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Globally unsolvability of integro-differential diffusion equation and system with exponential nonlinearities

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Abstract: In this paper, the Cauchy problem for an integro-differential diffusion equation and a system with nonlocal nonlinear sources are considered. The results on the existence of local integral solutions and the nonexistence of global weak solutions to the nonlinear integro-differential diffusion equation and system are presented.

Key words: Local existence, blow-up, global weak solution, integro-differential diffusion system

1. Introduction and statement of the problem

In this paper, we consider the integro-differential diffusion equation with the Cauchy data

$$\begin{cases} u_t(x,t) - \Delta D^{1-\alpha}_{0|t} u(x,t) = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} e^{u(x,s)} ds, \ x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 1,\ 0 < \alpha < 1,\ D^{1-\alpha}_{0|t}$ is the left-handed Riemann–Liouville fractional derivative of order $1-\alpha$.

Moreover, we consider local existence and global nonexistence of the solution for the integro-differential diffusion system

$$\begin{cases} u_t(x,t) - \Delta D^{1-\alpha}_{0|t} u(x,t) = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} e^{v(x,s)} ds, \\ v_t(x,t) - \Delta D^{1-\beta}_{0|t} v(x,t) = \dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_0^t (t-s)^{-\delta} e^{u(x,s)} ds, \end{cases} \tag{1.2}$$

for $(x,t) \in \mathbb{R}^N \times (0,T)$, subject to the initial conditions

$$u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0,\ x \in \mathbb{R}^N, \tag{1.3}$$

where $\alpha, \beta, \gamma, \delta \in (0,1)$.

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Before we present and prove our results, let us dwell on existing literature related to our problems. The Cauchy problem for the semilinear parabolic equation with a nonlinear memory (for $0 < \gamma < 1$)

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |u|^{p-1} u(x,s) ds, \; x \in \mathbb{R}^N, \; t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

has been considered by Cazenave et al. [6].

Then, Fino and Kirane in [7] and [8] studied the semilinear parabolic equation:

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |u|^{p-1} u(x,s) ds, \; x \in \mathbb{R}^N, \, t > 0, \\ u(x,0) = u_0(x), \; x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

and system

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |v|^{p-1} v(x,s) ds, \; x \in \mathbb{R}^N, \, t > 0, \\ v_t(x,t) - \Delta v(x,t) = \dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_0^t (t-s)^{-\delta} |u|^{p-1} u(x,s) ds, \; x \in \mathbb{R}^N, \, t > 0, \\ u(x,0) = u_0(x), \; v(x,0) = v_0(x), \; x \in \mathbb{R}^N. \end{cases} \tag{1.6}$$

In addition, Ahmad et al. [1] considered the semilinear nonlocal in time and space reaction diffusion equation:

$$\begin{cases} u_t(x,t) + (-\Delta)^{\beta/2} u(x,t) = I_{0|t}^{1-\gamma}[e^{u(x,t)}], \; x \in \mathbb{R}^N, \, t > 0, \\ u(x,0) = u_0(x), \; x \in \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $I_{0|t}^{1-\gamma}[e^{u(x,t)}]$ is the Riemann–Liouville fractional integral of order $1-\gamma \in (0,1)$ of $e^{u(x,t)}$.

Moreover, the system of (1.7) was investigated by Ahmad et al. [2] in the following form:

$$\begin{cases} u_t(x,t) + (-\Delta)^{\beta/2} u(x,t) = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} e^{v(x,s)} ds, \; x \in \mathbb{R}^N, \, t > 0, \\ v_t(x,t) + (-\Delta)^{\beta/2} v(x,t) = \dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_0^t (t-s)^{-\delta} e^{u(x,s)} ds, \; x \in \mathbb{R}^N, \, t > 0, \end{cases} \tag{1.8}$$

supplemented with the initial conditions

$$u(x,0) = u_0(x), \; v(x,0) = v_0(x), \; x \in \mathbb{R}^N, \tag{1.9}$$

where $u_0(x), v_0(x) \in C_0(\mathbb{R}^N)$, $0 < \beta \le 2$, $\gamma, \delta \in (0,1)$.

Recently, Kirane et al. in [4] treated the time-space fractional evolution equation with a time nonlocal nonlinearity of exponential growth

$$\begin{cases} \mathcal{D}^{\alpha}_{0|t}u(x,t) + (-\Delta)^{\beta/2}u(x,t) = I^{1-\alpha}_{0|t}[e^{u(x,t)}],\ x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^N, \end{cases} \tag{1.10}$$

where $N \geq 1,\ 0 < \alpha < 1,\ 0 < \beta \leq 2,\ \mathcal{D}^{\alpha}_{0|t}$ is the left-handed Caputo fractional derivative operator of order α.

Later on, Alsaedi et al. [3] studied the system of (1.10)

$$\begin{cases} \mathcal{D}^{\alpha_1}_{0|t}u_t(x,t) + (-\Delta)^{\beta/2}u(x,t) = I^{1-\alpha_1}_{0|t}(e^{v}),\ x \in \mathbb{R}^N,\ t > 0, \\ \mathcal{D}^{\alpha_2}_{0|t}v_t(x,t) + (-\Delta)^{\beta/2}v(x,t) = I^{1-\alpha_2}_{0|t}(e^{u}),\ x \in \mathbb{R}^N,\ t > 0, \end{cases} \tag{1.11}$$

subject to the initial conditions

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ x \in \mathbb{R}^N, \tag{1.12}$$

where $\alpha_1, \alpha_2 \in (0,1)$.

After that, in [5] the integro-differential diffusion system

$$\begin{cases} u_t(x,t) - \Delta D^{1-\alpha}_{0+,t}u(x,t) = |x|^{\rho_1}t^{\sigma_1}v^p(x,t),\ x \in \mathbb{R}^N,\ t > 0, \\ v_t(x,t) - \Delta D^{1-\beta}_{0+,t}v(x,t) = |x|^{\rho_2}t^{\sigma_2}u^q(x,t),\ x \in \mathbb{R}^N,\ t > 0, \end{cases} \tag{1.13}$$

with Cauchy data

$$u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0 \text{ for } x \in \mathbb{R}^N, \tag{1.14}$$

where $0 < \alpha, \beta < 1$ and $\sigma_1, \sigma_2 > -1,\ \rho_1, \rho_2 \geq 0$ was considered.

The here above cited papers treated the local existence, the blowing-up solutions and the blow-up profile of exploding solutions.

2. Preliminaries

Definition 2.1 *The left and right Riemann–Liouville fractional integrals* $I^{\alpha}_{0|t}f(t)$ *and* $I^{\alpha}_{t|T}f(t)$ *of order* $\alpha \in \mathbb{R}\,(\alpha > 0)$, *for all* $f(t) \in L^q(0,T))(1 \leq q \leq \infty)$, *we defined as [11, p. 69]*

$$I^{\alpha}_{0|t}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds, \quad a.e.\ \ t \in (0,T],$$

and

$$I^{\alpha}_{t|T}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}f(s)ds, \quad a.e.\ \ t \in [0,T),$$

respectively.

Definition 2.2 *If $f(t) \in C^1([0,T])$ the left-handed and right-handed Riemann–Liouville fractional derivatives $D^{\alpha}_{0|t}f(t)$ and $D^{\alpha}_{t|T}f(t)$ of order $\alpha \in (0,1)$ are defined by [11, p.70]*

$$D^{\alpha}_{0|t}f(t) = \frac{d}{dt}I^{1-\alpha}_{0|t}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_0^t (t-s)^{-\alpha}f(s)\,ds,\ \ t \in (0,T],$$

and

$$D^{\alpha}_{t|T}f(t) = -\frac{d}{dt}I^{1-\alpha}_{t|T}f(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_t^T (s-t)^{-\alpha}f(s)\,ds,\ \ t \in [0,T).$$

Definition 2.3 *The left and right Caputo fractional derivatives of order $\alpha \in \mathbb{R}$ $(0<\alpha<1)$, for $f \in C^1([0,T])$ are defined, respectively, by [11, p. 91]*

$$\mathcal{D}^{\alpha}_{0|t}f(t) = I^{1-\alpha}_{0|t}\frac{d}{dt}f(t) = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t (t-s)^{-\alpha}f'(s)\,ds,\ \ \forall t \in (0,T],$$

and

$$\mathcal{D}^{\alpha}_{t|T}f(t) = -I^{1-\alpha}_{t|T}\frac{d}{dt}f(t) = -\frac{1}{\Gamma(1-\alpha)}\int\limits_t^T (s-t)^{-\alpha}f'(s)\,ds,\ \ \forall t \in [0,T).$$

Lemma 2.4 *Let $\alpha > 0$, $p \geq 1$, $q \geq 1$ and $1/p + 1/q \leq 1+\alpha$. If $f(t) \in L^p(0,T)$, $g(t) \in L^q(0,T)$, then we have the formula of integration by parts (see [11], Lemma 2.7. p. 76)*

$$\int\limits_0^T I^{\alpha}_{0|t}[f(t)]g(t)dt = \int\limits_0^T f(t)I^{\alpha}_{t|T}[g(t)]dt.$$

Lemma 2.5 *[10] For any $p \geq 0$ and any $f \in C^2(\mathbb{R}^N)$ the pointwise inequality*

$$(p+1)f^p(x)(-\Delta)f(x) \geq (-\Delta)f^{p+1}(x),\ x \in \mathbb{R}^N,$$

holds.

Later on, we will use the following results (see [9]):

Suppose, $\varphi_2(t) = \left(1-\frac{t}{T}\right)^{\eta}$ and $t \geq 0,\ T>0,\ \eta>1,$ then

$$I^{1-\gamma}_{t|T}\varphi_2(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}T^{1-\gamma}\left(1-\frac{t}{T}\right)^{\eta-\gamma+1}, \tag{2.1}$$

$$\mathcal{D}^{1-\alpha}_{t|T}\varphi_2(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha)}T^{\alpha-1}\left(1-\frac{t}{T}\right)^{\eta+\alpha-1}, \tag{2.2}$$

for all $\alpha, \gamma \in (0,1)$.

Definition 2.6 *The Mittag–Leffler function is defined by [11, p. 42]*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ z\geq 0,$$

where $\beta\in\mathbb{R}$ *is an arbitrary constant.*

3. Fractional diffusion equation with exponential nonlinearity

3.1. Local existence of integral solution

Definition 3.1 (Integral solution) *Let* $u_0\in L^\infty(\mathbb{R}^N)$ *and* $T>0$. *We say that* $u\in L^\infty(\mathbb{R}^N;C[0,T])$ *is an integral solution (see [13, p. 78]) of (1.1) if* u *satisfies the following integral equation*

$$u\left(x,t\right)=\int_{\mathbb{R}^N}G\left(x-y,t\right)u_0\left(y\right)dy+\int_0^t\int_{\mathbb{R}^N}G\left(x-y,t-\tau\right)I_{0|s}^{1-\gamma}(e^u)dyd\tau, \tag{3.1}$$

for $t\in[0,T)$, $x\in\mathbb{R}^N$, *where*

$$G\left(x,t\right)=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}e^{-i<x,\xi>}E_{\alpha,1}\left(-\xi^2t^\alpha\right)d\xi. \tag{3.2}$$

Theorem 3.2 (Local existence) *For given* $u_0(x)\in L^\infty(\mathbb{R}^N)$, *there exists a maximal time* $T>0$ *such that Equation (1.1) has a unique integral solution* $u(x,t)\in L^\infty(\mathbb{R}^N;C[0,T))$.

Furthermore, either $T=\infty$ *or* $T<\infty$ *and* $\|u(t)\|_{L^\infty(\mathbb{R}^N)}\to\infty,$ *as* $t\to T$.

Proof. For arbitrary $T>0$, we define the ball

$$B_T=\{u\in L^\infty(\mathbb{R}^N;C[0,T));||u||_1\leq 2\parallel u_0\parallel_{L^\infty(\mathbb{R}^N)}\}, \tag{3.3}$$

where $\|\cdot\|_1=\|\cdot\|_{L^\infty(\mathbb{R}^N,L^\infty(0,T))}$.

Next, for every $u(x,t)\in B_T$, we define the map

$$\begin{aligned}\Psi(u)=&\int_{\mathbb{R}^N}G\left(x-y,t\right)u_0\left(y\right)dy\\&+\int_0^t\int_{\mathbb{R}^N}G\left(x-y,t-\tau\right)I_{0|s}^{1-\gamma}(e^u)dyd\tau,\quad t\in[0,T).\end{aligned}$$

The existence of a local solution will be proved as a fixed point of $\Psi(u)$ using the Banach fixed point theorem.

• $\Psi:B_T\to B_T$. According to Lemma 2.2 in [5], it holds

$$\begin{aligned}||\Psi(u)||_{B_T}&\leq\|u_0\|_\infty+\frac{1}{\Gamma(1-\gamma)}\left\|\int_0^t\int_0^s(s-\tau)^{-\gamma}\|e^u\|_\infty d\tau ds\right\|_{L^\infty(0,T)}\\&\leq\|u_0\|_\infty+\frac{1}{\Gamma(1-\gamma)}\left\|\int_0^t\int_\tau^t(s-\tau)^{-\gamma}\|e^u\|_\infty dsd\tau\right\|_{L^\infty(0,T)}\\&\leq\|u_0\|_\infty+CT^{2-\gamma}e^{\|u\|_1}\\&\leq\|u_0\|_\infty+CT^{2-\gamma}e^{2\|u_0\|_\infty},\end{aligned}$$

where

$$C := \frac{1}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} = \frac{1}{\Gamma(3-\gamma)}.$$

Now, we choose T so that

$$CT^{2-\gamma}e^{2\|u_0\|_\infty} \leq \|u_0\|_\infty,$$

to conclude that $\Psi(u) \in B_T$.

- Ψ is a contraction. For $(u,v) \in B_T$, we have

$$\begin{aligned}
||\Psi(u)-\Psi(v)||_{B_T} &\leq \frac{1}{\Gamma(1-\gamma)}\left\|\int_0^t\int_0^s (s-\tau)^{-\gamma}(e^{u(\tau)}-e^{v(\tau)})d\tau ds\right\|_{L^\infty(0,T)} \\
&\leq \frac{1}{\Gamma(1-\gamma)}\left\|\int_0^t\int_0^s (s-\tau)^{-\gamma}\|e^{u(\tau)}-e^{v(\tau)}\|_\infty d\tau ds\right\|_{L^\infty(0,T)} \\
&\leq CT^{2-\gamma}\|e^{u(s)}-e^{v(s)}\|_1 \\
&\leq CT^{2-\gamma}e^{2\|u_0\|_\infty}\|u-v\|_1 \\
&\leq \frac{1}{2}\|u-v\|_1,
\end{aligned}$$

thanks to

$$|e^{u(s)}-e^{v(s)}| = e^{\lambda u(s)+\mu v(s)}|u(s)-v(s)|, \;\; 0<\lambda,\mu<1, \;\; \lambda+\mu=1,$$

whenever T is chosen such that

$$CT^{2-\gamma}e^{2\|u_0\|_\infty} \leq \frac{1}{2}.$$

Then, Ψ is contractive on B_T. According to the Banach fixed point theorem, Equation (1.1) admits a unique integral solution $u(x,t) \in B_T$.

3.2. Global nonexistence of weak solution

Definition 3.3 (Weak solution) *Let $u_0(x) \in L^\infty_{loc}(\mathbb{R}^N)$ and $T>0$. We say that $u(x,t) \in L^\infty_{loc}(\mathbb{R}^N; L^p(0,T))$ is a weak solution of Equation (1.1) if it satisfies*

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} u_0(x)\varphi(x,0)dx + \int_0^T\int_{\mathbb{R}^N} I^{1-\gamma}_{0|t}[e^u]\varphi(x,t)dxdt \\
&= -\int_0^T\int_{\mathbb{R}^N} u(x,t)\Delta\mathcal{D}^{1-\alpha}_{t|T}\varphi(x,t)dxdt - \int_0^T\int_{\mathbb{R}^N} u(x,t)\varphi_t(x,t)dxdt,
\end{aligned} \tag{3.4}$$

for all compactly supported function $\varphi(x,t) \in C^2(\mathbb{R}^N; C^1[0,T])$, such that $\varphi(x,T)=0$.

Lemma 3.4 (Integral $\to$ Weak solution) *Assume $u_0(x) \in L^\infty(\mathbb{R}^N)$, let $u \in L^\infty(\mathbb{R}^N; C[0,T))$ be an integral solution of (1.1), then u is also a weak solution of (1.1).*

For the proof of Lemma 3.4, see [8, Lemma 4.2].

Theorem 3.5 *Let $u_0(x) \in L^\infty(\mathbb{R}^N)$ with $u_0(x) \geq 0$ and $u_0(x) \not\equiv 0$. Then problem (1.1) does not admit a global solution.*

Proof The proof is by contradiction. Assume that $u(x,t)$ is a global integral solution of (1.1), then u is a solution of (1.1) where $u(x,t) \in L^\infty(\mathbb{R}^N; C[0;T))$.

Let us choose as in [4]

$$\varphi(x,t) := \varphi_1^l(x)\varphi_2(t),$$

with

$$\varphi_1(x) := \Phi\left(\frac{|x|}{T^{\alpha/2}}\right),$$

and

$$\varphi_2(t) := \begin{cases} \left(1 - \dfrac{t}{T}\right)^\eta, \ t \leq T, \ \eta > 1, \\ 0, \ t > T. \end{cases}$$

The function Φ is the smooth nonnegative function

$$\Phi(x) = \begin{cases} 1 & \text{if } x \leq 1, \\ \searrow & \text{if } 1 < x \leq 2, \\ 0 & \text{if } x > 2. \end{cases}$$

According to Definition 3.3 and using Lemma 2.4 for set $\{x \in \mathbb{R}^N; |x| \leq 2T^{\alpha/2}\} \times [0,T]$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} u_0\varphi_1^l(x)dx + \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} I_{t|T}^{1-\gamma}\varphi(x,t)dxdt \\ &= -\int_0^T \int_{\mathbb{R}^N} u\Delta\mathcal{D}_{t|T}^{1-\alpha}\varphi(x,t)dxdt - \int_0^T \int_{\mathbb{R}^N} u\varphi_t(x,t)dxdt. \end{aligned}$$

Moreover, a use of Lemma 2.5 allows us to obtain, as $u > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^N} u_0\varphi_1^l(x)dx + \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} I_{t|T}^{1-\gamma}\varphi(x,t)dxdt \\ &\leq l\int_0^T \int_{\mathbb{R}^N} u|\varphi_1^{l-1}(x)(-\Delta)\varphi_1(x)\mathcal{D}_{t|T}^{1-\alpha}\varphi_2(t)dxdt \\ &- \int_0^T \int_{\mathbb{R}^N} u\varphi_1^l(x)\frac{d}{dt}\varphi_2(t)dxdt \\ &\leq l\int_0^T \int_{\mathbb{R}^N} u\Big|(-\Delta)\varphi_1(x)\Big|\mathcal{D}_{t|T}^{1-\alpha}\varphi_2(t)dxdt + \int_0^T \int_{\mathbb{R}^N} u\left|\frac{d}{dt}\varphi_2(t)\right|dxdt \\ &\leq lM + K, \end{aligned} \tag{3.5}$$

where

$$M := \int_0^T \int_{\mathbb{R}^N} u\Big|(-\Delta)\varphi_1(x)\Big|\mathcal{D}_{t|T}^{1-\alpha}\varphi_2(t)dxdt,$$

$$K := \int_0^T \int_{\mathbb{R}^N} u \left| \frac{d}{dt}\varphi_2(t) \right| dxdt.$$

Then, using Young's inequality

$$AB \le \varepsilon e^A + B\ln\frac{B}{\varepsilon e}, \quad \text{for } A, B > 0,\ \varepsilon > 0,$$

with $\varepsilon = \frac{1}{4l}\varphi(x,t), A = u(x,t)$ and $B = |(-\Delta)\varphi_1(x)|\mathcal{D}^{1-\alpha}_{t|T}\varphi_2(t)$ for M, we obtain

$$\begin{aligned} M \le & \int_0^T \int_{\mathbb{R}^N} \left|(-\Delta)\varphi_1(x)\right| \mathcal{D}^{1-\alpha}_{t|T}\varphi_2(t) \ln\left(\frac{4l|(-\Delta)\varphi_1(x)|\mathcal{D}^{1-\alpha}_{t|T}\varphi_2(t)}{e\varphi(x,t)} \right) dxdt \\ & + \frac{1}{4l}\int_0^T \int_{\mathbb{R}^N} e^{u(x,t)}\varphi(x,t)dxdt. \end{aligned}$$

Similarly, for K with $\varepsilon = \frac{1}{4}\varphi(x,t), A = u(x,t)$ and $B = \left|\frac{d}{dt}\varphi_2(t)\right|$, we have

$$K \le \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt}\varphi_2(t) \right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\varphi(x,t)} \right) dxdt + \frac{1}{4}\int_0^T \int_{\mathbb{R}^N} e^{u(x,t)}\varphi(x,t)dxdt.$$

Furthermore, using (2.2) we may write

$$\begin{aligned} M \le\ & lC_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\alpha-1}\left(1 - \frac{t}{T}\right)^{\eta+\alpha-1} \\ & \times \ln\left(\frac{4lC_1|(-\Delta)\varphi_1(x)|T^{\alpha-1}\left(1-\frac{t}{T}\right)^{\eta+\alpha-1}}{e\varphi_1^l(x)(1-\frac{t}{T})^\eta} \right) dxdt \\ & + \frac{1}{4}\int_0^T \int_{\mathbb{R}^N} e^{u(x,t)}\varphi(x,t)dxdt, \end{aligned}$$

and

$$\begin{aligned} K \le & \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt}\varphi_2(t) \right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\varphi_1^l(x)(1-\frac{t}{T})^\eta} \right) dxdt \\ & + \frac{1}{4}\int_0^T \int_{\mathbb{R}^N} e^{u(x,t)}\varphi(x,t)dxdt, \end{aligned}$$

where $C_1 := \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha)}$.

Consequently, according to (2.1) we finally obtain the estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^N} u_0 \varphi_1^l(x)dx + C_2 \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \varphi_1^l(x) T^{1-\gamma} \left(1-\frac{t}{T}\right)^{\eta-\gamma+1} dxdt \\
&\leq lC_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1-\frac{t}{T}\right)^{\eta+\alpha-1} \\
&\times \ln\left(\frac{4lC_1 |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1-\frac{t}{T}\right)^{\eta+\alpha-1}}{e\varphi_1^l(x)(1-\frac{t}{T})^\eta} \right) dxdt \\
&+ \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt}\varphi_2(t) \right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\varphi_1^l(x)(1-\frac{t}{T})^\eta} \right) dxdt \\
&+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \varphi(x,t) dxdt,
\end{aligned} \tag{3.6}
$$

where $C_2 := \dfrac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}$.

At this stage, let us introduce the scaled variables $\tau = \dfrac{t}{T}$ and $y = \dfrac{x}{T^{\alpha/2}}$, it follows that

$$dxdt = T^{\frac{\alpha N}{2}+1} dy d\tau,$$

$$\left| \frac{d}{dt}\varphi_2(t) \right| = \eta T^{-1}(1-\tau)^{\eta-1},$$

$$(-\Delta)\varphi_1(x) = T^{-\alpha}(-\Delta_y)\varphi_1(y).$$

Then, for $\{y \in \mathbb{R}^N,\ |y| \leq 2\} \times [0,1]$, the inequality (3.6) can be rewritten as

$$
\begin{aligned}
&\int_{\mathbb{R}^N} u_0 \varphi_1^l(x)dx + \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \varphi_1^l(x,t) \left[C_2 T^{1-\gamma} \left(1-\frac{t}{T}\right)^{1-\gamma} - \frac{1}{2} \right] dxdt \\
&\leq lC_1 T^{\frac{\alpha N}{2}} \int_0^1 \int_{\mathbb{R}^N} |(-\Delta_y)\varphi_1(y)| (1-\tau)^{\eta+\alpha-1} \\
&\times \ln\left(\frac{4lC_1 T^{\alpha-1} |(-\Delta)\varphi_1(y)| (1-\tau)^{\eta+\alpha-1}}{e\varphi_1^l(y)(1-\tau)^\eta} \right) dy d\tau \\
&+ \eta T^{\frac{\alpha N}{2}} \int_0^1 \int_{\mathbb{R}^N} (1-\tau)^{\eta-1} \ln\left(\frac{4\eta T^{-1}(1-\tau)^{\eta-1}}{e\varphi_1^l(y)(1-\tau)^\eta} \right) dy d\tau.
\end{aligned}
$$

Thus, we have two bounded functions $\varphi_1(y)$ and $(-\Delta)\varphi_1(y)$, also $\varphi_1(y) \to 1$ as $T \to \infty$. Using Lebesgue's dominated convergence theorem, we deduce that the right-hand side diverges to $-\infty$, while the left-hand side is positive when $T \to +\infty$.

This leads to a contradiction. $\square$

4. Fractional diffusion system with nonlinearities of exponential growth

In this section, we study the system:

$$\begin{cases} u_t(x,t) - D_{0|t}^{1-\alpha}\Delta u(x,t) = \dfrac{1}{\Gamma(1-\gamma)}\displaystyle\int_0^t (t-s)^{-\gamma} e^{v(x,s)} ds, \\ v_t(x,t) - D_{0|t}^{1-\beta}\Delta v(x,t) = \dfrac{1}{\Gamma(1-\delta)}\displaystyle\int_0^t (t-s)^{-\delta} e^{u(x,s)} ds, \end{cases} \tag{4.1}$$

for $x \in \mathbb{R}^N$, $t > 0$, subject to the initial conditions

$$u(x,0) = u_0(x) \geq 0, \;\; v(x,0) = v_0(x) \geq 0, \;\; x \in \mathbb{R}^N, \tag{4.2}$$

where $\alpha, \beta, \gamma, \delta \in (0,1)$.

4.1. Local existence of integral solution

Definition 4.1 (Integral solution) *Let $u_0(x), v_0(x) \in L^\infty(\mathbb{R}^N)$. We say that $(u,v) \in L^\infty(\mathbb{R}^N; C[0,T]) \times L^\infty(\mathbb{R}^N; C[0,T])$ is an integral solution of the system (4.1)–(4.2) if (u,v) satisfies:*

$$\begin{cases} u(x,t) = \displaystyle\int_{\mathbb{R}^N} G(x-y,t) u_0(y)\, dy + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-\tau) I_{0|s}^{1-\gamma}(e^v) dy d\tau, \\ v(x,t) = \displaystyle\int_{\mathbb{R}^N} G(x-y,t) v_0(y)\, dy + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-\tau) I_{0|s}^{1-\delta}(e^u) dy d\tau, \end{cases} \tag{4.3}$$

for $x \in \mathbb{R}^N$, $t \in [0,T)$, where

$$G(x,t) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i<x,\xi>} E_{\alpha,1}\left(-\xi^2 t^\alpha\right) d\xi \tag{4.4}$$

is a heat kernel of the problem (4.1)–(4.2) [5].

Theorem 4.2 (Local existence) *Given $u_0(x), v_0(x) \in L^\infty(\mathbb{R}^N)$. Then, there exists a maximal time $T > 0$ such that the system (4.1)–(4.2) has a unique integral solution $(u,v) \in L^\infty(\mathbb{R}^N; C[0,T)) \times L^\infty(\mathbb{R}^N; C[0,T))$.*

Proof. For arbitrary $T > 0$, we define the ball

$$B_T = \{(u,v) \in L^\infty(\mathbb{R}^N; C[0,T)) \times L^\infty(\mathbb{R}^N; C[0,T));$$
$$||(u,v)||_{B_T} \leq 2(\| u_0 \|_{L^\infty(\mathbb{R}^N)} + \| v_0 \|_{L^\infty(\mathbb{R}^N)})\},$$

where $\| \cdot \|_\infty = \| \cdot \|_{L^\infty(\mathbb{R}^N)}$ and $|| \cdot ||_{B_T}$ is the norm of B_T defined by

$$||(u,v)||_{B_T} = \| u \|_1 + \| v \|_1 = \| u \|_{L^\infty(\mathbb{R}^N, L^\infty(0,T))} + \| v \|_{L^\infty(\mathbb{R}^N, L^\infty(0,T))},$$

and

$$d(u,v) = \max_{t \in [0,T)} ||u(t) - v(t)||_{L^\infty(\mathbb{R}^N)} \;\text{ for }\; u, v \in B_T.$$

Since $L^\infty(\mathbb{R}^N; C[0,T))$ is a Banach space, $(B_T; d)$ is a complete metric space.

Next, for every $(u,v) \in B_T$, we introduce the map $\Psi(u,v)$ defined on B_T by

$$\Psi(u,v) := (\Psi_1(u,v), \Psi_2(u,v)),$$

where

$$\begin{aligned}\Psi_1(u,v) &= \int_{\mathbb{R}^N} G\,(x-y,t)u_0\,(y)\,dy \\ &+ \int_0^t \int_{\mathbb{R}^N} G\,(x-y,t-\tau) I_{0|s}^{1-\gamma}[e^v] dy d\tau,\ t \in [0,T),\end{aligned}$$

and

$$\begin{aligned}\Psi_2(u,v) &= \int_{\mathbb{R}^N} G\,(x-y,t)v_0\,(y)\,dy \\ &+ \int_0^t \int_{\mathbb{R}^N} G\,(x-y,t-\tau) I_{0|s}^{1-\delta}[e^u] dy d\tau,\ t \in [0,T).\end{aligned}$$

We are going to prove the existence of the local solution as a fixed point of Ψ via the Banach fixed point theorem.

- $\Psi : B_T \to B_T$.

If $(u,v) \in B_T$, using Lemma 2.2 in [5], we obtain

$$\begin{aligned}||\Psi(u,v)||_{B_T} &\le \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\tau)^{-\gamma} \|e^v\|_\infty d\tau ds \right\|_{L^\infty(0,T)} \\ &+ \|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_0^s (s-\tau)^{-\delta} \|e^u\|_\infty d\tau ds \right\|_{L^\infty(0,T)} \\ &\le \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\gamma} \|e^v\|_\infty ds d\tau \right\|_{L^\infty(0,T)} \\ &+ \|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\delta} \|e^u\|_\infty ds d\tau \right\|_{L^\infty(0,T)} \\ &\le \|u_0\|_\infty + \|v_0\|_\infty + \mathcal{C}_1 T^{2-\gamma} e^{\|v\|_1} + \mathcal{C}_2 T^{2-\delta} e^{\|u\|_1},\end{aligned}$$

where

$$\begin{aligned}\mathcal{C}_1 &:= \frac{1}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} = \frac{1}{\Gamma(3-\gamma)}, \\ \mathcal{C}_2 &:= \frac{1}{(1-\delta)(2-\delta)\Gamma(1-\delta)} = \frac{1}{\Gamma(3-\delta)}.\end{aligned}$$

Hence, using the fact that $(u,v) \in B_T$, we get

$$\begin{aligned}||\Psi(u,v)||_{B_T} &\le \|u_0\|_\infty + \|v_0\|_\infty + \mathcal{C}_1 T^{2-\gamma} e^{\|v\|_1} + T^{2-\delta} e^{\|u\|_1} \\ &\le (\|u_0\|_\infty + \|v_0\|_\infty) + \max\{\mathcal{C}_1 T^{2-\gamma} e^{\|v\|_1}; \mathcal{C}_2 T^{2-\delta} e^{\|u\|_1}\} \\ &\le (\|u_0\|_\infty + \|v_0\|_\infty) + T_{\gamma,\delta} e^{2(\|u_0\|_\infty + \|v_0\|_\infty)},\end{aligned}$$

where

$$T_{\gamma,\delta} = \max\{\mathcal{C}_1 T^{2-\gamma}; \mathcal{C}_2 T^{2-\delta}\}.$$

If we choose $T_{\gamma,\delta}$ such that

$$2T_{\gamma,\delta} \leq 1,$$

we conclude that $\|\Psi(u,v)\|_{B_T} \leq 2(\|u_0\|_\infty + \|v_0\|_\infty)$. Hence $\Psi(u,v) \in B_T$.

- Ψ is a contraction.

For $(u,v),(\tilde{u},\tilde{v}) \in B_T$, by the same computations as above, we have

$$\begin{aligned}
&||\Psi(u,v) - \Psi(\tilde{u},\tilde{v})||_{B_T} \\
&\quad \leq \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\tau)^{-\gamma} \|e^v - e^{\tilde{v}}\|_\infty d\tau ds \right\|_{L^\infty(0,T)} \\
&\quad + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_0^s (s-\tau)^{-\delta} \|e^u - e^{\tilde{u}}\|_\infty d\tau ds \right\|_{L^\infty(0,T)} \\
&\quad = \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\gamma} \|e^v - e^{\tilde{v}}\|_\infty ds d\tau \right\|_{L^\infty(0,T)} \\
&\quad + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\delta} \|e^u - e^{\tilde{u}}\|_\infty ds d\tau \right\|_{L^\infty(0,T)} \\
&\quad = \mathcal{C}_1 T^{2-\gamma} \|e^v - e^{\tilde{v}}\|_1 + \mathcal{C}_2 T^{2-\delta} \|e^u - e^{\tilde{u}}\|_1.
\end{aligned}$$

Now, thanks to

$$|e^{u(s)} - e^{v(s)}| = e^{\lambda u(s) + \mu v(s)} |u(s) - v(s)|, \;\; 0 < \lambda, \mu < 1, \quad \lambda + \mu = 1,$$

we have the estimate

$$\begin{aligned}
||\Psi(u,v) - \Psi(\tilde{u},\tilde{v})||_{B_T} &\leq \mathcal{C}_2 T^{2-\delta} \|e^u - e^{\tilde{u}}\|_1 + \mathcal{C}_1 T^{2-\gamma} \|e^v - e^{\tilde{v}}\|_1 \\
&\leq \mathcal{C}_1 T^{2-\gamma} e^{2(\|u_0\|_\infty + \|v_0\|_\infty)} \|v - \tilde{v}\|_1 \\
&+ \mathcal{C}_2 T^{2-\delta} e^{2(\|u_0\|_\infty + \|v_0\|_\infty)} \|u - \tilde{u}\|_1 \\
&\leq T_{\gamma,\delta} |||(u,v) - (\tilde{u},\tilde{v})||| \\
&\leq \frac{1}{2} |||(u,v) - (\tilde{u},\tilde{v})|||,
\end{aligned}$$

and the choice of $T_{\gamma,\delta}$

$$T_{\gamma,\delta} \leq \frac{1}{2}.$$

Hence, by the Banach fixed point theorem, the system (4.1)–(4.2) has an integral solution $(u,v) \in B_T$.

4.2. Global nonexistence of weak solution

Definition 4.3 (Weak solution) *Let* $u_0(x)$, $v_0(x) \in L^\infty_{loc}(\mathbb{R}^N)$ *and* $T > 0$. *Then, we say that* $(u,v) \in L^\infty_{loc}(\mathbb{R}^N; L^p(0,T)) \times L^\infty_{loc}(\mathbb{R}^N; L^p(0,T))$ *is a weak solution of the system* (4.1)-(4.2) *if* (u,v) *satisfies*

$$\begin{aligned}&\int\limits_{\mathbb{R}^N} u_0\varphi(x,0)dx + \int_0^T\int_{\mathbb{R}^N} I^{1-\gamma}_{0|t}[e^v]\varphi(x,t)dxdt \\ &= -\int_0^T\int_{\mathbb{R}^N} u\Delta\mathcal{D}^{1-\alpha}_{t|T}\varphi(x,t)dxdt - \int_0^T\int_{\mathbb{R}^N} u\varphi_t(x,t)dxdt,\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&\int\limits_{\mathbb{R}^N} v_0\varphi(x,0)dx + \int_0^T\int_{\mathbb{R}^N} I^{1-\delta}_{0|t}[e^u]\varphi(x,t)dxdt \\ &= -\int_0^T\int_{\mathbb{R}^N} v\Delta\mathcal{D}^{1-\beta}_{t|T}\varphi(x,t)dxdt - \int_0^T\int_{\mathbb{R}^N} v\varphi_t(x,t)dxdt,\end{aligned} \tag{4.6}$$

for all compactly supported function $\varphi(x,t) \in C^2(\mathbb{R}^N; C^1[0,T])$, *such that* $\varphi(x,T) = 0$.

Theorem 4.4 *Let* $u_0(x)$, $v_0(x) \in L^\infty(\mathbb{R}^N)$ *with* $u_0(x), v_0(x) \geq 0$ *and* $u_0(x), v_0(x) \not\equiv 0$. *Then, the system* (4.1)-(4.2) *does not admit global solutions.*

Proof The proof is by contradiction. Suppose that (u,v) is a global integral solution to (4.1)–(4.2), then (u,v) is a weak solution of (4.1)–(4.2) where $(u,v) \in L^\infty(\mathbb{R}^N; C[0,T)) \times L^\infty(\mathbb{R}^N; C[0,T))$.

Let us choose as in [3]

$$\varphi(x,t) := \varphi_1^l(x)\varphi_2(t),$$

with

$$\varphi_1(x) := \Phi\left(\frac{|x|}{T^{\theta/2}}\right),\ \theta = \min\{\alpha,\beta\},$$

and

$$\varphi_2(t) := \begin{cases}\left(1-\dfrac{t}{T}\right)^\eta,\ t\leq T,\ \eta>1,\\ 0,\ t>T.\end{cases}$$

The function Φ is a smooth nonnegative function such that

$$\Phi(x) = \begin{cases}1 & \text{if } x\leq 1,\\ \searrow & \text{if } 1<x\leq 2,\\ 0 & \text{if } x>2.\end{cases}$$

According to Definition 4.3 and using Lemma 2.4 we get

$$\begin{aligned}&\int_{\mathbb{R}^N} u_0\varphi_1^l(x)dx + \int_0^T\int_{\mathbb{R}^N} e^{v(x,t)}I^{1-\gamma}_{t|T}\varphi(x,t)dxdt \\ &= -\int_0^T\int_{\mathbb{R}^N} u\Delta\mathcal{D}^{1-\alpha}_{t|T}\varphi(x,t)dxdt - \int_0^T\int_{\mathbb{R}^N} u\varphi_t(x,t)dxdt,\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} v_0\varphi_1^l(x)dx+\int_0^T\int_{\mathbb{R}^N} e^{u(x,t)}I_{t|T}^{1-\delta}\varphi(x,t)dxdt\\&=-\int_0^T\int_{\mathbb{R}^N} v\Delta\mathcal{D}_{t|T}^{1-\beta}\varphi(x,t)dxdt-\int_0^T\int_{\mathbb{R}^N} v\varphi_t(x,t)dxdt.\end{aligned}\tag{4.8}$$

Furthermore, from Lemma 2.5 we obtain, as $u>0$,

$$\begin{aligned}&\int_{\mathbb{R}^N} u_0\varphi_1^l(x)dx+\int_0^T\int_{\mathbb{R}^N} e^{v(x,t)}I_{t|T}^{1-\gamma}\varphi(x,t)dxdt\\&\leq l\int_0^T\int_{\mathbb{R}^N} u\varphi_1^{l-1}(x)(-\Delta)\varphi_1(x)\mathcal{D}_{t|T}^{1-\alpha}\varphi_2(t)dxdt\\&-\int_0^T\int_{\mathbb{R}^N} u\varphi_1^l(x)\frac{d}{dt}\varphi_2(t)dxdt\\&\leq l\int_0^T\int_{\mathbb{R}^N} u\Big|(-\Delta)\varphi_1(x)\Big|\mathcal{D}_{t|T}^{1-\alpha}\varphi_2(t)dxdt+\int_0^T\int_{\mathbb{R}^N} u\Big|\frac{d}{dt}\varphi_2(t)\Big|dxdt\\&\leq lM_1+K_1,\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} u_0\varphi_1^l(x)dx+\int_0^T\int_{\mathbb{R}^N} e^{u(x,t)}I_{t|T}^{1-\delta}\varphi(x,t)dxdt\\&\leq l\int_0^T\int_{\mathbb{R}^N} v\varphi_1^{l-1}(x)(-\Delta)\varphi_1(x)\mathcal{D}_{t|T}^{1-\beta}\varphi_2(t)dxdt\\&-\int_0^T\int_{\mathbb{R}^N} v\varphi_1^l(x)\frac{d}{dt}\varphi_2(t)dxdt\\&\leq l\int_0^T\int_{\mathbb{R}^N} v\Big|(-\Delta)\varphi_1(x)\Big|\mathcal{D}_{t|T}^{1-\beta}\varphi_2(t)dxdt+\int_0^T\int_{\mathbb{R}^N} v\Big|\frac{d}{dt}\varphi_2(t)\Big|dxdt\\&\leq lM_2+K_2,\end{aligned}\tag{4.10}$$

where

$$M_1:=\int_0^T\int_{\mathbb{R}^N} u\Big|(-\Delta)\varphi_1(x)\Big|\mathcal{D}_{t|T}^{1-\alpha}\varphi_2(t)dxdt,$$

$$K_1:=\int_0^T\int_{\mathbb{R}^N} u\Big|\frac{d}{dt}\varphi_2(t)\Big|dxdt,$$

$$M_2:=\int_0^T\int_{\mathbb{R}^N} v\Big|(-\Delta)\varphi_1(x)\Big|\mathcal{D}_{t|T}^{1-\beta}\varphi_2(t)dxdt,$$

$$K_2:=\int_0^T\int_{\mathbb{R}^N} v\Big|\frac{d}{dt}\varphi_2(t)\Big|dxdt.$$

Hence, using Young's inequality

$$AB \leq \varepsilon e^{A} + B\ln\frac{B}{\varepsilon e}, \quad \text{for } A, B > 0, \ \varepsilon > 0,$$

with

$$\varepsilon = \frac{1}{4l}\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t),$$
$$A = u(x,t), B = |(-\Delta)\varphi_1(x)|\mathcal{D}^{1-\alpha}_{t|T}\varphi_2(t) \ \text{ for } \ M_1,$$

and

$$\varepsilon = \frac{1}{4l}\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t),$$
$$A = v(x,t), B = |(-\Delta)\varphi_1(x)|\mathcal{D}^{1-\beta}_{t|T}\varphi_2(t) \ \text{ for } \ M_2,$$

we obtain

$$M_1 \leq \int_0^T\int_{\mathbb{R}^N}\Big|(-\Delta)\varphi_1(x)\Big|\mathcal{D}^{1-\alpha}_{t|T}\varphi_2(t)\ln\left(\frac{4l|(-\Delta)\varphi_1(x)|\mathcal{D}^{1-\alpha}_{t|T}\varphi_2(t)}{e\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t)}\right)dxdt$$
$$+\frac{1}{4l}\int_0^T\int_{\mathbb{R}^N}e^{u(x,t)}\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t)dxdt,$$
$$M_2 \leq \int_0^T\int_{\mathbb{R}^N}\Big|(-\Delta)\varphi_1(x)\Big|\mathcal{D}^{1-\beta}_{t|T}\varphi_2(t)\ln\left(\frac{4l|(-\Delta)\varphi_1(x)|\mathcal{D}^{1-\beta}_{t|T}\varphi_2(t)}{e\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t)}\right)dxdt$$
$$+\frac{1}{4l}\int_0^T\int_{\mathbb{R}^N}e^{v(x,t)}\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t)dxdt,$$

respectively. Let

$$\varepsilon = \frac{1}{4}\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t),$$
$$A = u(x,t), B = \left|\frac{d}{dt}\varphi_2(t)\right| \ \text{ for } \ K_1,$$

and

$$\varepsilon = \frac{1}{4}\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t),$$
$$A = v(x,t), B = \left|\frac{d}{dt}\varphi_2(t)\right| \ \text{ for } \ K_2.$$

Then, for K_1 and K_2 we have the following estimates

$$K_1 \leq \int_0^T\int_{\mathbb{R}^N}\left|\frac{d}{dt}\varphi_2(t)\right|\ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t)}\right)dxdt$$
$$+\frac{1}{4}\int_0^T\int_{\mathbb{R}^N}e^{u(x,t)}\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t)dxdt,$$

and

$$K_2 \leq \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt}\varphi_2(t) \right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4}\int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t)dxdt.$$

Whereupon, (2.2) allows us to write

$$M_1 \leq l\mathcal{C}_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1-\frac{t}{T}\right)^{\eta+\alpha-1}$$

$$\times \ln\left(\frac{4l\mathcal{C}_1|(-\Delta)\varphi_1(x)|T^{\alpha-1}\left(1-\frac{t}{T}\right)^{\eta+\alpha-1}}{e\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4}\int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t)dxdt,$$

and

$$K_1 \leq \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt}\varphi_2(t) \right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4}\int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}(T-t)^{1-\delta}\varphi(x,t)dxdt,$$

where $\mathcal{C}_1 := \dfrac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha)}$.

On the other hand, we have

$$M_2 \leq l\mathcal{C}_2 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\beta-1} \left(1-\frac{t}{T}\right)^{\eta+\beta-1}$$

$$\times \ln\left(\frac{4l\mathcal{C}_2|(-\Delta)\varphi_1(x)|T^{\beta-1}\left(1-\frac{t}{T}\right)^{\eta+\beta-1}}{e\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4}\int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t)dxdt,$$

and

$$K_2 \leq \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt}\varphi_2(t) \right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4}\int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}(T-t)^{1-\gamma}\varphi(x,t)dxdt,$$

where $\mathcal{C}_2 := \dfrac{\Gamma(\eta+1)}{\Gamma(\eta+\beta)}$.

Collecting estimates and taking into account (2.1), we finally get

$$
\begin{aligned}
&\int_{\mathbb{R}^N} u_0\varphi_1^l(x)dx + \mathcal{C}_3\int_0^T\int_{\mathbb{R}^N} e^{v(x,t)}\varphi(x,t)\,(T-t)^{1-\gamma}\,dxdt\\
&\leq l\mathcal{C}_1\int_0^T\int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)|T^{\alpha-1}\left(1-\frac{t}{T}\right)^{\eta+\alpha-1}\\
&\times\ln\left(\frac{4l\mathcal{C}_1|(-\Delta)\varphi_1(x)|T^{\alpha-1}\left(1-\frac{t}{T}\right)^{\eta+\alpha-1}}{e\mathcal{C}_4(T-t)^{1-\delta}\varphi(x,t)}\right)dxdt\\
&+\int_0^T\int_{\mathbb{R}^N}\left|\frac{d}{dt}\varphi_2(t)\right|\ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\mathcal{C}_4(T-t)^{1-\delta}\varphi(x,t)}\right)dxdt\\
&+\frac{1}{2}\mathcal{C}_4\int_0^T\int_{\mathbb{R}^N} e^{u(x,t)}(T-t)^{1-\delta}\varphi(x,t)dxdt,
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} v_0\varphi_1^l(x)dx + \mathcal{C}_4\int_0^T\int_{\mathbb{R}^N} e^{u(x,t)}\,(T-t)^{1-\delta}\,\varphi(x,t)dxdt\\
&\leq l\mathcal{C}_2\int_0^T\int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)|T^{\beta-1}\left(1-\frac{t}{T}\right)^{\eta+\beta-1}\\
&\times\ln\left(\frac{4l\mathcal{C}_1|(-\Delta)\varphi_1(x)|T^{\beta-1}\left(1-\frac{t}{T}\right)^{\eta+\beta-1}}{e\mathcal{C}_3(T-t)^{1-\gamma}\varphi(x,t)}\right)dxdt\\
&+\int_0^T\int_{\mathbb{R}^N}\left|\frac{d}{dt}\varphi_2(t)\right|\ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\mathcal{C}_3(T-t)^{1-\gamma}\varphi(x,t)}\right)dxdt\\
&+\frac{1}{2}\mathcal{C}_3\int_0^T\int_{\mathbb{R}^N} e^{v(x,t)}(T-t)^{1-\gamma}\varphi(x,t)dxdt,
\end{aligned}
\tag{4.12}
$$

where

$$
\mathcal{C}_3 := \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)},\;\; \mathcal{C}_4 := \frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}.
$$

Now, combining (4.11) and (4.12), also reminding that $u_0, v_0 \geq 0$, these estimates lead to

$$\begin{aligned}
&\int_{\mathbb{R}^N} u_0 \varphi_1^l(x)dx + \frac{3}{4}\mathcal{C}_3 \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} (T-t)^{1-\gamma} \varphi(x,t)dxdt \\
&\leq l\mathcal{C}_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1-\frac{t}{T}\right)^{\eta+\alpha-1} \\
&\times \ln\left(\frac{4l\mathcal{C}_1 |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1-\frac{t}{T}\right)^{\eta+\alpha-1}}{e\mathcal{C}_4 (T-t)^{1-\delta}\varphi(x,t)}\right)dxdt \\
&+ \int_0^T \int_{\mathbb{R}^N} \left|\frac{d}{dt}\varphi_2(t)\right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\mathcal{C}_4 (T-t)^{1-\delta}\varphi(x,t)}\right)dxdt \\
&+ \frac{l}{2}\mathcal{C}_2 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\beta-1} \left(1-\frac{t}{T}\right)^{\eta+\beta-1} \\
&\times \ln\left(\frac{4l\mathcal{C}_1 |(-\Delta)\varphi_1(x)| T^{\beta-1} \left(1-\frac{t}{T}\right)^{\eta+\beta-1}}{e\mathcal{C}_3 (T-t)^{1-\gamma}\varphi(x,t)}\right)dxdt \\
&+ \frac{1}{2}\int_0^T \int_{\mathbb{R}^N} \left|\frac{d}{dt}\varphi_2(t)\right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\mathcal{C}_3 (T-t)^{1-\gamma}\varphi(x,t)}\right)dxdt,
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} v_0 \varphi_1^l(x)dx + \frac{3}{4}\mathcal{C}_4 \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} (T-t)^{1-\delta} \varphi(x,t)dxdt \\
&\leq l\mathcal{C}_2 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\beta-1} \left(1-\frac{t}{T}\right)^{\eta+\beta-1} \\
&\times \ln\left(\frac{4l\mathcal{C}_1 |(-\Delta)\varphi_1(x)| T^{\beta-1} \left(1-\frac{t}{T}\right)^{\eta+\beta-1}}{e\mathcal{C}_3 (T-t)^{1-\gamma}\varphi(x,t)}\right)dxdt \\
&+ \int_0^T \int_{\mathbb{R}^N} \left|\frac{d}{dt}\varphi_2(t)\right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\mathcal{C}_3 (T-t)^{1-\gamma}\varphi(x,t)}\right)dxdt \\
&+ \frac{l}{2}\mathcal{C}_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1-\frac{t}{T}\right)^{\eta+\alpha-1} \\
&\times \ln\left(\frac{4l\mathcal{C}_1 |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1-\frac{t}{T}\right)^{\eta+\alpha-1}}{e\mathcal{C}_4 (T-t)^{1-\delta}\varphi(x,t)}\right)dxdt \\
&+ \frac{1}{2}\int_0^T \int_{\mathbb{R}^N} \left|\frac{d}{dt}\varphi_2(t)\right| \ln\left(\frac{4|\frac{d}{dt}\varphi_2(t)|}{e\mathcal{C}_4 (T-t)^{1-\delta}\varphi(x,t)}\right)dxdt.
\end{aligned} \tag{4.14}$$

Let us take the scaled variables $\tau = \frac{t}{T}$ and $y = \frac{x}{T^{\theta/2}}, \theta := \{\alpha; \beta\}$. Then, inequalities (4.13) and (4.14) become

$$\begin{aligned}
&\int_{\mathbb{R}^N} u_0\varphi_1^l(x)dx + \frac{3}{4}\mathcal{C}_3\int_0^T\int_{\mathbb{R}^N} e^{v(x,t)}\,(T-t)^{1-\gamma}\,\varphi(x,t)dxdt \\
&\leq l\mathcal{C}_1T^{\frac{\alpha N}{2}}\int_0^1\int_{\mathbb{R}^N}|(-\Delta_y)\varphi_1(y)|\,(1-\tau)^{\eta+\alpha-1} \\
&\times\ln\left(\frac{4l\mathcal{C}_1|(-\Delta_y)\varphi_1(y)|T^{\alpha-1}\,(1-\tau)^{\eta+\alpha-1}}{e\mathcal{C}_4T^{1-\delta}(1-\tau)^{1-\delta}\varphi(y,\tau)}\right)dyd\tau \\
&+\eta T^{\frac{\alpha N}{2}}\int_0^1\int_{\mathbb{R}^N}(1-\tau)^{\eta-1}\ln\left(\frac{4\eta T^{-1}\,(1-\tau)^{\eta-1}}{e\mathcal{C}_4T^{1-\delta}(1-\tau)^{1-\delta}\varphi(y,\tau)}\right)dyd\tau \\
&+\frac{l}{2}\mathcal{C}_2T^{\frac{\beta N}{2}}\int_0^1\int_{\mathbb{R}^N}|(-\Delta_y)\varphi_1(y)|\,(1-\tau)^{\eta+\beta-1} \\
&\times\ln\left(\frac{4l\mathcal{C}_1|(-\Delta_y)\varphi_1(y)|T^{\beta-1}\,(1-\tau)^{\eta+\beta-1}}{e\mathcal{C}_3T^{1-\gamma}(1-\tau)^{1-\gamma}\varphi(y,\tau)}\right)dyd\tau \\
&+\frac{\eta}{2}T^{\frac{\beta N}{2}}\int_0^1\int_{\mathbb{R}^N}(1-\tau)^{\eta-1}\ln\left(\frac{4\eta T^{-1}\,(1-\tau)^{\eta-1}}{e\mathcal{C}_3T^{1-\gamma}(1-\tau)^{1-\gamma}\varphi(y,\tau)}\right)dyd\tau,
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} v_0\varphi_1^l(x)dx + \frac{3}{4}\mathcal{C}_4\int_0^T\int_{\mathbb{R}^N} e^{u(x,t)}\,(T-t)^{1-\delta}\,\varphi(x,t)dxdt \\
&\leq l\mathcal{C}_2T^{\frac{\beta N}{2}}\int_0^1\int_{\mathbb{R}^N}|(-\Delta_y)\varphi_1(y)|\,(1-\tau)^{\eta+\beta-1} \\
&\times\ln\left(\frac{4l\mathcal{C}_1|(-\Delta_y)\varphi_1(y)|T^{\beta-1}\,(1-\tau)^{\eta+\beta-1}}{e\mathcal{C}_3T^{1-\gamma}(1-\tau)^{1-\gamma}\varphi(y,\tau)}\right)dyd\tau \\
&+\eta T^{\frac{\beta N}{2}}\int_0^1\int_{\mathbb{R}^N}(1-\tau)^{\eta-1}\ln\left(\frac{4\eta T^{-1}\,(1-\tau)^{\eta-1}}{e\mathcal{C}_3T^{1-\gamma}(1-\tau)^{1-\gamma}\varphi(y,\tau)}\right)dyd\tau \\
&+\frac{l}{2}\mathcal{C}_1T^{\frac{\alpha N}{2}}\int_0^1\int_{\mathbb{R}^N}|(-\Delta_y)\varphi_1(y)|\,(1-\tau)^{\eta+\alpha-1} \\
&\times\ln\left(\frac{4l\mathcal{C}_1|(-\Delta_y)\varphi_1(y)|T^{\alpha-1}\,(1-\tau)^{\eta+\alpha-1}}{e\mathcal{C}_4T^{1-\delta}(1-\tau)^{1-\delta}\varphi(y,\tau)}\right)dyd\tau \\
&+\frac{\eta}{2}T^{\frac{\alpha N}{2}}\int_0^1\int_{\mathbb{R}^N}(1-\tau)^{\eta-1}\ln\left(\frac{4\eta T^{-1}\,(1-\tau)^{\eta-1}}{e\mathcal{C}_4T^{1-\delta}(1-\tau)^{1-\delta}\varphi(y,\tau)}\right)dyd\tau.
\end{aligned} \tag{4.16}$$

Similarly, using Lebesgue's dominated convergence theorem, we deduce that the right-hand sides of (4.15)-(4.16) diverge to $-\infty$ when $T \to +\infty$, while the left-hand sides are positive.

This leads to a contradiction. □

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