.13), we obtain:

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & \gamma>-2, p\ge 2 & m\ge 2, \\ n^{\frac{\gamma+2}{p\alpha}}, & 2\le p<1+\frac{\gamma+2}{\alpha} & m=1, \\ n^{1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha} & m=1, \\ n^{1-\frac{1}{p}}, & p>1+\frac{\gamma+2}{\alpha}, & m=1, \end{cases} \tag{4.14}$$

if $z\in\Omega(\delta)$ and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & 2\le p<\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, & \gamma>\alpha, \\ n^{1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, & \gamma>\alpha, \\ n^{1-\frac{1}{p}}, & p>\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, & \gamma>\alpha, \\ n^{1-\frac{1}{p}}, & p\ge 2, & -2<\gamma\le\alpha, \end{cases} \tag{4.15}$$

if $z\in\widehat{\Omega}(\delta)$.

1.2. If $\gamma<0$, for $w\in\Delta(w_1,\eta)\cap\Omega_R(\delta)$, such that $|\Psi(\tau)-\Psi(w_1)|\preceq|\Psi(\tau)-\Psi(w)|$, according to Lemma 3.1, analogously we have:

$$\begin{aligned}\left(I(E^{11}_{R_1,1})\right)^q &\preceq \int\limits_{E^{11}_{R_1,1}} \left(\frac{d(\Psi(\tau), L)}{|\tau|-1}\right)^{2-q} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \\ &\preceq n^{2-q}\int\limits_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\frac{qm-(2-q)}{\alpha}}} \preceq n^{2-q+\frac{qm-(2-q)}{\alpha}} mesE^{11}_{R_1} \\ &\preceq n^{2-q+\frac{qm-(2-q)}{\alpha}-1}; \\ I(E^{11}_{R_1,1}) &\preceq n^{\frac{m}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}.\end{aligned} \tag{4.16}$$

$$\begin{aligned}\left(I(E^{11}_{R_1,2})\right)^q &\asymp \int\limits_{E^{11}_{R_1,2}} \left(\frac{d(\Psi(\tau), L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \\ &\preceq n^{2-q}\int\limits_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha}}} \\ &\preceq n^{2-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}} mesE^{11}_{R_1,2} \preceq n^{2-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}-1}; \\ I(E^{11}_{R_1,2}) &\preceq n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}.\end{aligned} \tag{4.17}$$

For $\tau \in E^{12}_{R_1}$ we see that $|\tau - w_1| < \eta$ and from Lemma 3.1, $|\Psi(\tau)-\Psi(w_1)| \preceq 1$.
Then, for $w \in \Delta(w_1,\eta)\cap\Omega_R(\delta)$, such that $|\Psi(\tau)-\Psi(w_1)| \preceq |\Psi(\tau)-\Psi(w)|$, applying Lemma 3.3, we get:

$$\begin{aligned}\left(I(E^{12}_{R_1,1})\right)^q &= \int\limits_{E^{12}_{R_1,1}} \left(\frac{d(\Psi(\tau), L)}{|\tau|-1}\right)^{2-q} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \\ &\preceq n^{2-q}\int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\frac{qm-(2-q)}{\alpha}}} \preceq n^{2-q}\int\limits_{E^{12}_{R_1,1}} \frac{1}{|\tau-w|^{\frac{qm-(2-q)}{\alpha}}}|d\tau| \preceq n^{2-q+\frac{qm-(2-q)}{\alpha}-1}; \\ I(E^{12}_{R_1,1}) &\preceq n^{\frac{m}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}.\end{aligned} \tag{4.18}$$

$$\begin{aligned}\left(I(E^{12}_{R_1,2})\right)^q &\asymp \int\limits_{E^{12}_{R_1,2}} \left(\frac{d(\Psi(\tau), L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm+\gamma(q-1)}} \\ &\preceq n^{2-q}\int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^{\frac{qm+\gamma(q-1)-(2-q)}{\alpha}}} \\ &\preceq n^{2-q}\begin{cases} n^{\frac{qm+\gamma(q-1)-(2-q)}{\alpha}-1}, & qm+\gamma(q-1)-(2-q)>\alpha, \\ \ln n, & qm+\gamma(q-1)-(2-q)=\alpha, \\ 1, & qm+\gamma(q-1)-(2-q)<\alpha; \end{cases}\end{aligned} \tag{4.19}$$

$$I(E^{12}_{R_1,2}) \preceq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p\geq 2, \quad m\geq 2, \quad \gamma>-2, \\ n^{\left(\frac{\gamma}{p}+1\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p<1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad \gamma>-2+\alpha, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad \gamma>-2+\alpha, \\ n^{1-\frac{2}{p}}, & p>1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad \gamma>-2+\alpha, \\ \mathrm{n}^{1-\frac{2}{p}}, & \mathrm{p}\geq 2, \quad \mathrm{m}=1, \quad -2<\gamma\leq -2+\alpha. \end{cases}$$

For $\tau \in E^{13}_{R_1}$ and each $w \in \Delta(w_1,\eta)\cap\Omega_R(\delta)$ we see that $\eta < |\tau - w_1| < 2\pi \dot{R}_1$. Therefore, from Lemma 3.1 and applying (3.1), we get:

$$\begin{aligned} \left(I(E^{13}_{R_1})\right)^q \quad : &= \int\limits_{E^{13}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|\Psi'(\tau)|^{2-q}\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \qquad (4.20) \\ &\asymp \int\limits_{E^{13}_{R_1}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \\ &\preceq n^{2-q}\int\limits_{E^{13}_{R_1}} \frac{|d\tau|}{|\tau-w|^{\frac{qm-(2-q)}{\alpha}}} \preceq n^{1-q+\frac{qm-(2-q)}{\alpha}}; \\ I(E^{13}_{R_1}) &\preceq n^{\frac{m}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}. \end{aligned}$$

Therefore, combining (4.16)–(4.20) in case of $\gamma < 0$ for $z \in \Omega(\delta)$, we have:

$$\sum_{k=1}^{3} J_2^k(z) \preceq n^{\frac{m}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}} \qquad (4.21)$$

$$+\begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p\geq 2, \quad m\geq 2, \quad \gamma>-2, \\ n^{\left(\frac{\gamma}{p}+1\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p<1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad 0>\gamma>-2+\alpha, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad 0>\gamma>-2+\alpha, \\ n^{1-\frac{2}{p}}, & p>1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad 0>\gamma>-2+\alpha, \\ n^{1-\frac{2}{p}}, & p\geq 2, \quad m=1, \quad -2<\gamma\leq -2+\alpha, \end{cases}$$

$$\preceq \begin{cases} n^{\frac{m}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p\geq 2, \quad m\geq 2, \quad \gamma>-2, \\ n^{\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p<1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad 0>\gamma>-2+\alpha, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad 0>\gamma>-2+\alpha, \\ n^{1-\frac{2}{p}}, & p>1+\frac{\gamma+2}{\alpha}, \quad m=1, \quad 0>\gamma>-2+\alpha, \\ n^{1-\frac{2}{p}}, & p\geq 2, \quad m=1, \quad -2<\gamma\leq -2+\alpha. \end{cases}$$

If $z\in\widehat{\Omega}(\delta)$, then $|w-w_1|\geq\eta$, from Lemma 3.3 and from (3.1), we get:

$$\begin{aligned}\left(J_2^1(z)\right)^q &= \int\limits_{E_{R_1}^{11}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}|\Psi(\tau)-\Psi(w)|^{qm}} \\ &\preceq \int\limits_{E_{R_1}^{11}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} |\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}|d\tau| \\ &\preceq n^{2-q}\int\limits_{E_{R_1}^{11}} |\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)+2-q}|d\tau| \preceq n^{2-q} mes E_{R_1}^{11} \preceq n^{1-q}\preceq 1; \\ J_2^1(z) &\preceq 1.\end{aligned} \tag{4.22}$$

$$\begin{aligned}\left(J_2^2(z)\right)^q &\preceq \int\limits_{E_{R_1}^{12}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} |\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}|d\tau| \\ &\preceq \int\limits_{E_{R_1}^{12}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}|d\tau| \preceq n^{2-q}; \\ J_2^2(z) &\preceq n^{1-\frac{2}{p}}.\end{aligned}$$

$$\begin{aligned}\left(J_2^3(z)\right)^q &\preceq \int\limits_{E_{R_1}^{13}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preceq \int\limits_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}|d\tau| \preceq n^{2-q}; \\ J_2^3(z) &\preceq n^{1-\frac{2}{p}}.\end{aligned}$$

Combining the last three estimates, in case of $\gamma<0$ for $z\in\widehat{\Omega}(\delta)$, we have:

$$\sum_{k=1}^{3} J_2^k(z)\preceq n^{1-\frac{2}{p}}. \tag{4.23}$$

Then, for the $\gamma<0$, from (4.21–4.23), we obtain:

$$\sum_{k=1}^{3} J_2^k(z)\preceq \begin{cases} n^{\frac{m}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p\geq 2, & m\geq 2, & \gamma>-2, & z\in\Omega(\delta), \\ n^{\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p<1+\frac{\gamma+2}{\alpha}, & m=1, & \gamma>-2+\alpha, & z\in\Omega(\delta), \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, & m=1, & \gamma>-2+\alpha, & z\in\Omega(\delta), \\ n^{1-\frac{2}{p}}, & p>1+\frac{\gamma+2}{\alpha}, & m=1, & \gamma>-2+\alpha, & z\in\Omega(\delta), \\ n^{1-\frac{2}{p}}, & p\geq 2, & m=1, & -2<\gamma\leq -2+\alpha, & z\in\Omega(\delta), \\ n^{1-\frac{2}{p}}, & p\geq 2, & & & z\in\widehat{\Omega}(\delta), \end{cases}$$

and, consequently, in this case from (4.6), we have:

$$A_n(z) \quad \preceq \quad n^{\frac{1}{p}}\left\|P_n\right\|_p \cdot \begin{cases} n^{\frac{m}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p\geq 2, & m\geq 2, & \gamma>-2, & z\in\Omega(\delta), \\ n^{\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p<1+\frac{\gamma+2}{\alpha}, & m=1, & 0>\gamma>-2+\alpha, & z\in\Omega(\delta), \\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, & m=1, & 0>\gamma>-2+\alpha, & z\in\Omega(\delta), \\ n^{1-\frac{2}{p}}, & p>1+\frac{\gamma+2}{\alpha}, & m=1, & 0>\gamma>-2+\alpha, & z\in\Omega(\delta), \\ n^{1-\frac{2}{p}}, & p\geq 2, & m=1, & -2<\gamma\leq -2+\alpha, & z\in\Omega(\delta), \\ n^{1-\frac{2}{p}}, & p\geq 2, & & & z\in\widehat{\Omega}(\delta). \end{cases} \tag{4.24}$$

Therefore, combining (4.12) and (4.24), for any $\gamma>-2,\ p\geq 2,$ we obtain:

$$A_n(z)\preceq\left\|P_n\right\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+m-1\right)\frac{1}{\alpha}}, & p\geq 2, & m\geq 2, & \gamma>-2, \\ n^{\left(\frac{\gamma^*+2}{p}\right)\frac{1}{\alpha}}, & 2\leq p<1+\frac{\gamma+2}{\alpha}, & m=1, & \gamma>-2+\alpha, \\ \left(n\ln n\right)^{1-\frac{1}{p}}, & p=1+\frac{\gamma+2}{\alpha}, & m=1, & \gamma>-2+\alpha, \\ n^{1-\frac{1}{p}}, & p>1+\frac{\gamma+2}{\alpha}, & m=1, & \gamma>-2+\alpha, \\ n^{1-\frac{1}{p}}, & p\geq 2, & m=1, & -2<\gamma\leq -2+\alpha, \end{cases} \tag{4.25}$$

if $z\in\Omega(\delta)$, and

$$A_n(z)\preceq\left\|P_n\right\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & 2\leq p<\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, & \gamma>\alpha, \\ \left(n\ln n\right)^{1-\frac{1}{p}}, & p=\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, & \gamma>\alpha, \\ n^{1-\frac{1}{p}}, & p>\frac{\gamma+2}{1+\alpha}+\frac{\alpha}{1+\alpha}, & \gamma>\alpha, \\ n^{1-\frac{1}{p}}, & p\geq 2, & 0\leq\gamma\leq\alpha, \\ n^{1-\frac{1}{p}}, & p\geq 2, & -2<\gamma<0, \end{cases} \tag{4.26}$$

if $z\in\widehat{\Omega}(\delta)$.

Case 2. Let $q>2$ $(p<2)$. Then, $2-q<0$, and so

$$\begin{aligned} \left(I(E_{R_1,1}^{1k})\right)^q \quad &:= \begin{cases} \int\limits_{E_{R_1,1}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}}, & if\ \gamma\geq 0, \\ \int\limits_{E_{R_1,1}^{1k}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{qm}}, & if\ \gamma<0, \end{cases} \\ \left(I(E_{R_1,2}^{1k})\right)^q \quad &:= \int\limits_{E_{R_1,2}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma(q-1)+q}},\ k=1,2, \\ \left(J_2^3(z)\right)^q \quad &:= \int\limits_{E_{R_1}^{13}} \frac{|\Psi'(\tau)|^{2-q}\,|d\tau|}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma(q-1)}\left|\Psi(\tau)-\Psi(w)\right|^{qm}}. \end{aligned} \tag{4.27}$$

2.1. If $\gamma \geq 0$ and $z \in \Omega(\delta)$, applying Lemmas 3.1 and 3.3 to (4.27), we obtain:

$$\begin{aligned}
\left(I(E^{11}_{R_1,1})\right)^q &\preceq \int\limits_{E^{11}_{R_1,1}} \left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \\
&\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\frac{\gamma(q-1)+qm}{\alpha}}} \preceq n^{\frac{\gamma(q-1)+qm}{\alpha}+(\frac{1}{\alpha}-1)(q-2)} mes E^{11}_{R_1,1} \\
&\preceq n^{\frac{\gamma(q-1)+qm}{\alpha}+(\frac{1}{\alpha}-1)(q-2)-1}; \\
I(E^{11}_{R_1,1}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
\left(I(E^{11}_{R_1,2})\right)^q &\preceq \int\limits_{E^{11}_{R_1,2}} \left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \\
&\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)+qm}{\alpha}}} \preceq n^{\frac{\gamma(q-1)+qm}{\alpha}+(\frac{1}{\alpha}-1)(q-2)} mes E^{11}_{R_1,2} \\
&\preceq n^{\frac{\gamma(q-1)+qm}{\alpha}+(\frac{1}{\alpha}-1)(q-2)-1}; \\
I(E^{11}_{R_1,2}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
\left(I(E^{12}_{R_1,1})\right)^q &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\frac{\gamma(q-1)+qm}{\alpha}}} \preceq n^{\frac{\gamma(q-1)+qm}{\alpha}+(\frac{1}{\alpha}-1)(q-2)-1}; \\
I(E^{12}_{R_1,1}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned}$$

$$\left(I(E^{12}_{R_1,2})\right)^q \preceq \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \tag{4.30}$$

$$\begin{aligned}
&\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)+qm}{\alpha}}} \preceq n^{\frac{\gamma(q-1)+qm}{\alpha}+(\frac{1}{\alpha}-1)(q-2)-1}; \\
I(E^{12}_{R_1,2}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned}$$

For $\tau \in E^{13}_{R_1}$ we see that $\eta < |\tau - w_1| < 2\pi \dot{R}_1$. Therefore, from Lemma 3.1, we have $|\Psi(\tau)-\Psi(w_1)| \succeq 1$.For $w \in \Delta(w_1,\eta)$ $|\Psi(\tau)-\Psi(w)| \succeq |\tau-w|^{\frac{1}{\alpha}}$. Then, applying Lemma 3.3, we get:

$$\begin{aligned}
\left(J^3_2(z)\right)^q &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{13}_{R_1}} \frac{|d\tau|}{|\tau-w|^{\frac{qm}{\alpha}}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{qm}{\alpha}-1}; \\
J^3_2(z) &\preceq n^{\frac{m}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned} \tag{4.31}$$

If $z\in\widehat{\Omega}(\delta)$, then $|w-w_1|\geq\eta$, from (3.1) we have:

$$\begin{aligned}\left(J_2^1(z)\right)^q &\preceq \int\limits_{E_{R_1}^{11}}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}}\preceq n^{(\frac{1}{\alpha}-1)(q-2)}\int\limits_{E_{R_1}^{11}}\frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)}{\alpha}}}\\ &\preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{\gamma(q-1)}{\alpha}}mesE_{R_1}^{11}\preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{\gamma(q-1)}{\alpha}-1};\\ J_2^1(z) &\preceq n^{\frac{\gamma}{p\alpha}+\left(\frac{2}{p}-1\right)(\frac{1}{\alpha}-1)-\frac{1}{p}}.\end{aligned}$$

$$\begin{aligned}\left(J_2^2(z)\right)^q &\preceq \int\limits_{E_{R_1}^{12}}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}}\preceq n^{(\frac{1}{\alpha}-1)(q-2)}\int\limits_{E_{R_1}^{12}}\frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)}{\alpha}}}\\ &\preceq n^{(\frac{1}{\alpha}-1)(q-2)}\begin{cases} n^{\frac{\gamma(q-1)}{\alpha}-1}, & \gamma(q-1)>\alpha,\\ \ln n, & \gamma(q-1)=\alpha,\\ 1, & \gamma(q-1)<\alpha;\end{cases}\\ J_2^2(z) &\preceq \begin{cases} n^{\frac{\gamma}{p\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma>\alpha(p-1),\\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)}\left(\ln n\right)^{1-\frac{1}{p}}, & \gamma=\alpha(p-1),\\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)}, & \gamma<\alpha(p-1).\end{cases}\end{aligned}$$

$$\begin{aligned}\left(J_2^3(z)\right)^q &= \int\limits_{E_{R_1}^{13}}\frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}|\Psi(\tau)-\Psi(w)|^{qm}}\\ &\preceq \int\limits_{E_{R_1}^{13}}\frac{|d\tau|}{|\Psi'(\tau)|^{q-2}}\preceq\int\limits_{E_{R_1}^{13}}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}|d\tau|\preceq n^{(\frac{1}{\alpha}-1)(q-2)};\\ J_2^3(z) &\preceq n^{(\frac{1}{\alpha}-1)\left(\frac{2}{p}-1\right)}.\end{aligned}$$

From (4.28–4.31) and (4.6), for $\gamma\geq 0, 1<p<2, m\geq 1,$ we have:

$$A_n(z)\preceq\|P_n\|_p\, n^{(\frac{\gamma+2}{p}+m-1)\frac{1}{\alpha}}, \tag{4.32}$$

if $z\in\Omega(\delta)$ and

$$A_n(z)\preceq\|P_n\|_p\begin{cases} n^{(\frac{\gamma+2}{p}+m-1)\frac{1}{\alpha}}, & \gamma>(p-1)\,\alpha,\\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & \gamma=(p-1)\,\alpha,\\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}, & \gamma<(p-1)\,\alpha,\end{cases} \tag{4.33}$$

if $z\in\widehat{\Omega}(\delta)$.

2.2. Let $\gamma < 0$. For $z \in \Omega(\delta)$, according to Lemma 3.1, we have:

$$\begin{aligned}
\left(I(E^{11}_{R_1,1})\right)^q &= \int\limits_{E^{11}_{R_1,1}} \left|\frac{\Psi(\tau)-\Psi(w_1)}{\Psi(\tau)-\Psi(w)}\right|^{-\gamma(q-1)} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^{qm+\gamma(q-1)}} \\
&\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{11}_{R_1,1}} \left(\frac{|\tau-w_1|}{|\tau-w|}\right)^{-\gamma(q-1)\frac{1}{c_1}} \frac{|d\tau|}{|\tau-w|^{\frac{qm+\gamma(q-1)}{\alpha}}} \\
&\preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{\gamma(q-1)+qm}{\alpha}} mes E^{11}_{R_1,1} \preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{\gamma(q-1)+qm}{\alpha}-1}; \\
I(E^{11}_{R_1,1}) &\preceq n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
\left(I(E^{11}_{R_1,2})\right)^q &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{11}_{R_1}} \frac{|d\tau|}{|\tau-w_1|^{\frac{qm+\gamma(q-1)}{\alpha}}} \\
&\preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{qm+\gamma(q-1)}{\alpha}} mes E^{11}_{R_1,2} \preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{qm+\gamma(q-1)}{\alpha}-1}; \\
I(E^{11}_{R_1,2}) &\preceq n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned}$$

$$\begin{aligned}
\left(I(E^{12}_{R_1,1})\right)^q &\preceq \int\limits_{E^{12}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^{qm}} \preceq \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^{qm}} \\
&\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\frac{qm}{\alpha}}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{qm}{\alpha}-1}; \\
I(E^{12}_{R_1,1}) &\preceq n^{\frac{m}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
\left(I(E^{12}_{R_1,2})\right)^q &\preceq \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^{qm}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^{\frac{qm}{\alpha}}} \\
&\preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{qm}{\alpha}-1}; \\
I(E^{12}_{R_1,2}) &\preceq n^{\frac{m}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned}$$

$$\begin{aligned}
\left(J^3_2(z)\right)^q &\preceq \int\limits_{E^{13}_{R_1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^{qm}} \\
&\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E^{13}_{R_1}} \frac{|d\tau|}{|\tau-w|^{\frac{qm}{\alpha}}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)+\frac{qm}{\alpha}-1}; \\
J^3_2(z) &\preceq n^{\frac{m}{\alpha}+\left(\frac{2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.
\end{aligned} \tag{4.36}$$

For the $z \in \widehat{\Omega}(\delta)$, analogously we obtain:

$$J_2^i(z) \preceq n^{\left(\frac{2}{p}-1\right)\left(\frac{1}{\alpha}-1\right)}, i = 1, 2, 3. \tag{4.37}$$

So, for $\gamma < 0$, from (4.6), we have:

$$A_n(z) \quad \preceq \quad \|P_n\|_p \begin{cases} n^{\left(\frac{2}{p}+m\right)\frac{1}{\alpha}}, & \text{if } z \in \Omega(\delta), \\ n^{\left(\frac{2}{p}-1\right)\left(\frac{1}{\alpha}-1\right)+\frac{1}{p}}, & \text{if } z \in \widehat{\Omega}(\delta). \end{cases} \tag{4.38}$$

Therefore, for any $\gamma \geq -2, 1 < p < 2$, $m \geq 1$, from (4.32), (4.33) and (4.38), we get:

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & \gamma \geq 0, \\ n^{\left(\frac{2}{p}+m-1\right)\frac{1}{\alpha}}, & \gamma < 0, \end{cases} \tag{4.39}$$

if $z \in \Omega(\delta)$, and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & \gamma > (p-1)\,\alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+1-\frac{1}{p}} \left(\ln n\right)^{1-\frac{1}{p}}, & \gamma = (p-1)\,\alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+1-\frac{1}{p}}, & -2 < \gamma < (p-1)\,\alpha, \end{cases} \tag{4.40}$$

if $z \in \widehat{\Omega}(\delta)$.

B) Now, we begin to estimate the $\left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right|$. Since $\Phi(\infty) = \infty$, then Cauchy integral representation for the region Ω_R gives:

$$\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} \frac{1}{\Phi^{n+1}(\zeta)} \frac{d\zeta}{(\zeta - z)^{j+1}}\ ,\ z \in \Omega_R.$$

If we take the modulus of both sides, we get:

$$\left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right| \leq \frac{1}{2\pi} \int\limits_{L_{R_1}} \frac{1}{|\Phi^{n+1}(\zeta)|} \frac{|d\zeta|}{|\zeta - z|^{j+1}} \leq \frac{1}{2\pi} \int\limits_{L_{R_1}} \frac{|d\zeta|}{|\zeta - z|^{j+1}}.$$

Replacing the variable $\tau = \Phi(\zeta)$ and according to (3.1), we obtain:

$$\left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right| \quad \preceq \quad \int\limits_{|\tau|=R_1} \frac{d(\Psi(\tau), L)}{|\tau| - 1} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{j+1}} \tag{4.41}$$

$$\preceq n \int\limits_{|\tau|=R_1} \frac{|d\tau|}{|\tau - w|^{\frac{j}{\alpha}}} \preceq \begin{cases} n^{\frac{j}{\alpha}}, & \text{if } z \in \Omega(\delta), \\ n, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases} \quad j = \overline{1, m}.$$

Combining estimates (4.1)–(4.6), (4.25), (4.26), (4.39), (4.40) and (4.41), we get:

$$\left|P_n^{(m)}(z)\right| \leq \left|\Phi^{n+1}(z)\right| \cdot \left[\frac{A_n(z)}{d(z, L)} + \sum_{j=1}^{m} C_m^j \left|P_n^{(m-j)}(z)\right| \right\} \begin{matrix} n^{\frac{j}{\alpha}}, & \text{if } z \in \Omega(\delta), \\ n, & \text{if } z \in \widehat{\Omega}(\delta), \end{matrix} \Bigg] \tag{4.42}$$

where for any $\gamma > -2, p \geq 2, m \geq 1$

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+m-1\right)\frac{1}{\alpha}}, & p \geq 2, & m \geq 2, & \gamma > -2, \\ n^{\left(\frac{\gamma^*+2}{p}\right)\frac{1}{\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2+\alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2+\alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & m = 1, & -2 < \gamma \leq -2+\alpha, \end{cases}$$

if $z \in \Omega(\delta)$ and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & 2 \leq p < \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma \leq \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$; for any $\gamma \geq -2, 1 < p < 2, m \geq 1$,

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+m-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma > -2, & \text{if } z \in \Omega(\delta), \\ n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}+\frac{1}{p}}, & \gamma > (p-1)\alpha, & \text{if } z \in \widehat{\Omega}(\delta), \\ n^{\left(\frac{1}{\alpha}-1\right)\left(\frac{2}{p}-1\right)}(\ln n)^{1-\frac{1}{p}}, & \gamma = (p-1)\alpha, & \text{if } z \in \widehat{\Omega}(\delta), \\ n^{\left(\frac{1}{\alpha}-1\right)\left(\frac{2}{p}-1\right)}, & -2 < \gamma < (p-1)\alpha, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases}$$

and $\gamma^* := \max\{0; \gamma\}$. Therefore, the proofs of Theorems 2.2 and 2.3 are completed.

Now, we start to evaluate $|P_n(z)|$. For this, we will make the necessary evaluations by writing $m = 0$ in the above proof. Since the function $H_n(z) = \frac{P_n(z)}{\Phi^{n+1}(z)}$, $H_n(\infty) = 0$, is analytic in Ω, continuous on $\overline{\Omega}$, then Cauchy integral representation for the region Ω_{R_1} gives

$$H_n(z) = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} H_n(\zeta) \frac{d\zeta}{\zeta - z}, z \in \Omega_R.$$

Then,

$$\left| \frac{P_n(z)}{\Phi^{n+1}(z)} \right| \leq \frac{1}{2\pi} \int\limits_{L_{R_1}} \left| \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta - z|} \leq \frac{1}{2\pi d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)| |d\zeta| \tag{4.43}$$

and so,

$$|P_n(z)| \preceq \frac{\left|\Phi^{n+1}(z)\right|}{d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)| |d\zeta|.$$

Denote by

$$A_n := \int\limits_{L_{R_1}} |P_n(\zeta)| |d\zeta|, \tag{4.44}$$

and estimate this integral. To estimate A_n, first of all replacing the variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the integrand by $|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}$ $|\Psi'(\tau)|^{\frac{2}{p}}$ and applying the Hölder inequality, we obtain:

$$\begin{aligned} A_n &= \int\limits_{L_{R_1}} |P_n(\zeta)|\,|d\zeta| = \sum_{i=1}^{2} \int\limits_{F^i_{R_1}} \frac{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma_j}{p}} \left| P_n(\Psi(\tau))\,(\Psi'(\tau))^{\frac{2}{p}} \right| |\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}} |d\tau| \\ &\le \sum_{i=1}^{2} \left(\int\limits_{F^i_{R_1}} |\Psi(\tau) - \Psi(w_1)|^{\gamma} |P_n(\Psi(\tau))|^p |\Psi'(\tau)|^2 |d\tau| \right)^{\frac{1}{p}} \times \left(\int\limits_{F^i_{R_1}} \left(\frac{|\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}} \right)^q |d\tau| \right)^{\frac{1}{q}} \\ &=: \sum_{i=1}^{2} A^i_n, \end{aligned}$$

where $F^1_{R_1} := \Phi(L^1_{R_1}) = \Delta'_1 \cap \{\tau : |\tau| = R_1\}$, $F^2_{R_1} := \Phi(L_{R_1}) \backslash F^1_{R_1}$ and

$$\begin{aligned} A^i_n &:= \left(\int\limits_{F^i_{R_1}} |f_{n,p}(\tau)|^p |d\tau| \right)^{\frac{1}{p}} \left(\int\limits_{F^i_{R_1}} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} |d\tau| \right)^{\frac{1}{q}} \\ &=: J^i_{n,1} \cdot J^i_{n,2}, \\ & f_{n,p}(\tau) := (\Psi(\tau) - \Psi(w_1))^{\frac{\gamma}{p}} P_n(\Psi(\tau))\,(\Psi'(\tau))^{\frac{2}{p}}, \ |\tau| = R_1. \end{aligned}$$

Applying Lemma 3.4, we get:

$$J^i_{n,1} \preceq n^{\frac{1}{p}} \|P_n\|_p, \ i = 1, 2.$$

Therefore, we need to evaluate the following integrals:

$$\left(J^i_{n,2}\right)^q = \int\limits_{F^i_{R_1}} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} |d\tau|, i = 1, 2.$$

For the estimation of the integral $J^i_{n,2}$, for $i = 1, 2$, we use notations (4.5) and (4.6) and, consequently, we need to evaluate the following statement:

$$A_n = n^{\frac{1}{p}} \|P_n\|_p \cdot (J^1_2 + J^2_2 + J^3_2), \tag{4.45}$$

where

$$\left(J^k_2\right)^q := \int\limits_{E^{1k}_{R_1}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}}, \ k = 1, 2, 3.$$

So, for any $k = 1, 2, 3$ we will estimate the integrals J^k_2 . Given possible values of q ($q > 2$ and $q \le 2$) and γ ($-2 < \gamma < 0$ and $\gamma \ge 0$), we will consider the cases separately.

Case 1. Let $1 < q \leq 2$ $(p \geq 2)$.

1.1. Let $\gamma \geq 0$. Applying Lemma 3.3, we get:

$$\begin{aligned} \left(J_2^1\right)^q &= \int\limits_{E_{R_1}^{11}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preceq \int\limits_{E_{R_1}^{11}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \\ &\preceq n^{2-q} \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)-(2-q)}{\alpha}}} \preceq n^{2-q+\frac{\gamma(q-1)-(2-q)}{\alpha}} mes E_{R_1}^{11} \\ &\preceq n^{2-q+\frac{\gamma(q-1)-(2-q)}{\alpha}-1}; \\ J_2^1 &\preceq n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}. \end{aligned} \tag{4.46}$$

$$\begin{aligned} \left(J_2^2\right)^q &\leq \int\limits_{E_{R_1}^{12}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(2-q)} \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)-(2-q)}{\alpha}}} \\ &\preceq n^{2-q} \begin{cases} n^{\frac{\gamma(q-1)-(2-q)}{\alpha}-1}, & \gamma(q-1)-(2-q) > \alpha, \\ \ln n, & \gamma(q-1)-(2-q) = \alpha, \\ 1, & \gamma(q-1)-(2-q) < \alpha; \end{cases} \end{aligned} \tag{4.47}$$

$$J_2^2 \preceq \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & 2 \leq p < \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p > \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & -2 < \gamma \leq \alpha. \end{cases}$$

For $\tau \in E_{R_1}^{13}$ we see that $\eta < |\tau - w_1| < 2\pi\dot{R}_1$. Therefore, $|\Psi(\tau)-\Psi(w_1)| \succeq 1$, from Lemma 3.1 and applying (3.1), we get:

$$\begin{aligned} \left(J_2^3\right)^q &\preceq \int\limits_{E_{R_1}^{13}} |\Psi'(\tau)|^{2-q}|d\tau| \preceq \int\limits_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}|d\tau| \preceq n^{(\frac{1}{\alpha}-1)(2-q)}; \\ J_2^3 &\preceq n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})}. \end{aligned} \tag{4.48}$$

Combining (4.46–4.48), for $p \geq 2, \gamma \geq 0$ and $z \in \Omega_R$, we get:

$$\sum_{k=1}^{3} J_2^k \preceq \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & 2 \leq p < \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p > \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & 0 < \gamma \leq \alpha. \end{cases} \tag{4.49}$$

From (4.45)–(4.49), we obtain:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ (n\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2, 0 < \gamma \leq \alpha. \end{cases} \tag{4.50}$$

1.2. If $\gamma < 0$, analogously we have:

$$\left(J_2^1\right)^q \preceq \int\limits_{E_{R_1}^{11}} |\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} |d\tau| \tag{4.51}$$

$$\begin{aligned} &\preceq n^{2-q} \int\limits_{E_{R_1}^{11}} |\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)+(2-q)} |d\tau| \preceq n^{2-q} \int\limits_{E_{R_1}^{11}} |d\tau| \\ &\preceq n^{2-q} mes E_{R_1}^{11} \preceq n^{1-q}; \\ J_2^1 &\preceq 1. \end{aligned}$$

For $\tau \in E_{R_1}^{12}$, we get:

$$\left(J_2^2\right)^q \preceq \int\limits_{E_{R_1}^{12}} |\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(2-q)}; \tag{4.52}$$

$$J_2^2 \preceq n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})}.$$

$$\left(J_2^3\right)^q \preceq \int\limits_{E_{R_1}^{12}} |\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(2-q)}; \tag{4.53}$$

$$J_2^3 \preceq n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})}.$$

Therefore, combining (4.51)–(4.53) in case of $\gamma < 0$ for $z \in \Omega_R$, we have:

$$\sum_{k=1}^{3} J_2^k \preceq n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})},$$

and, consequently, in this case from (4.45), we have:

$$A_n \preceq \|P_n\|_p \cdot n^{(\frac{1}{\alpha}-1)\left(1-\frac{2}{p}\right)+\frac{1}{p}}, \; z \in \Omega_R. \tag{4.54}$$

Therefore, combining (4.49) and (4.54), for any $\gamma > -2, p \geq 2, z \in \Omega_R$, we obtain:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & 0 < \gamma \leq \alpha, \\ n^{(\frac{1}{\alpha}-1)\left(1-\frac{2}{p}\right)+\frac{1}{p}}, & p \geq 2, & -2 < \gamma < 0. \end{cases} \tag{4.55}$$

Case 2. Let $q>2$ $(p<2)$. Then, $2-q<0$, and so

$$\left(J_2^k(z)\right)^q := \int\limits_{E_{R_1}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}}, k=1,2,3. \tag{4.56}$$

2.1. If $\gamma\geq 0$, applying Lemmas 3.1, 3.3 and (3.1), we obtain:

$$\begin{aligned}\left(J_2^1\right)^q &\preceq \int\limits_{E_{R_1,2}^{11}} \left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E_{R_1,2}^{11}} \frac{|d\tau|}{|\tau-w_1|^{\gamma(q-1)}} \\ &\preceq n^{\gamma(q-1)\frac{1}{\alpha}+(\frac{1}{\alpha}-1)(q-2)} mesE_{R_1,2}^{11} \preceq n^{\gamma(q-1)\frac{1}{\alpha}+(\frac{1}{\alpha}-1)(q-2)-1}; \\ J_2^1 &\preceq n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}.\end{aligned} \tag{4.57}$$

$$\begin{aligned}\left(J_2^2\right)^q &\preceq \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)}{\alpha}}} \\ &\preceq \begin{cases} n^{\frac{\gamma(q-1)}{\alpha}+(\frac{1}{\alpha}-1)(q-2)-1}, & \gamma(q-1)>\alpha, \\ n^{(\frac{1}{\alpha}-1)(q-2)}\ln n, & \gamma(q-1)=\alpha, \\ n^{(\frac{1}{\alpha}-1)(q-2)}, & \gamma(q-1)<\alpha; \end{cases} \\ J_2^2 &\preceq \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & 1<p<1+\frac{\gamma}{\alpha}, \quad 0<\gamma<\alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)}\left(\ln n\right)^{1-\frac{1}{p}}, & p=1+\frac{\gamma}{\alpha}, \quad 0<\gamma<\alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)}, & 1+\frac{\gamma}{\alpha}<p<2, \quad 0\leq\gamma<\alpha, \\ n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & 1<p<2, \quad \gamma\geq\alpha. \end{cases}\end{aligned} \tag{4.58}$$

For $\tau\in E_{R_1}^{13}$ $\eta<|\tau-w_1|<2\pi\dot{R}_1$ and from Lemma 3.1 $|\Psi(\tau)-\Psi(w_1)|\asymp 1$. Then, we get:

$$\begin{aligned}\left(J_2^3\right)^q &= \int\limits_{E_{R_1}^{13}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int\limits_{E_{R_1}^{13}} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(q-2)}; \\ J_2^3 &\preceq n^{(\frac{1}{\alpha}-1)\left(\frac{2}{p}-1\right)}.\end{aligned} \tag{4.59}$$

From (4.56–4.59) and (4.45), for $\gamma\geq 0, 1<p<2, z\in\Omega_R$, we have:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}}, & 1<p<1+\frac{\gamma}{\alpha}, \quad 0<\gamma<\alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=1+\frac{\gamma}{\alpha}, \quad 0<\gamma<\alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}}, & 1+\frac{\gamma}{\alpha}<p<2, \quad 0\leq\gamma<\alpha, \\ n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}}, & 1<p<2, \quad \gamma\geq\alpha. \end{cases} \tag{4.60}$$

2.2. Let $\gamma<0$. For $z\in\Omega_R$, according to Lemma 3.1, we have:

$$\begin{aligned}\left(J_2^1\right)^q &\preceq \int\limits_{E_{R_1}^{11}} |\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}|d\tau| \\ &\preceq n^{(\frac{1}{\alpha}-1)(q-2)}\int\limits_{E_{R_1}^{11}}|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}|d\tau|\preceq n^{(\frac{1}{\alpha}-1)(q-2)}\int\limits_{E_{R_1}^{11}}|d\tau| \\ &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} mes E_{R_1}^{11}\preceq n^{(\frac{1}{\alpha}-1)(q-2)-1}; \\ J_2^1 &\preceq n^{\left(\frac{2}{p}-1\right)(\frac{1}{\alpha}-1)-1+\frac{1}{p}}.\end{aligned} \tag{4.61}$$

$$\begin{aligned}\left(J_2^2\right)^q &\preceq \int\limits_{E_{R_1}^{12}} |\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}|d\tau| \\ &\preceq n^{(\frac{1}{\alpha}-1)(q-2)}\int\limits_{E_{R_1}^{12}}|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}|d\tau|\preceq n^{(\frac{1}{\alpha}-1)(q-2)}; \\ J_2^2 &\preceq n^{\left(\frac{2}{p}-1\right)(\frac{1}{\alpha}-1)}.\end{aligned} \tag{4.62}$$

$$\begin{aligned}\left(J_2^3\right)^q &\preceq \int\limits_{E_{R_1}^{13}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}}\preceq n^{(\frac{1}{\alpha}-1)(q-2)}; \\ J_2^3 &\preceq n^{\left(\frac{2}{p}-1\right)(\frac{1}{\alpha}-1)}.\end{aligned} \tag{4.63}$$

So, for $\gamma<0$, from (4.45), we have:

$$A_n\preceq n^{\left(\frac{2}{p}-1\right)(\frac{1}{\alpha}-1)+\frac{1}{p}}\left\|P_n\right\|_p\ ,\ z\in\Omega_R. \tag{4.64}$$

Therefore, for any $\gamma\geq-2, 1<\ p<2$, from (4.60) and (4.64), we get:

$$A_n \preceq \left\|P_n\right\|_p\begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}}, & 1<p<1+\frac{\gamma}{\alpha},\quad 0<\gamma<\alpha,\\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma}{\alpha},\quad 0<\gamma<\alpha,\\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}}, & 1+\frac{\gamma}{\alpha}<p<2,\quad 0\leq\gamma<\alpha,\\ n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}}, & 1<p<2,\quad \gamma\geq\alpha,\\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}}, & 1<p<2,\quad -2<\gamma<0.\end{cases} \tag{4.65}$$

Combining estimates (4.1)–(4.45), (4.55), (4.55), (4.65), we get:

$$\left|P_n\left(z\right)\right|\preceq\frac{\left|\Phi^{n+1}(z)\right|}{d(z,L_{R_1})}A_n,$$

where for $p \geq 2$

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}}, & p > \max\left\{2; \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}\right\}, & \gamma \geq 0, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

and for $1 < p < 2$

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}}, & 1 < p < \min\left\{2; 1 + \frac{\gamma}{\alpha}\right\}, & \gamma > 0, \\ n^{\left(\frac{1}{\alpha}-1\right)\left(\frac{2}{p}-1\right)+\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{\left(\frac{1}{\alpha}-1\right)\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & \max\left\{1; 1 + \frac{\gamma}{\alpha}\right\} < p < 2, & -2 < \gamma < \alpha, \end{cases}$$

and therefore, the proof of Theorem 2.4 is completed. □

Proof [of Theorems 2.6 and 2.7.] From (4.25), (4.26) and (4.42) ,we get:

$$|P'_n(z)| \preceq \left|\Phi^{n+1}(z)\right| \cdot \left[\frac{A_n(z)}{d(z,L)} + |P_n(z)| \begin{cases} n^{\frac{1}{\alpha}} & \text{if } z \in \Omega(\delta), \\ n, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases}\right] \tag{4.66}$$

where for any $\gamma > -2, p \geq 2, m = 1$

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\frac{\gamma^*+2}{p\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & \gamma > -2+\alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & \gamma > -2+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & \gamma > -2+\alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma \leq -2+\alpha, \end{cases}$$

if $z \in \Omega(\delta)$, and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{\left(1-\frac{2}{p}\right)} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & 0 \leq \gamma \leq \alpha, \\ n^{\left(\frac{1}{\alpha}-1\right)\left(1-\frac{2}{p}\right)}, & p \geq 2, & \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$; for any $\gamma \geq -2, 1 < p < 2$, $m = 1$,

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\frac{\gamma^*+2}{p}\frac{1}{\alpha}-\frac{1}{p}}, & \gamma > -2, & \text{if } z \in \Omega(\delta), \\ n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma > \alpha(p-1), & \text{if } z \in \widehat{\Omega}(\delta), \\ n^{\left(\frac{1}{\alpha}-1\right)\left(\frac{2}{p}-1\right)} (\ln n)^{1-\frac{1}{p}}, & \gamma = \alpha(p-1), & \text{if } z \in \widehat{\Omega}(\delta), \\ n^{\left(\frac{1}{\alpha}-1\right)\left(\frac{2}{p}-1\right)}, & -2 < \gamma < \alpha(p-1), & \text{if } z \in \widehat{\Omega}(\delta), \end{cases}$$

and $\gamma^* := \max\{0; \gamma\}$. Taking into account the estimates for $|P_n(z)|$ from (4.39), and combining with (4.40) give a proof of the needed estimates. □

Proof [of Theorems 2.8 and 2.9.] According to the Theorems 2.2, 2.4, 2.6 and estimates (4.2), (4.3), (4.25), (4.26), (4.42), for $m=2$ and $p\geq 2$ from (4.1), we have:

$$\begin{aligned}\left|P_n^{''}(z)\right| &\leq \left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{''}\right| + \sum_{j=1}^{2} C_2^j \left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right| \left|P_n^{(2-j)}(z)\right| \\ &\leq \left|\Phi^{n+1}(z)\right| \left[\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{''}\right| + C_2^1 B_{n,1}^1 \left|P_n^{'}(z)\right| + C_2^2 B_{n,2}^1 \left|P_n(z)\right|\right] \\ &\preceq \left|\Phi^{n+1}(z)\right| \left[\frac{\|P_n\|_p}{d(z,L)} A_n^1(z,2) + C_2^1 B_{n,1}^1 \left|P_n^{'}(z)\right| + C_2^2 B_{n,2}^1 \left|P_n(z)\right|\right].\end{aligned}$$

Substituting estimates for the $B_{n,j}^1$, $j=1,2$, $|P_n(z)|$ and $\left|P_n^{'}(z)\right|$ from Theorems 2.2, 2.4 and 2.6 correspondingly, we get:

$$\left|P_n^{''}(z)\right| \preceq \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L_{R_1})} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 2\leq p<\frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma\geq\alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}}(\ln n)^{1-\frac{1}{p}}, & p=\frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma\geq\alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}}, & p>\frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma\geq\alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}}, & 2\leq p<1+\frac{\gamma+2}{\alpha}, & 0\leq\gamma<\alpha, \\ n^{\left(3-\frac{2}{p}\right)\frac{1}{\alpha}+\frac{3}{p}-1} & p\geq 2, & -2<\gamma<0, \end{cases}$$

for $z\in\Omega(\delta)$, and

$$\left|P_n^{''}(z)\right| \preceq \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L_{R_1})} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 2\leq p<\frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma\geq\alpha, \\ n^{3-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=\frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma\geq\alpha, \\ n^{3-\frac{1}{p}}, & p>\frac{\alpha}{1+\alpha}+\frac{\gamma+2}{1+\alpha}, & \gamma\geq\alpha, \\ n^{3-\frac{1}{p}}, & p\geq 2, & 0\leq\gamma<\alpha \\ n^{(1-\frac{2}{p})\frac{1}{\alpha}+\frac{3}{p}+1}, & p\geq 2, & -2<\gamma<0, \end{cases}$$

for $z\in\widehat{\Omega}(\delta)$.
Analogously, from Theorems 2.3, 2.4 and 2.7 , for $1<p<2$, we have:

$$\left|P_n^{''}(z)\right| \preceq \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L_{R_1})} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1<p<2, & \gamma\geq\alpha, \\ n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1<p<1+\frac{\gamma}{\alpha}, & 0<\gamma<\alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}+1)+\frac{1}{p}+2}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma}{\alpha}, & 0<\gamma<\alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}+1)+\frac{1}{p}+2}, & 1+\frac{\gamma}{\alpha}<p<2, & 0\leq\gamma<\alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}+1)+\frac{1}{p}+2}, & 1<p<2, & -2<\gamma<0, \end{cases}$$

for $z\in\Omega(\delta)$, and

$$\left|P_n^{''}(z)\right| \preceq \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L_{R_1})} \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 1<p<2, & \gamma\geq\alpha, \\ n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 1<p<1+\frac{\gamma}{\alpha}, & 0<\gamma<\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+3-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma}{\alpha}, & 0<\gamma<\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+3-\frac{1}{p}}, & 1+\frac{\gamma}{\alpha}<p<2, & 0\leq\gamma<\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+3-\frac{1}{p}}, & 1<p<2, & -2<\gamma<0, \end{cases}$$

for $z \in \widehat{\Omega}(\delta)$. Therefore, the proofs of Theorems 2.8 and 2.9 are completed.

In conclusion, note that, there is a quantity $d(z, L_{R_1})$ everywhere in proofs. We show that $d(z, L_{R_1}) \succeq d(z, L)$ holds for all $z \in \Omega_R$. For the points $z \notin \Omega(L_{R_1}, d(L_{R_1}, L_R))$, we have: $d(z, L_{R_1}) \succeq \delta \succeq d(z, L)$. Now, let $z \in \Omega(L_{R_1}, d(L_{R_1}, L_R))$. Denote by $\xi_1 \in L_{R_1}$ the point for which $d(z, L_{R_1}) = |z - \xi_1|$, and point $\xi_2 \in L$, such that $d(z, L) = |z - \xi_2|$, and for $w = \Phi(z)$, $t_1 = \Phi(\xi_1), t_2 = \Phi(\xi_2)$, we have: $|w - w_1| \geq ||w - w_2| - |w_2 - w_1|| \geq \left||w - w_2| - \frac{1}{2}|w - w_2|\right| \geq \frac{1}{2}|w - w_2|$. Then, according to Lemma 3.1, we obtain: $d(z, L_{R_1}) \succeq d(z, L)$. □

4.0.1. Proof of Remark 2.5

Proof Let $G \in Q_\alpha$, $\frac{1}{2} \leq \alpha \leq 1$, $h(z) \equiv 1$ and $\{Q_n(z)\}$, $\deg Q_n = n$, $n = 0, 1, 2, ...$, be a system of Bergman polynomials for region G, i.e. system of polynomials, $Q_n(z) := a_n z^n + a_{n-1} z^{n-1} + ... + a_0$, $a_n > 0$ satisfying the conditions

$$\iint\limits_G Q_n(z)\overline{Q_m(z)} d\sigma_z = \begin{cases} 1, & n = m, \\ 0 & n \neq m. \end{cases}$$

According to [2], for arbitrary quasidiscs, we have

$$Q_n(z) = a_n \rho^{n+1} \Phi^n(z) \Phi'(z) A_n(z), \ z \in F \Subset \Omega,$$

where $\rho = \rho(G) > 1$ and

$$\sqrt{\frac{n+1}{\pi}} \leq a_n \rho^{n+1} \leq c_1 \sqrt{\frac{n+1}{\pi}},$$

for some $c_1 = c_1(G) > 1$ and

$$c_2 \leq |A_n(z)| \leq 1 + \frac{c_3}{\sqrt{|\Phi(z)| - 1}},$$

for some $c_i = c_i(G) > 0$, $i = 2, 3$. Therefore, since $\|Q_n\|_{A_2(G)} = 1$, then:

$$\begin{aligned} |Q_n(z)| &\geq c_2 \sqrt{\frac{n+1}{\pi}} |\Phi(z)|^n \frac{|\Phi(z)| - 1}{d(z, L)} \\ &\geq c_3 \frac{\sqrt{n}}{d(z, L)} |\Phi(z)|^{n+1} \left(1 - \frac{1}{|\Phi(z)|}\right) \\ &\geq c_4 \frac{\sqrt{n}}{d(z, L)} |\Phi(z)|^{n+1} \|Q_n\|_{A_2(G)}, \end{aligned}$$

and the proof is completed. □

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments and corrections.

References

[1] Abdullayev FG, Andrievskii VV. On the orthogonal polynomials in the domains with K-quasiconformal boundary. Izvestiya Akademii Nauk Azerbaidzhanskoi SSR Seriya Fiziko-Tekhnicheskikh I Matematicheskikh Nauk 1983; 4 (1): 7-11 (in Russian).

[2] Abdullayev FG. On the orthogonal polynomials in domains with quasiconformal boundary. PhD, Institute of Applied Mathematics and Mechanics National Academy of Science of Ukraine, Donetsk, Ukraine, 1986 (in Russian).

[3] Abdullayev FG. On some properties of the orthogonal polynomials over the region of the complex plane (Part I). Ukrainian Mathematical Journal 2000; 52 (12): 1807-1821.

[4] Abdullayev FG. On some properties of the orthogonal polynomials over the region of the complex plane (Part II). Ukrainian Mathematical Journal 2001; 53 (1): 1-14.

[5] Abdullayev FG. On some properties of the orthogonal polynomials over the region of the complex plane (Part III). Ukrainian Mathematical Journal 2001; 53 (12): 1934-1948.

[6] Abdullayev FG. On the interference of the weight and boundary contour for orthogonal polynomials over the region. Journal of Computational Analysis and Applications 2004; 6 (1): 31-42.

[7] Abdullayev FG. The properties of the orthogonal polynomials with weight having singularity on the boundary contour. Journal of Computational Analysis and Applications 2004; 6 (1): 43-59.

[8] Abdullayev FG, Aral D. On the Bernstein-Walsh type lemmas in regions of the complex plane. Ukrainian Mathematical Journal 2011; 63 (3): 337-350.

[9] Abdullayev FG, Özkartepe P. An analogue of the Bernstein-Walsh lemma in Jordan regions of the complex plane. Journal of Inequalities and Applications 2013; 2013 (570): 1-7. doi: 10.1186/1029-242X-2013-570

[10] Abdullayev FG, Gün CD. On the behavior of the algebraic polynomials in regions with piecewise smooth boundary without cusps. Annales Polonici Mathematici 2014; 111: 39-58.

[11] Abdullayev FG, Özkartepe P. On the behavior of the algebraic polynomial in unbounded regions with piecewise Dini-smooth boundary. Ukrainian Mathematical Journal 2014; 66 (5): 645-665.

[12] Abdullayev FG, Özkartepe P. On the growth of algebraic polynomials in the whole complex plane. Journal of the Korean Mathematical Society 2015; 52 (4): 699-725.

[13] Abdullayev FG, Tunc T, Abdullayev GA. Polynomial inequalities in quasidisks on weighted Bergman space. Ukrainian Mathematical Journal 2017; 69 (5): 675-695.

[14] Abdullayev FG, Gün CD. Bernstein-Nikolskii type inequalities for algebraic polynomials in Bergman space in regions of complex plane. Ukrainian Mathematical Journal 2021; 73 (4): 439-454.

[15] Ahlfors L. Lectures on Quasiconformal Mappings. Princeton, NJ: Van Nostrand, 1966.

[16] Anderson JM, Gehring FW, Hinkkanen A. Polynomial approximation in quasidisks. Differential Geometry and Complex Analysis 1985; A Volume Dedicated to the Memory of Harry Ernest Rauch: 75-86.

[17] Andrievskii VV, Belyi VI, Dzyadyk VK. Conformal Invariants in Constructive Theory of Functions of Complex Plane. Atlanta: World Federation Publishers Company, 1995.

[18] Andrievskii VV. Weighted polynomial inequalities in the complex plane. Journal of Approximation Theory 2012; 164 (9): 1165-1183.

[19] Andrievskii VV, Blatt HP. Discrepancy of Signed Measures and Polynomial Approximation. New York: Springer-Verlag, 2010.

[20] Balcı S, Imashkyzy M, Abdullayev FG. Polynomial inequalities in regions with interior zero angles in the Bergman space. Ukrainian Mathematical Journal 2018; 70 (3): 362-384.

[21] Benko D, Dragnev P, Totik V. Convexity of harmonic densities. Revista MatemÃ¡tica Iberoamericana, 2012; 28 (4): 1-14.

[22] Bernstein SN. Sur la limitation des derivees des polnomes. Comptes rendus de l'AcadÃ©mie des Sciences 1930; 190: 338-341.

[23]] Bernstein SN. On the best approximation of continuos functions by polynomials of given degree. Izvestiia Akademii Nauk SSSR I. 1952; II; 1954 (O nailuchshem problizhenii nepreryvnykh funktsii posredstrvom mnogochlenov dannoi stepeni), Sobraniye sochinenii. 1912; I (4): 11-96.

[24] Belinskii PP. General Properties of Quasiconformal Mappings. Novosibirsk: Nauka, 1974 (in Russian).

[25] Becker J, Pommerenke C. Über die quasikonforme fortsetzung schlichter funktionen. Mathematische Zeitschrift 1978; 161: 69-80.

[26] Ditzian Z, Tikhonov S. Ul'yanov and Nikol'skii-type inequalities. Journal of Approximation Theory 2005; 133 (1): 100-133. doi:10.1016/j.jat.2004.12.008

[27] Ditzian Z, Prymak A. Nikol'skii inequalities for Lorentz spaces. Rocky Mountain Journal of Mathematics 2010; 40 (1): 209-223.

[28] Dzjadyk VK. Introduction to the Theory of Uniform Approximation of Function by Polynomials. Moscow: Nauka, 1977.

[29] Dyn'kin EM. Nonanalytic symmetry principle and conformal mappings. St. Petersburg Mathematical Journal 1994; 5: 523-544.

[30] Gutlyanskii V, Ryazanov V. On asymptotically conformal curves. Complex Variables and Elliptic Equations 1994; 25 (4): 357-366.

[31] Gutlyanskii V, Ryazanov V. On the local behaviour of quasi-conformal mappings. Izvestiya: Mathematics 1995; 59 (3): 471-498.

[32] Gutlyanskii VYa, Ryazanov VI. On quasicircles and asymptotically conformal curves. Doklady Akademii Nauk 1993; 330 (5): 546-548. English translation: Russian Academy of Sciences 1993; 47: 563-566.

[33] Hille E, Szegö G, Tamarkin JD. On some generalization of a theorem of A.Markoff. Duke Mathematical Journal 1937; 3: 729-739.

[34] Lehto O, Virtanen KI. Quasiconformal Mapping in the Plane. Berlin: Springer-Verlag, 1973.

[35] Lesley FD. Hölder continuity of conformal mappings at the boundary via the strip method. Indiana University Mathematics Journal 1982; 31: 341-354.

[36] Mamedhanov DI. Inequalities of S.M.Nikol'skii type for polynomials in the complex variable on curves. Soviet Mathematics - Doklady 1974; 15: 34-37.

[37] Milovanovic GV, Mitrinovic DS, Rassias ThM. Topics in Polynomials: Extremal Problems, Inequalities, Zeros. Singapore: World Scientific, 1994.

[38] Nevai P, Totik V. Sharp Nikolskii inequalities with exponential weights. Analysis Mathematica 1987; 13 (4): 261-267.

[39] Nikol'skii SM. Approximation of Function of Several Variable and Imbeding Theorems. New York: Springer-Verlag, 1975.

[40] Özkartepe P. Pointwise Bernstein-Walsh-type inequalities in regions with piecewise Dini-smooth boundary. Manas Journal of Engineering 2017; 5 (3): 35-47.

[41] Pommerenke Ch. Univalent Functions. Göttingen: Vandenhoeck Ruprecht, 1975.

[42] Pommerenke Ch. Boundary Behaviour of Conformal Maps. Berlin: Springer-Verlag, 1992.

[43] Pommerenke Ch, Warschawski SE. On the quantitative boundary behavior of conformal maps. Commentarii Mathematici Helvetici 1982; 57: 107-129.

[44] Pritsker I. Comparing norms of polynomials in one and several variables. Journal of Mathematical Analysis and Applications 1997; 216: 685-695. doi:10.1006/jmaa.1997.5699

[45] Stylianopoulos N. Strong asymptotics for Bergman polynomials over domains with corners and applications. Constructive Approximation 2012; 38 (1): 59-100.

[46] Szegö G, Zygmund A. On certain mean values of polynomials. Journal d'Analyse Mathematique. 1953; 3 (1): 225-244.

[47] Warschawski SE. On differentiability at the boundary in conformal mapping. Proceedings of the American Mathematical Society 1961; 12: 614-620.

[48] Warschawski SE. On Hölder continuity at the boundary in conformal maps. Journal of Mathematics and Mechanics 1968; 18: 423-427.

[49] Walsh JL. Interpolation and Approximation by Rational Functions in the Complex Domain. AMS Colloquium Publications, 1960.