Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 3025 – 3044
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doi:10.55730/1300-0098.3316

Solving a class of ordinary differential equations and fractional differential equations with conformable derivative by fractional Laplace transform

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Received: 12.04.2021 • **Accepted/Published Online:** 15.08.2022 • **Final Version:** 05.09.2022

Abstract: In this paper, we use the fractional Laplace transform to solve a class of second-order ordinary differential equations (ODEs), as well as some conformable fractional differential equations (CFDEs), including the Laguerre conformable fractional differential equation. Specifically, we apply the transform to convert the differential equations into first-order, linear differential equations. This is done by using the fractional Laplace transform of order $\alpha+\beta$ or $\alpha+\beta+\gamma$. Also, we investigate some more results on the fractional Laplace transform, obtained by Abdeljawad.

Key words: Fractional Laplace transform, conformable fractional derivative, conformable fractional differential equation, Laguerre conformable fractional differential equation

1. Introduction

Fractional calculus has been of interest to many researchers of the last and present centuries. Many researchers have dealt with the discrete version of fractional calculus, which benefits from the theory of time scales. However, in recent decades, fractional calculus and fractional differential equations have gained great development in both theory and applications, because of their powerful, potential applications. Fractional differential equations are sometimes referred to as extraordinary differential equations, because of their nature and the fact that they can be easily found in various fields of applied sciences [2, 10]. For example, fractional-order differential equations have been applied to the modeling of real-world phenomena in such diverse fields as physics, engineering, mechanics, control theory, economics, medical sciences, finance, etc. See [1–8, 15, 17].

In recent years, scientists have proposed many efficient and powerful methods to obtain exact or numerical solutions of fractional differential equations [15, 18]. In addition, many researchers have tried to propose new definitions of fractional derivatives. Such a definition usually uses an integral form for the fractional derivative. There are many types of differential derivatives in fractional calculus, including the derivatives of Grunwald–Letnikov, Riemann–Liouville, Caputo [7], Caputo–Fabrizio [11] and Atangana–Baleanu [2]. Recently, some authors introduced the concept of nonlocal derivative.

In [15], Khalil et al. proposed a new notion of derivative, prominently compatible with the classical one. This operator is called a ”conformable derivative”. The chain rule, which is an applicable and useful rule in calculus, holds authentically only for conformable fractional derivatives. This derivative satisfies some

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2010 *AMS Mathematics Subject Classification:* 34A08, 44A10 conventional properties and can be used to solve conformable differential equations. In [18], the author used the conformable derivative to find the exact solutions of time heat differential equations. The conformable fractional derivative has many advantages in its properties. The impact of this fractional derivative in both pure and applied branches of sciences and engineering has increased substantially during the last decade. As a result, it is now widely used in many research fields.

This paper is organized as follows. In Section 2, we prove some important theorems based on the conformable fractional derivative. In Section 3, we present some conformable fractional Laplace theorems and provide some examples which are solved by taking the fractional Laplace transformation of order α. In Section 4, we develop some new methods for solving a class of conformable fractional differential equations using fractional Laplace transformations of order $\alpha+\beta$. In Section 5, we present some new methods for solving a class of second-order ODEs using fractional Laplace transformations. Finally, in Section 6, we solve the Laguerre conformable fractional differential equation and some other conformable fractional differential equations by taking the fractional Laplace transformation of order $\alpha+\beta$.

2. Basic definitions and tools related to the conformable fractional derivative

Despite becoming a popular topic in recent years, the concept of the fractional derivative emerged in the late 17th century. There are several definitions for fractional derivatives. Recently, the conformable fractional derivative was introduced by Khalil et al.[15], using a limit operator. After that, Abdeljawad [17] has also presented fractional versions of the chain rule, exponential functions, Gronwall's inequality, Taylor power series expansions, and fractional Laplace transform for the conformable derivative.

In [15], Khalil et al. introduced a new kind of fractional derivatives as follows.

Definition 2.1 *The left conformable fractional derivative of order $0<\alpha\leq 1$ of a function $u:[a,+\infty)\to\mathbb{R}$, starting from $a\in\mathbb{R}$, is defined by*

$$({}_tT^a_\alpha u)(t)=\lim_{\epsilon\to 0}\frac{u\left(t+\epsilon(t-a)^{1-\alpha}\right)-u(t)}{\epsilon}. \tag{2.1}$$

When $a=0$,

$$({}_tT^0_\alpha u)(t)={}_tT_\alpha u(t)=\lim_{\epsilon\to 0}\frac{u\left(t+\epsilon t^{1-\alpha}\right)-u(t)}{\epsilon}.$$

If $({}_tT^a_\alpha u)(t)$ exists on $(a,+\infty)$, then $({}_tT^a_\alpha u)(a)=\lim_{t\to a^+}({}_tT^a_\alpha u)(t)$. If $({}_tT^a_\alpha u)(z_0)$ exists and is finite, then we say that u is left $\alpha-$differentiable at z_0. See [17].

Theorem 2.2 *Suppose that $0<\alpha\leq 1$, and that u_1 and u_2 are left $\alpha-$differentiable functions. Then, the following statements are true.*

1) $\forall\omega_1,\omega_2\in\mathbb{R}$, $\left({}_tT^a_\alpha(\omega_1u_1+\omega_2u_2)\right)(t)=\omega_1\left({}_tT^a_\alpha u_1\right)(t)+\omega_2\left({}_tT^a_\alpha u_2\right)(t)$.

2) $\forall p\in\mathbb{R}$, ${}_tT^a_\alpha\left((t-a)^p\right)=p(t-a)^{p-\alpha}$.

3) *If C is a constant, then* ${}_tT^a_\alpha(C)=0$.

4) *(Product rule for the left CF derivatives)* $\left({}_tT^a_\alpha(u_1u_2)\right)(t)=u_1(t)\left({}_tT^a_\alpha u_2\right)(t)+u_2(t)\left({}_tT^a_\alpha u_1\right)(t)$.

5) (Quotient rule for the left CF derivatives) $\left({}_tT^a_\alpha(\frac{u_1}{u_2})\right)(t) = \frac{u_2(t)({}_tT^a_\alpha u_1)(t) - u_1(t)({}_tT^a_\alpha u_2)(t)}{(u_2(t))^2}, \quad u_2(t) \neq 0.$

6) $({}_tT^a_\alpha u)(t) = (t-a)^{1-\alpha}u'(t),$ *where* $u'(t) = \lim_{\epsilon\to 0}\left(\frac{u(t+\epsilon)-u(t)}{\epsilon}\right).$

Proof See [17]. □

The chain rule is valid for conformable fractional derivatives.

3. The fractional Laplace transform

In this section, we recall the basic definitions and some useful facts related to the fractional Laplace transform. The transform, introduced in [17], helps us to solve some conformable fractional differential equations and second-order ODEs. The conformable fractional Laplace transform is defined as follows.

Definition 3.1 *(Abdeljawad [17]) The conformable fractional Laplace transform (CFLT) of order* $0 < \alpha \leq 1$ *of a function* $u : [0,\infty) \to \mathbb{R}$*, starting from* a*, is defined by*

$$L^a_\alpha\{u(t)\} = \int_a^\infty e^{-s\frac{(t-a)^\alpha}{\alpha}} u(t) d_\alpha(t,a) = \int_a^\infty e^{-s\frac{(t-a)^\alpha}{\alpha}} u(t)(t-a)^{\alpha-1}dt = U^a_\alpha(s). \tag{3.1}$$

If a=0, then

$$L^0_\alpha\{u(t)\} = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} u(t)t^{\alpha-1}dt = U^0_\alpha(s) = U_\alpha(s). \tag{3.2}$$

In particular, if $\alpha = 1,$ *then equation (3.2) is reduced to the definition of the Laplace transform:*

$$L\{u(t)\} = \int_0^\infty e^{-st}u(t)dt = U(s). \tag{3.3}$$

Theorem 3.2 *[14]. If u(t) is piecewise continuous for* $t > t_0$ *if* $|u(t)| \leq v(t)$ *when* $t \geq t_0$ *for some positive constant* t_0 *and if* $\int_{t_0}^\infty v(t)d_\alpha t$ *converges then* $\int_{t_0}^\infty u(t)d_\alpha t$ *also converges. On the other hand if* $u(t) \geq v(t) \geq 0$ *for* $t \geq t_0$ *and if* $\int_{t_0}^\infty v(t)d_\alpha t$ *diverges then* $\int_{t_0}^\infty u(t)d_\alpha t$ *also diverges.*

Theorem 3.3 *[14]. Let an exponential order function u(t) be piecewise continuous on the interval* $0 \leq t \leq A$ *for any positive* A *and* $e^{-a\frac{t^\alpha}{\alpha}}|u(t)| < M$ *when* $t > t_0$*. In this equality* $M,\ a,\ t_0$ *are positive real constants and* $0 < \alpha \leq 1$*. Then the conformable Laplace transform defined by 3.1 exists for any* $s > a$*.*

Theorem 3.4 *Let* $u : [a,\infty) \to \mathbb{R}$ *be a differentiable real-valued function, and* $0 < \alpha \leq 1$*. Then,*

$$L^a_\alpha\left\{{}_tT^a_\alpha(u)(t)\right\} = sU^a_\alpha(s) - u(a). \tag{3.4}$$

Proof See [17]. □

Theorem 3.5 *If* u *is piecewise continuous on* $[0,\infty)$ *and* $L^a_\alpha\{u(t)\} = U^a_\alpha(s)$*, then*

$$L^0_\alpha\left\{t^{n\alpha}u(t)\right\} = (-1)^n\alpha^n\frac{d^n}{ds^n}\left[U^0_\alpha(s)\right], \quad n \in \mathbb{N}. \tag{3.5}$$

Proof See [12]. □

Example 3.6 *We calculate the fractional Laplace transform of* $f(t) = \frac{1}{t}$*, for* $0 < \alpha \leq 1$ *and* $\frac{1}{\alpha} \notin \mathbb{N}$*.*
By Definition 3.1,

$$L^0_\alpha\left\{\frac{1}{t}\right\} = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}\frac{1}{t}t^{\alpha-1}dt.$$

Hence, using the change of variable $z = \frac{t^\alpha}{\alpha}$ *we obtain*

$$L^0_\alpha\left\{\frac{1}{t}\right\} = \int_0^\infty e^{-sz}\frac{1}{\alpha^{\frac{1}{\alpha}}z^{\frac{1}{\alpha}}}dz = \frac{\alpha^{-\frac{1}{\alpha}}}{s^{1-\frac{1}{\alpha}}}\Gamma\left(1-\frac{1}{\alpha}\right).$$

Example 3.7 *If* $0 < \alpha \leq 1,$ *then the following equalities hold.*

$$1)\ \int_0^\infty e^{-2\sqrt{t}}sin2\sqrt{t}dt = \frac{1}{4}.$$

$$2)\ L^0_\alpha\{\ln(t)\} = -\frac{1}{\alpha s}\Big(\ln(\frac{s}{\alpha})+\gamma\Big), \qquad \gamma = .5772157...$$

Using Theorem 3.5 we can write

$$\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}f(t)t^{2\alpha-1}dt = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}\Big(t^\alpha f(t)\Big)t^{\alpha-1}dt = -\alpha F'_\alpha(s).$$

So, letting $f(t) = sin(1\frac{t^\alpha}{\alpha})$ *we find that*

$$\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}sin(\frac{t^\alpha}{\alpha})t^{2\alpha-1}dt = \frac{2\alpha s}{(s^2+1)^2}.$$

Finally, we let $s = 1$ *and* $\alpha = \frac{1}{2}$ *to obtain*

$$\int_0^\infty e^{-2\sqrt{t}}sin2\sqrt{t}dt = \frac{1}{4}.$$

By Definition 3.1 and the change of variable $z = t^\alpha$ *we can write*

$$L^0_\alpha\{\ln(t)\} = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}\ln(t)t^{\alpha-1}dt = \int_0^\infty e^{-\frac{s}{\alpha}z}\Big(\frac{1}{\alpha}\ln(z)\Big)\frac{1}{\alpha}dz = -\frac{1}{\alpha s}\Big(\ln(\frac{s}{\alpha})+\gamma\Big).$$

4. Basic properties of converting CFDEs to first order ODEs

In this section, we use the fractional Laplace transform for solving a class of conformable fractional differential equations. This is done by taking the fractional Laplace transform of such equations and converting them into first-order ODEs.

Theorem 4.1 *[13]. Let* $u : [a,\infty) \to \mathbb{R}$ *be twice differentiable on* $(a,\infty),$ $\alpha,\ \beta > 0$ *and* $\alpha+\beta \leq 1.$ *Then,*

$${}_tT_\beta[{}_tT_\alpha u(t)] = (1-\alpha)t^{1-(\alpha+\beta)}u'(t) + t^{2-(\alpha+\beta)}u''(t). \tag{4.1}$$

Theorem 4.2 *[13]. If $u:[a,\infty)\to\mathbb{R}$ is twice differentiable on (a,∞), $\alpha,\ \beta>0$ and $\alpha+\beta\le 1$, then the following equalities hold.*

1) $\int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}\Big(tu''(t)\Big)dt = u(0)+s\int_0^\infty t^{\alpha+\beta}u'(t)e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}dt - sU_{(\alpha+\beta)}(s).$

2) $\int_0^\infty \Big(1-\alpha+st^{\alpha+\beta}\Big)u'(t)e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}dt = (\alpha-1)u(0)-(\alpha+\beta)sU_{(\alpha+\beta)}(s)+(1-\alpha)sU_{(\alpha+\beta)}(s)-(\alpha+\beta)s^2U'_{(\alpha+\beta)}(s).$

3) $L^0_{(\alpha+\beta)}\Big\{{}_tT_\beta\Big(\ {}_tT_\alpha u(t)\Big)\Big\} = \alpha u(0)-(2\alpha+\beta)sU_{(\alpha+\beta)}(s)-(\alpha+\beta)s^2U'_{(\alpha+\beta)}(s).$

4) $L^0_{(\alpha+\beta)}\Big\{{}_tT_\beta\Big(\ {}_tT_\alpha u(t)\Big)+\ {}_tT_\alpha\Big(\ {}_tT_\beta u(t)\Big)\Big\} = (\alpha+\beta)u(0)-(3\alpha+3\beta)sU_{(\alpha+\beta)}(s)-(2\alpha+2\beta)s^2U'_{(\alpha+\beta)}(s).$

Theorem 4.3 *[13]. If $s,\ \alpha,\ \beta>0$ and $\alpha+\beta\le 1$, then the following are true.*

1) $L^0_{\alpha+\beta}\Big\{t^{-\beta}\Big(\ {}_tT_\alpha u(t)\Big)\Big\} = -u(0)+sU_{(\alpha+\beta)}(s).$

2) $L^0_{\alpha+\beta}\Big\{t^{\alpha}\Big(\ {}_tT_\alpha u(t)\Big)\Big\} = L^0_{\alpha+\beta}\Big\{tu'(t)\Big\} = L^0_{\alpha+\beta}\Big\{t^{\alpha+\beta}\Big(\ {}_tT_{\alpha+\beta}u(t)\Big)\Big\} = -(\alpha+\beta)U_{(\alpha+\beta)}(s)-(\alpha+\beta)sU'_{(\alpha+\beta)}(s).$

Proposition 4.4 *If $s>0$ and $0<\alpha\le 1$ then*

$$\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}\Big(\frac{{}_tT_\alpha f(t)}{t}\Big)t^{2\alpha}dt = -\alpha F_\alpha(s)-\alpha sF'_\alpha(s).$$

Proof By Theorem 3.4 and Theorem 3.5,

$$\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}\Big(\frac{{}_tT_\alpha f(t)}{t}\Big)t^{2\alpha}dt = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}\Big(t^\alpha\ {}_tT_\alpha f(t)\Big)t^{\alpha-1}dt = -\alpha\frac{d}{ds}\Big(sF_\alpha(s)-f(0)\Big) = -\alpha F_\alpha(s)-\alpha sF'_\alpha(s).$$

□

Proposition 4.5 *Let an exponential order function $u(t)$ be twice differentiable on $(0,\infty)$ and $\alpha,\beta>0$ such that $\alpha+\beta\le 1$, then, the solution of the CFDE*

$$ {}_tT_\beta\Big(\ {}_tT_\alpha u(t)\Big)+\Big(t^\alpha+t^{-\beta}\Big)\Big(\ {}_tT_\alpha u(t)\Big) = q(t) \tag{4.2}$$

is given by

$$u(t) = (L^0_{\alpha+\beta})^{-1}\left\{\Big(\frac{(s+1)^{\frac{1-\alpha}{\alpha+\beta}}}{s}\Big)\Big(\int\frac{(\alpha-1)u(0)-Q_{(\alpha+\beta)}(s)}{(\alpha+\beta)(s+1)^{\frac{\beta+1}{\alpha+\beta}}}ds+C\Big)\right\}.$$

Proof Applying $L^0_{\alpha+\beta}$ to the both sides of (4.2), and using Theorem 4.2 and Theorem 4.3, we obtain

$$\alpha u(0)-(2\alpha+\beta)sU_{(\alpha+\beta)}(s)-(\alpha+\beta)s^2U'_{(\alpha+\beta)}(s)-(\alpha+\beta)U_{(\alpha+\beta)}(s)-(\alpha+\beta)sU'_{(\alpha+\beta)}(s)-u(0)+sU_{(\alpha+\beta)}(s)$$
$$= Q_{(\alpha+\beta)}(s).$$

Therefore, we can write

$$U'_{(\alpha+\beta)}(s)+\Big(\frac{\alpha-1}{(\alpha+\beta)(s+1)}+\frac{1}{s}\Big)U_{(\alpha+\beta)}(s)=\frac{(\alpha-1)u(0)-Q_{(\alpha+\beta)}(s)}{s(s+1)(\alpha+\beta)}.$$

By solving this first-order differential equation we obtain

$$U_{(\alpha+\beta)}(s)=\Big(e^{-\int(\frac{\alpha-1}{(\alpha+\beta)(s+1)}+\frac{1}{s})ds}\Big)\Big(\int\frac{(\alpha-1)u(0)-Q_{(\alpha+\beta)}(s)}{s(s+1)(\alpha+\beta)}e^{\int(\frac{\alpha-1}{(\alpha+\beta)(s+1)}+\frac{1}{s})ds}ds+C\Big).$$

Hence,

$$U_{(\alpha+\beta)}(s)=\Big(\frac{(s+1)^{\frac{1-\alpha}{\alpha+\beta}}}{s}\Big)\Big(\int\frac{(\alpha-1)u(0)-Q_{(\alpha+\beta)}(s)}{(\alpha+\beta)(s+1)^{\frac{\beta+1}{\alpha+\beta}}}ds+C\Big).$$

Now, the solution $u(t)$ can be found by applying the CF inverse transform. □

In particular, when $\alpha=\beta=\frac{1}{2}$ and u(0)=0, the solution of the CFDE

$${}_tT_{\frac{1}{2}}\Big({}_tT_{\frac{1}{2}}u(t)\Big)+\Big(\sqrt{t}+\frac{1}{\sqrt{t}}\Big)\Big({}_tT_{\frac{1}{2}}u(t)\Big)=5t+2t^2$$

is given by

$$u(t)=(L^0_{(\frac{1}{2}+\frac{1}{2})})^{-1}\{U_{(\frac{1}{2}+\frac{1}{2})}(s)\}=L^{-1}\left\{\frac{2}{s^3}+C(\frac{\sqrt{s+1}}{s})\right\}=t^2+C\Big(\frac{e^{-t}}{\sqrt{\pi t}}+erf(\sqrt{t})\Big).$$

Since $u(0)=0$, $u(t)=t^2$.

Theorem 4.6 *Let $s>0$, $0<\alpha\leq 1$ and an exponential order function g(t) be twice differentiable on $(0,\infty)$ such that $L^0_\alpha\{g(t)\}=G_\alpha(s)$, then*

$$L^0_\alpha\left\{{}_tT_\alpha g(t)+t^{2-\alpha}g''(t)\right\}=-\alpha sG_\alpha(s)-\alpha s^2G'_\alpha(s). \tag{4.3}$$

Proof By (3.5),

$$L^0_\alpha\left\{f(t)+\frac{t}{\alpha}f'(t)\right\}=L^0_\alpha\left\{f(t)\right\}+\frac{1}{\alpha}L^0_\alpha\left\{t^\alpha\Big({}_tT_\alpha f(t)\Big)\right\}=F_\alpha(s)+\frac{1}{\alpha}(-\alpha)\Big(sF_\alpha(s)-f(0)\Big)'=-sF'_\alpha(s).$$

Now, letting $f(t)={}_tT_\alpha g(t)$ and using Theorem 3.4 we can write

$$L^0_\alpha\left\{{}_tT_\alpha g(t)+\frac{t}{\alpha}\Big({}_tT_\alpha g(t)\Big)'\right\}=-s\Big(sG_\alpha(s)-g(0)\Big)'=-sG_\alpha(s)-s^2G'_\alpha(s).$$

Since ${}_tT_\alpha g(t)+\frac{t}{\alpha}\Big({}_tT_\alpha g(t)\Big)'=\frac{1}{\alpha}t^{1-\alpha}g'(t)+\frac{1}{\alpha}t^{2-\alpha}g''(t)$,

$$L^0_\alpha\left\{{}_tT_\alpha g(t)+t^{2-\alpha}g''(t)\right\}=L^0_\alpha\left\{t^{1-\alpha}g'(t)+t^{2-\alpha}g''(t)\right\}=-\alpha sG_\alpha(s)-\alpha s^2G'_\alpha(s).$$

□

Therefore, if $\alpha, \beta > 0$ *and* $\alpha + \beta \leq 1$, *then*

$$L^0_{\alpha+\beta}\left\{\Big({}_tT_{(\alpha+\beta)}u(t)\Big) + t^{2-(\alpha+\beta)}u^{''}(t)\right\} = L^0_{\alpha+\beta}\left\{t^{1-(\alpha+\beta)}u^{'}(t) + t^{2-(\alpha+\beta)}u^{''}(t)\right\}$$

$$= -(\alpha+\beta)sU_{(\alpha+\beta)}(s) - (\alpha+\beta)s^2U^{'}_{(\alpha+\beta)}(s).$$

Proposition 4.7 *Suppose that* $u(t)$ *is twice differentiable on* $(0,\infty)$, $\alpha, \beta > 0$ *and* $\alpha + \beta \leq 1$. *Then, the solution of the CFDE*

$$ {}_tT_\beta\Big({}_tT_\alpha u(t)\Big) + \Big({}_tT_{\alpha+\beta}u(t)\Big) + t^{2-(\alpha+\beta)}u^{''}(t) = q(t) \tag{4.4}$$

is given by

$$u(t) = \Big(L^0_{\alpha+\beta}\Big)^{-1}\left\{\Big(\frac{1}{s^{\frac{3\alpha+2\beta}{2\alpha+2\beta}}}\Big)\Big(\int \frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{2(\alpha+\beta)s^{\frac{\alpha+2\beta}{2\alpha+2\beta}}}ds + C\Big)\right\}.$$

Proof Applying $L^0_{\alpha+\beta}$ to the both sides of (4.4), and using Theorem 4.2 and Theorem 4.6, we find that

$$\alpha u(0) - (2\alpha+\beta)sU_{(\alpha+\beta)}(s) - s^2(\alpha+\beta)U'_{(\alpha+\beta)}(s) - (\alpha+\beta)sU_{(\alpha+\beta)}(s) - (\alpha+\beta)s^2U^{'}_{(\alpha+\beta)}(s) = Q_{(\alpha+\beta)}(s).$$

This differential equation can be written as

$$2(\alpha+\beta)s^2U^{'}_{(\alpha+\beta)}(s) + (3\alpha+2\beta)sU_{(\alpha+\beta)}(s) = \alpha u(0) - Q_{(\alpha+\beta)}(s).$$

Therefore, we can write

$$U^{'}_{(\alpha+\beta)}(s) + \Big(\frac{\alpha}{(2\alpha+2\beta)s} + \frac{1}{s}\Big)U_{(\alpha+\beta)}(s) = \frac{\alpha u(0) - Q_{(\alpha+\beta)}(s)}{(2\alpha+2\beta)s^2}.$$

By solving this first-order ODE we obtain

$$U_{\alpha+\beta}(s) = \Big(e^{-\int(\frac{\alpha}{(2\alpha+2\beta)s}+\frac{1}{s})ds}\Big)\Big(\int \frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{(2\alpha+2\beta)s^2}e^{\int(\frac{\alpha}{(2\alpha+2\beta)s}+\frac{1}{s})ds}ds + C\Big)$$

$$= \frac{1}{s^{\frac{3\alpha+2\beta}{2\alpha+2\beta}}}\Big(\int \frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{2(\alpha+\beta)s^{\frac{\alpha+2\beta}{2\alpha+2\beta}}}ds + C\Big).$$

Now, the solution $u(t)$ can be found by applying the CF inverse transform. □

In particular, when $\alpha = \beta = \frac{1}{2}$ *and u(0)=1, the solution of the CFDE*

$$ {}_tT_{\frac{1}{2}}\Big({}_tT_{\frac{1}{2}}u(t)\Big) + {}_tT_{\frac{1}{2}+\frac{1}{2}}u(t) + t^{2-(\frac{1}{2}+\frac{1}{2})}u^{''}(t) = (\frac{3}{2} + 2t)e^t$$

is given by

$$u(t) = L^{-1}\left\{\frac{1}{s^{1+\frac{1}{4}}}\Big(\frac{s^{\frac{5}{4}}}{s-1} + C\Big)\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} + CL^{-1}\left\{\frac{1}{s^{1+\frac{1}{4}}}\right\} = e^t + C\Big(\frac{2\sqrt{2}\Gamma(\frac{3}{4})t^{\frac{1}{4}}}{\pi}\Big).$$

Theorem 4.8 *If $s, \alpha > 0$ and $\alpha \leq 1$, then*

$$L^0_\alpha\{t^{\alpha+1}u^{'}(t)\} = 2\alpha^2 U^{'}_\alpha(s) + \alpha^2 s U^{''}_\alpha(s). \tag{4.5}$$

Proof By Theorem 4.3,

$$L^0_\alpha\{tu^{'}(t)\} = -\alpha U_\alpha(s) - \alpha s U^{'}_\alpha(s).$$

Now, Theorem 3.5 allows us to write

$$L^0_\alpha\{t^{\alpha+1}u^{'}(t)\} = L^0_\alpha\{t^\alpha(tu^{'}(t))\} = -\alpha\frac{d}{ds}\Big(-\alpha U_\alpha(s) - \alpha s U^{'}_\alpha(s)\Big) = 2\alpha^2 U^{'}_\alpha(s) + \alpha^2 s U^{''}_\alpha(s).$$

□

Proposition 4.9 *Consider the CFDE*

$$_tT_\beta\Big(\,_tT_\alpha u(t)\Big) - \,_tT_\alpha\Big(\,_tT_\beta u(t)\Big) = 0. \tag{4.6}$$

If $\alpha,\ \beta > 0$ and $\alpha + \beta \leq 1$, then $\alpha = \beta$ or $u(t)$ is constant.

Proof Applying $L^0_{\alpha+\beta}$ and using Theorem 4.2 we obtain

$$\alpha u(0) - (2\alpha+\beta)sU_{(\alpha+\beta)}(s) - (\alpha+\beta)s^2U^{'}_{(\alpha+\beta)}(s) - \beta u(0) + (2\beta+\alpha)sU_{(\alpha+\beta)}(s) + (\alpha+\beta)s^2U^{'}_{(\alpha+\beta)}(s) = 0.$$

Therefore,

$$\Big(\alpha - \beta\Big)\Big(u(0) - sU_{(\alpha+\beta)}(s)\Big) = 0.$$

Therefore,

$$\Big(\alpha = \beta\Big) \quad or \quad U_{(\alpha+\beta)}(s) = \frac{u(0)}{s}.$$

Finally, we apply $(L^0_{\alpha+\beta})^{-1}$ to obtain

$$u(t) = \Big(L^0_{\alpha+\beta}\Big)^{-1}\Big\{U_{(\alpha+\beta)}(s)\Big\} = \Big(L^0_{\alpha+\beta}\Big)^{-1}\Big\{\frac{u(0)}{s}\Big\} = u(0).$$

□

Proposition 4.10 *Assume that $u(t)$ is twice differentiable on $(0,\infty)$, $k, m \in \mathbb{R}$, $\alpha, \beta > 0$ and $\alpha + \beta \leq 1$. Then, the solution of the CFDE*

$$_tT_\beta[_tT_\alpha u(t)] + \,_tT_\alpha[_tT_\beta u(t)] + ktu^{'}(t) + mu(t) = q(t) \tag{4.7}$$

is given by

$$u(t) = \Big(L^0_{\alpha+\beta}\Big)^{-1}\left\{\Big(\frac{s^{\frac{m}{k(\alpha+\beta)}-1}}{(2s+k)^{\frac{m}{k(\alpha+\beta)}+\frac{1}{2}}}\Big)\Big(\int \frac{(\alpha+\beta)u(0) - Q_{(\alpha+\beta)}(s)}{(\alpha+\beta)s^{\frac{m}{k(\alpha+\beta)}}}(2s+k)^{\frac{2m-k(\alpha+\beta)}{2k(\alpha+\beta)}}ds + C\Big)\right\}.$$

Proof Applying the CF Laplace transform to the both sides of (4.7), and using Theorem 4.2 and Theorem 4.3, we obtain

$$(\alpha+\beta)u(0)-(3\alpha+3\beta)sU_{\alpha+\beta}(s)-(2\alpha+2\beta)s^2U'_{\alpha+\beta}(s)-k(\alpha+\beta)\Big(U_{(\alpha+\beta)}(s)+sU'_{(\alpha+\beta)}(s)\Big)+mU_{(\alpha+\beta)}(s)=Q_{(\alpha+\beta)}(s).$$

Equivalently,

$$U'_{(\alpha+\beta)}(s)+\Big(\frac{3(\alpha+\beta)s+k(\alpha+\beta)-m}{(\alpha+\beta)s(2s+k)}\Big)U_{(\alpha+\beta)}(s)=\frac{(\alpha+\beta)u(0)-Q_{(\alpha+\beta)}(s)}{(\alpha+\beta)s(2s+k)}.$$

By solving this first-order ODE we can write

$$U_{(\alpha+\beta)}(s)=\Big(e^{-\int(\frac{3(\alpha+\beta)s+k(\alpha+\beta)-m}{(\alpha+\beta)s(2s+k)})ds}\Big)\Big(\int\frac{(\alpha+\beta)u(0)-Q_{(\alpha+\beta)}(s)}{(\alpha+\beta)s(2s+k)}e^{\int(\frac{3(\alpha+\beta)s+k(\alpha+\beta)-m}{(\alpha+\beta)s(2s+k)})ds}ds+C\Big)$$

$$=\Big(\frac{s^{\frac{m}{k(\alpha+\beta)}-1}}{(2s+k)^{\frac{m}{k(\alpha+\beta)}+\frac{1}{2}}}\Big)\Big(\int\frac{(\alpha+\beta)u(0)-Q_{(\alpha+\beta)}(s)}{(\alpha+\beta)s^{\frac{m}{k(\alpha+\beta)}}}(2s+k)^{\frac{2m-k(\alpha+\beta)}{2k(\alpha+\beta)}}ds+C\Big).$$

Now, the solution $u(t)$ can be found by applying the CF inverse transform. □

Therefore the solution may not be unique. For example, if $\alpha=\frac{1}{3}$, $\beta=\frac{2}{3}$ and $k=m=1$, then the conformable-type problem

$${}_tT_{\frac{2}{3}}\Big({}_tT_{\frac{1}{3}}u(t)\Big)+{}_tT_{\frac{1}{3}}\Big({}_tT_{\frac{2}{3}}u(t)\Big)+tu'(t)+u(t)=4t^3+15t^2,\quad u(0)=0$$

has the solution given by

$$u(t)=(L^0_{\frac{1}{3}+\frac{2}{3}})^{-1}\left\{\Big(\frac{s^{\frac{1}{1(\frac{1}{3}+\frac{2}{3})}-1}}{(2s+1)^{\frac{1}{1(\frac{1}{3}+\frac{2}{3})}+\frac{1}{2}}}\Big)\Big(\int\frac{(\frac{1}{3}+\frac{2}{3})(0)-\frac{24}{s^4}-\frac{30}{s^3}}{(\frac{1}{3}+\frac{2}{3})s^{\frac{1}{1(\frac{1}{3}+\frac{2}{3})}}}(2s+1)^{\frac{2(1)-1(\frac{1}{3}+\frac{2}{3})}{2(1)(\frac{1}{3}+\frac{2}{3})}}ds+C\Big)\right\}$$

$$=(L_1)^{-1}\left\{\frac{1}{(2s+1)^{\frac{3}{2}}}\Big(\int(\frac{-24}{s^5}-\frac{30}{s^4})(2s+1)^{\frac{1}{2}}ds+C\Big)\right\}$$

$$=L^{-1}\Big\{\frac{1}{(2s+1)^{\frac{3}{2}}}\Big(\frac{6(2s+1)^{\frac{3}{2}}}{s^4}+C\Big)\Big\}$$

$$=L^{-1}\Big\{\frac{6}{s^4}\Big\}+CL^{-1}\Big\{\frac{1}{(2s+1)^{\frac{3}{2}}}\Big\}$$

$$=t^3+C\Big(\frac{\sqrt{2}}{2}\sqrt{\frac{t}{\pi}}e^{-\frac{1}{2}t}\Big).$$

5. Solving a class of second-order ODEs via the fractional Laplace transform

In this section, we use the fractional Laplace transform to solve some second-order ODEs.

Proposition 5.1 *The solution of the second-order ODE*

$$2tu^{''}(t) + (t+1)u^{'}(t) + u(t) = q(t) \tag{5.1}$$

is given by

$$u(t) = L^{-1}\left\{\frac{1}{(2s+1)^{\frac{3}{2}}}\Big(\int \frac{u(0)-Q(s)}{s}(2s+1)^{\frac{1}{2}}ds + C\Big)\right\}.$$

Proof Equation (5.1) can be rewritten as

$$\frac{2}{3}u^{'}(t) + tu^{''}(t) + \frac{1}{3}u^{'}(t) + tu^{''}(t) + tu^{'}(t) + u(t) = q(t),$$

or equivalently,

$$(1-\frac{1}{3})t^{1-(\frac{1}{3}+\frac{2}{3})}u^{'}(t) + t^{2-(\frac{1}{3}+\frac{2}{3})}u^{''}(t) + (1-\frac{2}{3})t^{1-(\frac{2}{3}+\frac{1}{3})}u^{'}(t) + t^{2-(\frac{2}{3}+\frac{1}{3})}u^{''}(t) + tu^{'}(t) + u(t) = q(t).$$

Using (4.1) we can write

$${}_tT_{\frac{2}{3}}\Big({}_tT_{\frac{1}{3}}u(t)\Big) + {}_tT_{\frac{1}{3}}\Big({}_tT_{\frac{2}{3}}u(t)\Big) + 1tu^{'}(t) + 1u(t) = q(t).$$

Now, (4.7) gives us

$$\begin{aligned} u(t) &= (L_1^0)^{-1}\left\{\frac{1}{(2s+1)^{\frac{3}{2}}}\Big(\int \frac{u(0)-Q_1(s)}{s}(2s+1)^{\frac{1}{2}}ds + C\Big)\right\} \\ &= L^{-1}\left\{\frac{1}{(2s+1)^{\frac{3}{2}}}\Big(\int \frac{u(0)-Q(s)}{s}(2s+1)^{\frac{1}{2}}ds + C\Big)\right\}. \end{aligned}$$

□

For example, the solution of the differential equation

$$2tu''(t) + (t+1)u^{'}(t) + u(t) = te^{-t}, \quad u(0) = 1$$

is given by

$$\begin{aligned} u(t) &= (L_1^0)^{-1}\left\{\frac{1}{(2s+1)^{\frac{3}{2}}}\Big(\int \frac{1-\frac{1}{(s+1)^2}}{s}(2s+1)^{\frac{1}{2}}ds + C\Big)\right\} = L^{-1}\left\{\frac{1}{(2s+1)^{\frac{3}{2}}}\Big(\frac{(2s+1)^{\frac{3}{2}}}{s+1} + C\Big)\right\} \\ &= L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{C}{(2s+1)^{\frac{3}{2}}}\right\} \\ &= e^{-t} + C\Big(\frac{\sqrt{2}}{2}\sqrt{\frac{t}{\pi}}e^{-\frac{1}{2}t}\Big). \end{aligned}$$

Proposition 5.2 *If $k,\ m \in \mathbb{R}$, then the solution of the second-order ODE*

$$2tu^{''}(t) + (kt+1)u^{'}(t) + mu(t) = q(t) \tag{5.2}$$

is given by

$$u(t) = L^{-1}\left\{\Big(\frac{s^{\frac{m-k}{k}}}{(2s+k)^{\frac{2m+k}{2k}}}\Big)\Big(\int \frac{u(0)-Q(s)}{s^{\frac{m}{k}}}(2s+k)^{\frac{2m-k}{2k}}ds + C\Big)\right\}.$$

Proof Equation (5.2) can be rewritten as

$$\frac{2}{3}u^{'}(t) + tu^{''}(t) + \frac{1}{3}u^{'}(t) + tu^{''}(t) + ktu^{'}(t) + mu(t) = q(t),$$

or equivalently,

$$(1-\frac{1}{3})t^{1-(\frac{1}{3}+\frac{2}{3})}u^{'}(t) + t^{2-(\frac{1}{3}+\frac{2}{3})}u^{''}(t) + (1-\frac{2}{3})t^{1-(\frac{2}{3}+\frac{1}{3})}u^{'}(t) + t^{2-(\frac{2}{3}+\frac{1}{3})}u^{''}(t) + ktu^{'}(t) + mu(t) = q(t).$$

Using (4.1) we can write

$${}_tT_{\frac{2}{3}}\Big({}_tT_{\frac{1}{3}}u(t)\Big) + {}_tT_{\frac{1}{3}}\Big({}_tT_{\frac{2}{3}}u(t)\Big) + ktu^{'}(t) + mu(t) = q(t).$$

Finally, (4.7) gives us

$$\begin{aligned}
u(t) &= (L^0_{(\frac{1}{3}+\frac{2}{3})})^{-1}\left\{U_{(\frac{1}{3}+\frac{2}{3})}(s)\right\} \\
&= \Big(L^0_{(\frac{1}{3}+\frac{2}{3})}\Big)^{-1}\left\{\Big(\frac{s^{\frac{m}{k(\frac{1}{3}+\frac{2}{3})}-1}}{(2s+k)^{\frac{m}{k(\frac{1}{3}+\frac{2}{3})}+\frac{1}{2}}}\Big)\Big(\int \frac{(\frac{1}{3}+\frac{2}{3})u(0) - Q_{\frac{1}{3}+\frac{2}{3}}(s)}{(\frac{1}{3}+\frac{2}{3})s^{\frac{m}{k(\frac{1}{3}+\frac{2}{3})}}}(2s+k)^{\frac{2m-k(\frac{1}{3}+\frac{2}{3})}{2k(\frac{1}{3}+\frac{2}{3})}}ds + C\Big)\right\} \\
&= L^{-1}\left\{\Big(\frac{s^{\frac{m-k}{k}}}{(2s+k)^{\frac{2m+k}{2k}}}\Big)\Big(\int \frac{u(0)-Q(s)}{s^{\frac{m}{k}}}(2s+k)^{\frac{2m-k}{2k}}ds + C\Big)\right\}.
\end{aligned}$$

□

Proposition 5.3 *Let* $a,\ b,\ p \in \mathbb{R}$, *and* $a,\ b \neq 0$. *Then, the solution of the second-order ODE*

$$2atu^{''}(t) + (bt+a)u^{'}(t) + pu(t) = q(t) \tag{5.3}$$

is given by

$$u(t) = L^{-1}\left\{\Big(\frac{s^{\frac{p}{b}-1}}{(2s+\frac{b}{a})^{\frac{p}{b}+\frac{1}{2}}}\Big)\Big(\int \frac{u(0)-\frac{1}{a}Q(s)}{s^{\frac{p}{b}}}(2s+\frac{b}{a})^{\frac{p}{b}-\frac{1}{2}}ds + C\Big)\right\}.$$

Proof Equation (5.3) can be rewritten as

$$\frac{2}{3}u^{'}(t) + tu^{''}(t) + \frac{1}{3}u^{'}(t) + tu^{''}(t) + \frac{b}{a}tu^{'}(t) + \frac{p}{a}u(t) = \frac{1}{a}q(t),$$

or equivalently

$$(1-\frac{1}{3})t^{1-(\frac{1}{3}+\frac{2}{3})}u^{'}(t) + t^{2-(\frac{1}{3}+\frac{2}{3})}u^{''}(t) + (1-\frac{2}{3})t^{1-(\frac{2}{3}+\frac{1}{3})}u^{'}(t) + t^{2-(\frac{2}{3}+\frac{1}{3})}u^{''}(t) + \frac{b}{a}tu^{'}(t) + \frac{p}{a}u(t) = \frac{1}{a}q(t).$$

Using (4.1) we can write

$$ {}_tT_{\frac{2}{3}}\Big({}_tT_{\frac{1}{3}}u(t)\Big) + {}_tT_{\frac{1}{3}}\Big({}_tT_{\frac{2}{3}}u(t)\Big) + \frac{b}{a}tu^{'}(t) + \frac{p}{a}u(t) = \frac{1}{a}q(t). $$

Letting $k = \frac{b}{a},\ m = \frac{p}{a}$ *and* $q(t) = \frac{1}{a}q(t)$ in (4.7) we get

$$ u(t) = L^{-1}\Bigg\{\Big(\frac{s^{\frac{p}{b}-1}}{(2s+\frac{b}{a})^{\frac{p}{b}+\frac{1}{2}}}\Big)\Big(\int \frac{u(0)-\frac{1}{a}Q(s)}{s^{\frac{p}{b}}}(2s+\frac{b}{a})^{\frac{p}{b}-\frac{1}{2}}ds + C\Big)\Bigg\}. $$

□

In particular, if $a = b = 2,\ p = 1$ *and* $u(0) = 0,$ *then the solution of the ODE*

$$ 4tu^{''}(t) + (2t+2)u^{'}(t) + u(t) = \big(2 - 17t + 12t^2\big)e^{-2t} $$

is given by

$$ u(t) = te^{-2t} + C\Big(\frac{-\sqrt{2}}{2}Ie^{-\frac{1}{2}t}erf(\frac{\sqrt{2}}{2}I\sqrt{t})\Big). $$

Since $u(0) = 0,$ *C=0. So, we conclude that*

$$ u(t) = te^{-2t}. $$

6. Solving some conformable fractional differential equations via the fractional Laplace transform

In this section, we use the fractional Laplace transform to solve the Laguerre fractional differential equation. Also, we obtain some new results on some second-order ODEs and some other conformable fractional differential equations.

Theorem 6.1 *If* $s,\ \alpha > 0,\ \alpha \leq 1$ *and* $L^0_\alpha\{u(t)\} = U_\alpha(s),$ *then the following equalities hold.*

$$ 1)\ L^0_\alpha\Big\{t^{2-\alpha}u^{''}(t)\Big\} = \big(-1-\alpha\big)sU_\alpha(s) + u(0) - \alpha s^2 U^{'}_\alpha(s). $$

$$ 2)\ L^0_\alpha\Big\{t^2u^{''}(t)\Big\} = \big(\alpha+\alpha^2\big)U_\alpha(s) + \alpha s\big(1+3\alpha\big)U^{'}_\alpha(s) + \alpha^2 s^2 U^{''}_\alpha(s). $$

Proof By Theorem 4.6,

$$ L^0_\alpha\Big\{ {}_tT_\alpha u(t) + t^{2-\alpha}u^{''}(t)\Big\} = -\alpha sU_\alpha(s) - \alpha s^2 U^{'}_\alpha(s). $$

Therefore, Theorem 3.4 gives us the first formula:

$$ L^0_\alpha\Big\{t^{2-\alpha}u^{''}(t)\Big\} = \big(-1-\alpha\big)sU_\alpha(s) + u(0) - \alpha s^2 U^{'}_\alpha(s). $$

Now, using Theorem 3.5 we can write

$$ L^0_\alpha\Big\{t^\alpha\big(t^{2-\alpha}u^{''}(t)\big)\Big\} = -\alpha\frac{d}{ds}\Big((-1-\alpha)sU_\alpha(s) + u(0) - \alpha s^2 U^{'}_\alpha(s)\Big). $$

So, we obtain

$$L^0_\alpha\Big\{t^2u^{''}(t)\Big\} = \big(\alpha+\alpha^2\big)U_\alpha(s) + \alpha s\big(1+3\alpha\big)U^{'}_\alpha(s) + \alpha^2 s^2 U^{''}_\alpha(s),$$

which completes the proof.

□

Therefore, if $\alpha,\ \beta > 0,\ \alpha+\beta \le 1$ *and* $m\in\mathbb{R}$, then we obtain the following equalities.

1) $L^0_{\alpha+\beta}\left\{t^{1-(\alpha+\beta)}u^{'}(t)\right\} = L^0_{\alpha+\beta}\left\{T_{\alpha+\beta}u(t)\right\} = -u(0) + sU_{\alpha+\beta}(s).$

2) $L^0_{\alpha+\beta}\left\{t^{2-(\alpha+\beta)}u^{''}(t)\right\} = \big(-1-(\alpha+\beta)\big)sU_{\alpha+\beta}(s) + u(0) - \big(\alpha+\beta\big)s^2U^{'}_{\alpha+\beta}(s).$

3) $L^0_{\alpha+\beta}\left\{(m+1)t^{1-(\alpha+\beta)}u^{'}(t) + t^{2-(\alpha+\beta)}u^{''}(t)\right\} = -mu(0) - \big(\alpha+\beta-m\big)sU_{\alpha+\beta}(s) - \big(\alpha+\beta\big)s^2U^{'}_{\alpha+\beta}(s).$

Theorem 6.2 *If* $s,\alpha,\beta,\gamma>0$ *and* $\alpha+\beta+\gamma\le 1$, *then the following equalities hold.*

I) $\displaystyle\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big(t^2u^{'''}(t)\Big)dt = -2\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big(tu^{''}(t)\Big)dt + s\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big(t^{\alpha+\beta+\gamma+1}u^{''}(t)\Big)dt.$

II) $\displaystyle\int_0^\infty \Big(\big(1-2\alpha-\beta\big)t + st^{\alpha+\beta+\gamma+1}\Big)e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{''}(t)dt$

$$= -\int_0^\infty\Big(\big(1-2\alpha-\beta\big) + s\big(\alpha+\beta+\gamma+1\big)t^{\alpha+\beta+\gamma}\Big)\Big(e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{'}(t)\Big)dt$$

$$+s\big(1-2\alpha-\beta\big)\int_0^\infty t^{\alpha+\beta+\gamma}e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{'}(t)dt + s^2\int_0^\infty t^{2\alpha+2\beta+2\gamma}e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{'}(t)dt.$$

Proof Using integration by parts we can write

$$\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}(t^2u^{'''}(t))dt$$

$$= t^2e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{''}(t)\Big]_0^\infty - \int_0^\infty u^{''}(t)\Big(2t - st^{\alpha+\beta+\gamma+1}\Big)e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}dt$$

$$= 0 - 2\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big(tu^{''}(t)\Big)dt + s\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big(t^{\alpha+\beta+\gamma+1}u^{''}(t)\Big)dt.$$

Also,

$$\int_0^\infty\Big((1-2\alpha-\beta)t + st^{\alpha+\beta+\gamma+1}\Big)e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{''}(t)dt$$

$$= \Big((1-2\alpha-\beta)t + st^{\alpha+\beta+\gamma}\Big)e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{'}(t)\Big]_0^\infty$$

$$-\int_0^\infty\Big\{(1-2\alpha-\beta) + s(\alpha+\beta+\gamma+1)t^{\alpha+\beta+\gamma}\Big\}\Big(e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{'}(t)\Big)dt$$

$$+s\Big(1-2\alpha-\beta\Big)\int_0^\infty t^{\alpha+\beta+\gamma}\Big(e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big)u^{'}(t)dt+s^2\int_0^\infty t^{2\alpha+2\beta+2\gamma}\Big(e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big)u^{'}(t)dt$$

$$=0-\int_0^\infty\Big\{\big(1-2\alpha-\beta\big)+s\big(\alpha+\beta+\gamma+1\big)t^{\alpha+\beta+\gamma}\Big\}\Big(e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{'}(t)\Big)dt$$

$$+s\big(1-2\alpha-\beta\big)\int_0^\infty t^{\alpha+\beta+\gamma}\Big(e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big)u^{'}(t)dt+s^2\int_0^\infty t^{2\alpha+2\beta+2\gamma}\Big(e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big)u^{'}(t)dt.$$

□

Theorem 6.3 *If* $s,\alpha,\beta,\gamma>0$ *and* $\alpha+\beta+\gamma\leq 1,$ *then*

$$\int_0^\infty\Big\{(\alpha^2+\alpha\beta)-(3\alpha+2\beta+\gamma)st^{\alpha+\beta+\gamma}+s^2t^{2\alpha+2\beta+2\gamma}\Big\}\Big\{e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{'}(t)\Big\}dt$$

$$=-\big(\alpha^2+\alpha\beta\big)u(0)+\big(3\alpha+2\beta+\gamma\big)\big(\alpha+\beta+\gamma\big)sU_{\alpha+\beta+\gamma}(s)$$

$$+2\big(\alpha+\beta+\gamma\big)^2s^2U^{'}_{\alpha+\beta+\gamma}(s)+\big(\alpha^2+\alpha\beta\big)sU_{\alpha+\beta+\gamma}(s)$$

$$+\big(3\alpha+2\beta+\gamma\big)\big(\alpha+\beta+\gamma\big)s^2U^{'}_{\alpha+\beta+\gamma}(s)+\big(\alpha+\beta+\gamma\big)^2s^3U^{''}_{\alpha+\beta+\gamma}(s).$$

Proof Using integration by parts and Theorem 3.5 we obtain

$$\int_0^\infty\Big\{(\alpha^2+\alpha\beta)-(3\alpha+2\beta+\gamma)st^{\alpha+\beta+\gamma}+s^2t^{2\alpha+2\beta+2\gamma}\Big\}\Big\{e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{'}(t)\Big\}dt$$

$$=\Big\{(\alpha^2+\alpha\beta)-(3\alpha+2\beta+\gamma)st^{\alpha+\beta+\gamma}+s^2t^{2\alpha+2\beta+2\gamma}\Big\}\Big(e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u(t)\Big)\Big]_0^\infty$$

$$-\int_0^\infty\Big\{(-3\alpha-2\beta-\gamma)s(\alpha+\beta+\gamma)t^{\alpha+\beta+\gamma-1}+s^2(2\alpha+2\beta+2\gamma)t^{\alpha+\beta+\gamma-1}(t^{\alpha+\beta+\gamma})\Big\}e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}u(t)dt$$

$$+\int_0^\infty\Big\{st^{\alpha+\beta+\gamma-1}\Big(\alpha^2+\alpha\beta-s(3\alpha+2\beta+\gamma)\Big)(t^{\alpha+\beta+\gamma})+s^3t^{\alpha+\beta+\gamma-1}(t^{(2\alpha+2\beta+2\gamma)})\Big\}e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}u(t)dt$$

$$=(-\alpha^2-\alpha\beta)u(0)$$

$$+\int_0^\infty(3\alpha+2\beta+\gamma)s(\alpha+\beta+\gamma)t^{\alpha+\beta+\gamma-1}e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}u(t)dt$$

$$-s^2\int_0^\infty(2\alpha+2\beta+2\gamma)(t^{\alpha+\beta+\gamma}u(t))t^{\alpha+\beta+\gamma-1}e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}dt$$

$$+s(\alpha^2+\alpha\beta)\int_0^\infty t^{\alpha+\beta+\gamma-1}e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}u(t)dt$$

$$-s^2(3\alpha+2\beta+\gamma)\int_0^\infty(t^{\alpha+\beta+\gamma}u(t))t^{\alpha+\beta+\gamma-1}e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}dt$$

$$+s^3\int_0^\infty(t^{2\alpha+2\beta+2\gamma}u(t))t^{\alpha+\beta+\gamma-1}e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}dt$$

$$= -(\alpha^2 + \alpha\beta)u(0)$$
$$+(3\alpha + 2\beta + \gamma)(\alpha + \beta + \gamma)sU_{\alpha+\beta+\gamma}(s)$$
$$+2s^2(\alpha + \beta + \gamma)^2 U^{'}_{\alpha+\beta+\gamma}(s) + (\alpha^2 + \alpha\beta)sU_{\alpha+\beta+\gamma}(s)$$
$$+(3\alpha + 2\beta + \gamma)(\alpha + \beta + \gamma)s^2 U^{'}_{\alpha+\beta+\gamma}(s)$$
$$+s^3(\alpha + \beta + \gamma)^2 U^{''}_{\alpha+\beta+\gamma}(s).$$

□

Theorem 6.4 *If* $\alpha,\ \beta,\ \gamma > 0$ *and* $\alpha + \beta + \gamma \leq 1$, *then*

$${}_tT_\gamma\Big({}_tT_\beta\big({}_tT_\alpha u(t)\big)\Big) = (1-\alpha)\big(1-\alpha-\beta\big)t^{1-(\alpha+\beta+\gamma)}u^{'}(t) + \big(3-(2\alpha+\beta)\big)t^{2-(\alpha+\beta+\gamma)}u^{''}(t) + t^{3-(\alpha+\beta+\gamma)}u^{'''}(t).$$

Proof By Theorem 2.2 and Theorem 4.1,

$${}_tT_\gamma\Big({}_tT_\beta({}_tT_\alpha u(t))\Big) = {}_tT_\gamma\Big((1-\alpha)t^{1-(\alpha+\beta)}u^{'}(t) + t^{2-(\alpha+\beta)}u^{''}(t)\Big)$$
$$= t^{1-\gamma}\frac{d}{dt}\Big\{(1-\alpha)t^{1-(\alpha+\beta)}u^{'}(t) + t^{2-(\alpha+\beta)}u^{''}(t)\Big\}$$
$$= (1-\alpha)\big(1-\alpha-\beta\big)t^{1-(\alpha+\beta+\gamma)}u^{'}(t) + \big(3-(2\alpha+\beta)\big)t^{2-(\alpha+\beta+\gamma)}u^{''}(t) + t^{3-(\alpha+\beta+\gamma)}u^{'''}(t).$$

□

Theorem 6.5 *If* $s,\ \alpha,\ \beta,\ \gamma > 0$ *and* $\alpha + \beta + \gamma \leq 1$, *then*

$$L^0_{\alpha+\beta+\gamma}\Big\{{}_tT_\gamma\Big({}_tT_\beta\big({}_tT_\alpha u(t)\big)\Big)\Big\}$$
$$= -\Big(\alpha^2 + \alpha\beta\Big)u(0)$$
$$+\Big\{(3\alpha + 2\beta + \gamma)\big(\alpha + \beta + \gamma\big) + \alpha^2 + \alpha\beta\Big\}sU_{\alpha+\beta+\gamma}(s)$$
$$+\Big(5\alpha + 4\beta + 3\gamma\Big)\Big(\alpha + \beta + \gamma\Big)s^2 U^{'}_{\alpha+\beta+\gamma}(s)$$
$$+\big(\alpha + \beta + \gamma\big)^2 s^3 U^{''}_{\alpha+\beta+\gamma}(s).$$

Proof By Theorem 6.4,

$$L^0_{\alpha+\beta+\gamma}\Big\{{}_tT_\gamma\Big({}_tT_\beta\big({}_tT_\alpha u(t)\big)\Big)\Big\}$$
$$= L^0_{\alpha+\beta+\gamma}\Big\{(1-\alpha)(1-(\alpha+\beta))t^{1-(\alpha+\beta+\gamma)}u^{'}(t)\Big\}$$
$$+L^0_{\alpha+\beta+\gamma}\Big\{\big(3-(2\alpha+\beta)\big)t^{2-(\alpha+\beta+\gamma)}u^{''}(t)\Big\}$$

$$
\begin{aligned}
&+L^0_{\alpha+\beta+\gamma}\Big\{t^{3-(\alpha+\beta+\gamma)}u^{'''}(t)\Big\}\\
&=\Big((1-\alpha)^2+\beta(\alpha-1)\Big)\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u^{'}(t)dt\\
&+\Big(3-2\alpha-\beta\Big)\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big(tu^{''}(t)\Big)dt\\
&+\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big(t^2u^{'''}(t)\Big)dt.
\end{aligned}
$$

Now, the proof is straightforward using Theorem 6.2 and Theorem 6.3.

□

Proposition 6.6 *If $\alpha,\ \beta>0$, $\alpha+\beta\leq 1$, $k,\ m\in\mathbb{R}$ and $k\neq 0$, then the solution of the second-order ODE*

$$t^{2-(\alpha+\beta)}u^{''}(t)+ktu^{'}(t)+mu(t)=q(t) \tag{6.1}$$

is given by

$$u(t)=\Big(L^0_{\alpha+\beta}\Big)^{-1}\Bigg\{\Big(\frac{s^{\frac{m}{k(\alpha+\beta)}-1}}{(s+k)^{\frac{m+k}{k(\alpha+\beta)}}}\Big)\Big(\int\frac{u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{m}{k(\alpha+\beta)}}}(s+k)^{\frac{m+k}{k(\alpha+\beta)}-1}ds+C\Big)\Bigg\}.$$

Proof Applying the CF Laplace transform $L^0_{\alpha+\beta}$ to the both sides of (6.1), and using Theorem 4.3 and Theorem 6.1, we obtain

$$\Big(-1-(\alpha+\beta)\Big)sU_{\alpha+\beta}(s)+u(0)-(\alpha+\beta)s^2U^{'}_{\alpha+\beta}(s)+k\Big(-(\alpha+\beta)U_{\alpha+\beta}(s)-(\alpha+\beta)sU^{'}_{\alpha+\beta}(s)\Big)+mU_{\alpha+\beta}(s)=Q_{\alpha+\beta}(s).$$

Equivalently,

$$U^{'}_{\alpha+\beta}(s)+\Big(\frac{s+(\alpha+\beta)(s+k)-m}{(\alpha+\beta)s(s+k)}\Big)U_{\alpha+\beta}(s)=\frac{u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s(s+k)}.$$

By solving this first-order ODE we can write

$$
\begin{aligned}
U_{\alpha+\beta}(s)&=\Big(e^{-\int(\frac{s+(\alpha+\beta)(s+k)-m}{(\alpha+\beta)s(s+k)})ds}\Big)\Big(\int\frac{u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s(s+k)}e^{\int(\frac{s+(\alpha+\beta)(s+k)-m}{(\alpha+\beta)s(s+k)})ds}ds+C\Big)\\
&=\Big(\frac{s^{\frac{m}{k(\alpha+\beta)}-1}}{(s+k)^{\frac{m+k}{k(\alpha+\beta)}}}\Big)\Big(\int\frac{u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{m}{k(\alpha+\beta)}}}(s+k)^{\frac{m+k}{k(\alpha+\beta)}-1}ds+C\Big).
\end{aligned}
$$

Now, the solution u(t) can be found by applying the CF inverse transform. □

For example, we use the fractional Laplace transform to find the solution of the second-order ODE

$$tu^{''}(t)+ktu^{'}(t)+mu(t)=q(t),$$

where $k,\ m\in\mathbb{R}$ and $k\neq 0$. To do so, we let $\alpha=\beta=\frac{1}{2}$ in (6.1) to obtain

$$t^{2-(\frac{1}{2}+\frac{1}{2})}u^{''}(t)+ktu^{'}(t)+mu(t)=q(t).$$

Therefore,

$$u(t) = (L_1^0)^{-1}\Big\{U_1(s)\Big\} = L^{-1}\left\{\Big(\frac{s^{\frac{m}{k}-1}}{(s+k)^{\frac{m}{k}+1}}\Big)\Big(\int\Big(u(0)-Q(s)\Big)\Big(1+\frac{k}{s}\Big)^{\frac{m}{k}}ds + C\Big)\right\}.$$

In particular, if $k=m=1$ *and* $u(0)=1$*, then the solution of the differential equation*

$$tu^{''}(t) + 1tu^{'}(t) + 1u(t) = 2te^t + e^t$$

is given by

$$\begin{aligned} u(t) &= L^{-1}\Big\{\Big(\frac{1}{(s+1)^2}\Big)\Big(\int(1-\frac{2}{(s-1)^2}-\frac{1}{s-1})(1+\frac{1}{s})ds + C\Big)\Big\} \\ &= e^t + (C-3)te^{-t}. \end{aligned}$$

Proposition 6.7 *If* $\alpha,\ \beta>0$, $\alpha+\beta\le 1$, $a,\ b,\ m\in\mathbb{R}$ *and* $a\neq 0$*, then the solution of the second-order ODE*

$$at^{2-(\alpha+\beta)}u^{''}(t) + bt^{1-(\alpha+\beta)}u^{'}(t) + mu(t) = q(t) \tag{6.2}$$

is given by

$$u(t) = \Big(L_{\alpha+\beta}^0\Big)^{-1}\left\{\Big(\frac{e^{\frac{-m}{a(\alpha+\beta)s}}}{s^{1-\frac{b-a}{a(\alpha+\beta)}}}\Big)\Big(\int\frac{(a-b)u(0)-Q_{\alpha+\beta}(s)}{a(\alpha+\beta)s^{1+\frac{b-a}{a(\alpha+\beta)}}}e^{\frac{m}{a(\alpha+\beta)s}}ds + C\Big)\right\}.$$

Proof Applying the CF Laplace transform $L_{\alpha+\beta}^0$ to the both sides of (6.2) and using Theorem 6.1 we find that

$$a\Big\{\Big(-1-(\alpha+\beta)\Big)sU_{\alpha+\beta}(s) + u(0) - (\alpha+\beta)s^2U_{\alpha+\beta}^{'}(s)\Big\} + b\Big\{-u(0) + sU_{\alpha+\beta}(s)\Big\} + mU_{\alpha+\beta}(s) = Q_{\alpha+\beta}(s).$$

Therefore,

$$U_{\alpha+\beta}^{'}(s) + \Big(\frac{a(\alpha+\beta)s - bs + as - m}{a(\alpha+\beta)s^2}\Big)U_{\alpha+\beta}(s) = \frac{(a-b)u(0) - Q_{\alpha+\beta}(s)}{a(\alpha+\beta)s^2}.$$

By solving this first-order ODE we obtain

$$U_{\alpha+\beta}(s) = \Big(\frac{e^{\frac{-m}{a(\alpha+\beta)s}}}{s^{1-\frac{b-a}{a(\alpha+\beta)}}}\Big)\Big(\int\frac{(a-b)u(0)-Q_{\alpha+\beta}(s)}{a(\alpha+\beta)s^{1+\frac{b-a}{a(\alpha+\beta)}}}e^{\frac{m}{a(\alpha+\beta)s}}ds + C\Big).$$

Now, the solution $u(t)$ can be found by applying the CF inverse transform. □

In particular, letting $\alpha=\beta=\frac{1}{2}$ *in (6.2) we observe that the solution of the second-order ODE*

$$atu''(t) + bu'(t) + mu(t) = q(t)$$

is given by

$$u(t) = L^{-1}\left\{\Big(\frac{e^{\frac{-m}{a(\frac{1}{2}+\frac{1}{2})s}}}{s^{1-\frac{b-a}{a(\frac{1}{2}+\frac{1}{2})}}}\Big)\Big(\int\frac{(a-b)u(0)-Q_{\frac{1}{2}+\frac{1}{2}}(s)}{a(\frac{1}{2}+\frac{1}{2})s^{1+\frac{b-a}{a(\frac{1}{2}+\frac{1}{2})}}}e^{\frac{m}{a(\frac{1}{2}+\frac{1}{2})s}}ds + C\Big)\right\}$$

$$= L^{-1}\left\{\frac{e^{\frac{-m}{as}}}{s^{2-\frac{b}{a}}}\Big(\int \frac{(a-b)u(0)-Q(s)}{as^{\frac{b}{a}}}e^{\frac{m}{as}}ds + C\Big)\right\}.$$

For example, if $a=b=m=1$*, then the solution of the differential equation*

$$tu''(t)+u'(t)+u(t)=te^{-t}$$

is given by

$$u(t)=L^{-1}\Big\{\frac{e^{\frac{-1}{s}}}{s}\Big(\int -\frac{1}{(s+1)^2 s}e^{\frac{1}{s}}ds+C\Big)\Big\}=L^{-1}\Big\{\frac{e^{\frac{-1}{s}}}{s}\Big(\frac{se^{\frac{1}{s}}}{s+1}+C\Big)\Big\}=L^{-1}\Big\{\frac{1}{s+1}\Big\}+L^{-1}\Big\{C\frac{e^{\frac{-1}{s}}}{s}\Big\}$$

$$= e^{-t}+CBesselJ\big(0,2\sqrt{t}\big).$$

Proposition 6.8 *(The Laguerre conformable fractional differential equation) Suppose that* $u(t)$ *is twice differentiable on* $(0,\infty)$*,* $\alpha,\beta>0$*,* $\alpha+\beta\le 1$*,* $k,m,n\in\mathbb{R}$ *and* $k\neq 0$*. Then, the solution of the CFDE*

$$ {}_tT_{\beta}\Big({}_tT_{\alpha}u(t)\Big)+\Big(\alpha+m\Big)\Big({}_tT_{\alpha+\beta}u(t)\Big)+ktu'(t)+\Big(n-m\Big)u(t)=q(t) \tag{6.3}$$

is given by

$$u(t)=\Big(L^0_{\alpha+\beta}\Big)^{-1}\left\{\Big((s+k)^{\frac{(k+1)m-n}{k(\alpha+\beta)}}s^{\frac{n-m}{k(\alpha+\beta)}-1}\Big)\Big(\int\frac{-mu(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{n-m}{k(\alpha+\beta)}}}(s+k)^{\frac{n-m(k+1)}{k(\alpha+\beta)}-1}ds+C\Big)\right\}.$$

Proof By Theorem 2.2 and Theorem 4.1, equation (6.3) can be rewritten as

$$t^{2-(\alpha+\beta)}u''(t)+(m+1)t^{1-(\alpha+\beta)}u'(t)+ktu'(t)+(n-m)u(t)=q(t). \tag{6.4}$$

Now, by applying $L^0_{\alpha+\beta}$ to the both sides of (6.4), and using Theorem 4.3 and Theorem 6.1, we obtain

$$-mu(0)-(\alpha+\beta-m)sU_{\alpha+\beta}(s)-(\alpha+\beta)s^2U'_{\alpha+\beta}(s)+k\Big\{-(\alpha+\beta)U_{\alpha+\beta}(s)-(\alpha+\beta)sU'_{\alpha+\beta}(s)\Big\}$$

$$+(n-m)U_{\alpha+\beta}(s)=Q_{\alpha+\beta}(s).$$

Therefore,

$$U'_{\alpha+\beta}(s)+\Big(\frac{(\alpha+\beta-m)s+(\alpha+\beta)k+m-n}{(\alpha+\beta)s(s+k)}\Big)U_{\alpha+\beta}(s)=\frac{-mu(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)(s)(s+k)}.$$

By solving this first-order ODE we obtain

$$U_{\alpha+\beta}(s)=\Big(e^{-\int(\frac{(\alpha+\beta-m)s+(\alpha+\beta)k+m-n}{(\alpha+\beta)s(s+k)})ds}\Big)\Big(\int\frac{-mu(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s(s+k)}e^{\int(\frac{(\alpha+\beta-m)s+(\alpha+\beta)k+m-n}{(\alpha+\beta)s(s+k)})ds}ds+C\Big)$$

$$=\Big((s+k)^{\frac{(k+1)m-n}{k(\alpha+\beta)}}s^{\frac{n-m}{k(\alpha+\beta)}-1}\Big)\Big(\int\frac{-mu(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{n-m}{k(\alpha+\beta)}}}(s+k)^{\frac{n-m(k+1)}{k(\alpha+\beta)}-1}ds+C\Big).$$

Now, the solution $u(t)$ can be found by applying the CF inverse transform. □

In particular, if $\alpha+\beta=1,\ k=-1$ *and* $m,\ n\in\mathbb{N}\bigcup\{0\}$ *in (6.4), then we obtain the solution of the differential equation*

$$tu^{''}(t)+(m+1-t)u^{'}(t)+(n-m)u(t)=0 \tag{6.5}$$

as

$$u(t)=L^{-1}\left\{(s-1)^n s^{m-n-1}\Big(\int\frac{-mu(0)}{s^{m-n}}(s-1)^{-n-1}ds+C\Big)\right\}.$$

On the other hand, we can find the Laguerre dependent function using equation (6.5):

$$\psi_n^m(t)=\frac{d^m}{dt^m}\psi_n(t)=\frac{d^m}{dt^m}\Big(e^t\frac{d^n}{dt^n}(t^ne^{-t})\Big)=L^{-1}\Big\{(s-1)^n s^{m-n-1}\Big(\int\frac{-mu(0)}{s^{m-n}}(s-1)^{-n-1}ds+C\Big)\Big\}.$$

For example, letting m=0 in (6.5) we find that the solution of the Laguerre differential equation

$$tu^{''}(t)+(1-t)u^{'}(t)+nu(t)=0$$

is given by

$$u(t)=L^{-1}\Big\{C\frac{(s-1)^n}{s^{n+1}}\Big\}.$$

So, we obtain the Laguerre polynomial of degree n*, that is,*

$$\psi_n(t)=L^{-1}\Big\{C\frac{(s-1)^n}{s^{n+1}}\Big\}.$$

7. Conclusion

The conformable fractional derivative is a new kind of fractional derivative that still needs to be further investigated. We discussed the fractional Laplace transform, as a transform compatible with this type of fractional derivatives. The conformable fractional derivative behaves well in the product rule and the chain rule, while complicated formulas appear in case of the usual fractional calculus. Some new results were reported, which were shown to be useful in the theory of conformable fractional differential equations. The method of fractional Laplace transform, as a powerful approach for extracting the exact solutions of differential equations, was developed for the conformable time fractional differential equations. Using the fractional Laplace transform, we were able to convert some ordinary differential equations and conformable fractional differential equations into first-order ordinary differential equations.

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